Marshall and Hicks
Understanding the Ordinary and Compensated Demand

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UTILITY MAXIMIZATION AND THE DEMAND FUNCTIONS

Consider a consumer with the utility function $U = xy$, who faces a budget constraint of $B = P_x x + P_y y$, where B, P_x and P_y are the budget and prices, which are given.

The choice problem is

Maximize

$$U = xy \tag{1}$$

Subject to

$$B = P_x x + P_y y \tag{2}$$

The Lagrangian for this problem is

$$Z = xy + \lambda(B - P_x x - P_y y) \tag{3}$$

The first order conditions are

$$\begin{aligned} Z_x &= y - \lambda P_x = 0 \\ Z_y &= x - \lambda P_y = 0 \\ Z_\lambda &= B - P_x x - P_y y = 0 \end{aligned} \tag{4}$$

Solving the first order conditions yield the following solutions

$$x^M = \tfrac{B}{2P_x} \qquad y^M = \tfrac{B}{2P_y} \tag{5}$$

where x^M and y^M are the consumer's Marshallian demand functions.

Indirect Utility and the Expenditure Function

The Utility function

$$U = U(x, y) = xy$$

is the DIRECT utility function. The name refers to the fact that utility depends directly upon the quantities of x and y the consumer chooses. However, from the Lagrange problem above, we derived the Marshallian (ordinary) demand functions

$$x^M = \tfrac{B}{2P_x} \qquad y^M = \tfrac{B}{2P_x}$$

where the optimal quanities are in, fact, determined by prices (P_x, P_y) and income (B)

If we substitute our answers to the lagrange problem back into the utility function

$$\begin{aligned} U &= xy \\ U &= \left(\frac{B}{2P_x}\right)\left(\frac{B}{2P_y}\right) \end{aligned}$$

$$U = \frac{B^2}{4P_x P_y} \tag{6}$$

this is called the INDIRECT Utility function. The name refers to the fact that, since money and prices determine quantities purchased, and the quantities determine utility, it holds that utility indirectly depends on money an prices.

The Expenditure Function

If we take the indirect utility function an re-arrange it to isolate the budget, B,

$$\begin{aligned} U &= \frac{B^2}{4P_xP_y} \\ 4P_xP_yU &= B^2 \\ B &= \sqrt{4P_xP_yU} \end{aligned}$$

$$B = 2P_x^{1/2}P_y^{1/2}U^{1/2} \tag{7}$$

we have the expenditure function, which is sometimes written as

$$E(P_x, P_y, U) = 2P_x^{1/2}P_y^{1/2}U^{1/2} \tag{8}$$

What is the Expenditure Function?

The expenditure function tells us the minimum budget needed for a consumer to achieve a target level of utility. For example, if $P_x = 4$ and $P_y = 1$ and we want the consumer to have a utility level of $U = 900$, then

$$\begin{aligned} B &= 2P_x^{1/2}P_y^{1/2}U^{1/2} \\ B &= 2\,(4)^{1/2}\,(1)^{1/2}\,(900)^{1/2} \\ B &= 2(2)(1)(30) = 120 \end{aligned}$$

The consumer will need a minimum of \$120 to achieve the utility of 900.

Now suppose the price of x falls to $P_x = 1$. How much budget will the consumer now need to achieve $U = 900$? Using the expenditure function

$$\begin{aligned} B &= 2P_x^{1/2}P_y^{1/2}U^{1/2} \\ B &= 2\,(1)^{1/2}\,(1)^{1/2}\,(900)^{1/2} \\ B &= 2(1)(1)(30) = 60 \end{aligned}$$

Therefore the consumer only needs $60 of budget to achieve the same utility.

CV and EV

The expenditure function is also used to solve for CV and EV. Using our example above, if the consumer initially had an income of $120 and the price of x, $P_x = 4$, then our consumer would have utility of $U = 900$. If the consumer gave up $60 of budget to have the price of x lowered to $P_x = 1$, the consumer would still have a utility of $U = 900$. Therefore the consumer would be indifferent between the old price and $120 or the new price and $60. The difference between the old and new budget when $U = 900$ is the most the consumer would pay to have the price lowered. This is the consumer's compensating variation or CV

Compensated Demand from Expenditure Function

The compensated, or Hicksian, demand function can be derived from the expenditure function by use of a relationship called Shephard's Lemma[1], which states the following:

if

$$E(P_x, P_y U) \tag{9}$$

is the expenditure function, then the compensated demand for x is found by taking the derivative with respect to P_x

$$\frac{\partial E}{\partial P_x} = x^H \tag{10}$$

where x^H denotes "Hicksian Demand". The compensated demand for good y can similarly be found by differentiating the expenditure function with respect to P_y

[1] A *Lemma* is a shortcut used in derivations. Lemma's, like theorems, are results or relationships that have been proven mathematicially.

Illustration:.—

Using our expenditure function from above

$$E(P_x, P_y, U) = 2P_x^{1/2}P_y^{1/2}U^{1/2}$$

We can find x^H

$$\begin{aligned}
\frac{\partial E}{\partial P_x} &= 2\left(\frac{1}{2}P_x^{-1/2}\right)P_y^{1/2}U^{1/2} \\
\frac{\partial E}{\partial P_x} &= P_x^{-1/2}P_y^{1/2}U^{1/2} \\
\frac{\partial E}{\partial P_x} &= \frac{P_y^{1/2}U^{1/2}}{P_x^{1/2}}
\end{aligned}$$

$$x^H = \frac{P_y^{1/2}U^{1/2}}{P_x^{1/2}} \tag{11}$$

HICKS VERSUS MARSHALL

From the utility maximization problem with $U = xy$ we derived the **Marshallian Demand** function as

$$x^M = \frac{B}{2P_x}$$

and from the Expenditure Function we derived the **Hicksian Demand** function as

$$x^H = \frac{P_y^{1/2}U^{1/2}}{P_x^{1/2}}$$

Both are the consumer's demand function for X.

The question you may ask is *"How do they differ?"*and "*Why do we need two demand functions?*".

First, notice that the Marshallian demand is a function of prices and **BUDGET** while the Hicksian demand is a function of prices and **UTILITY.** This is the key distinction.

- Marshall measured changes in demand when MONEY INCOME is held constant. Marshall also measures the total (or net) effect of a change in price.
- Hicks measures the change in demand when UTILITY (or REAL INCOME) is held constant. It measures the change in demand along an indifference curve.
- Hicks measures the substitution effect where as marshall measures the total effect. The difference between Marshall and Hicks should be the income effect.

Example 1 *Suppose the budget is initially $B = 24$, the price of x is $P_x = 2$ and the price of y is $P_y = 1$. From marshall we get*

$$\begin{aligned} x^M &= \frac{B}{2P_x} = \frac{24}{(2)(2)} = 6 \\ y^M &= \frac{B}{2P_y} = \frac{24}{(2)(1)} = 12 \end{aligned}$$

and utility would be $U = xy = 72$. Substituting prices and utility into the Hicksian demand for x

$$x^H = \frac{P_y^{1/2}U^{1/2}}{P_x^{1/2}} = \frac{(1)^{1/2}(72)^{1/2}}{(2)^{1/2}} = 6$$

So we see, at the initial point $x^H = x^M = 6$. This is what we should expect as both Hicks and Marshall have the same starting point.

Now let's see what happens when the price of x falls to $P_x = 1$. According to the Marshallian demand

$$x^M = \frac{B}{2P_x} = \frac{24}{(2)(1)} = 12$$

but according to Hicksian demand

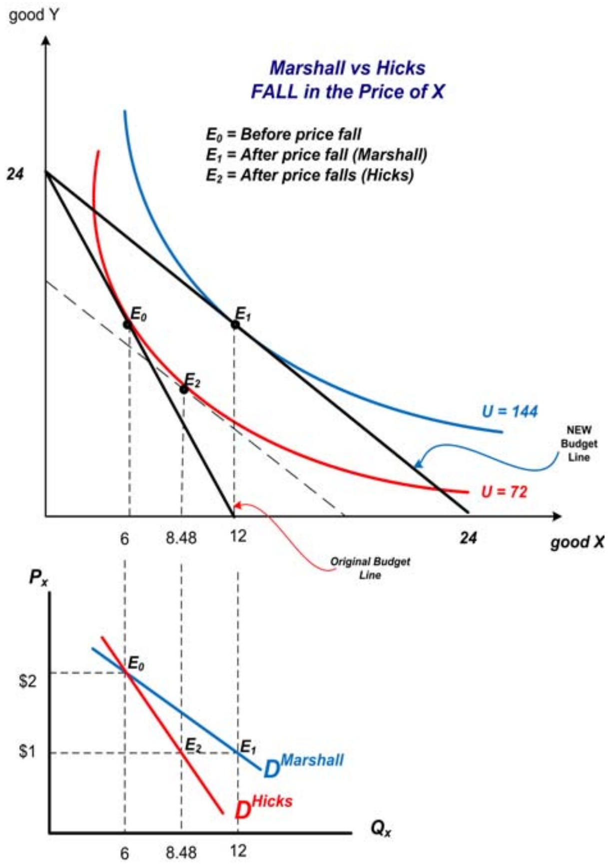
$$x^H = \frac{P_y^{1/2}U^{1/2}}{P_x^{1/2}} = \frac{(1)^{1/2}(72)^{1/2}}{(1)^{1/2}} = 8.48$$

For the fall in the price of X,

Marshall says the demand for x will go from 6 to 12

Hicks says the demand for x will go from 6 to 8.48

Hicks represents the "pure substitution effect" while Marshall is the "total effect". The difference between the two $(12 - 8.48 = 3.52)$ *is therefore the income effect. The results are illustrated in figure 1*



FINDING HICKS FROM LAGRANGE MINIMIZATION PROBLEM

Above we used Shephard's lemma to find the compensated demand function from the expenditure function. However, the compensated, or Hicksian", demands can be found by using Lagrange to minimize expenditure subject to a utility constraint. The following example essentially "proves" shephard's lemma.

Minimize expenditure

$$E = P_x x + P_y y \tag{12}$$

Subject to

$$U_0 = xy \tag{13}$$

The Lagrangian for the problem is

$$Z = P_x x + P_y y + \lambda(U_0 - xy) \tag{14}$$

The first order conditions are

$$\begin{aligned} Z_x &= P_x - \lambda y = 0 \\ Z_y &= P_y - \lambda x = 0 \\ Z_\lambda &= U_0 - xy = 0 \end{aligned} \tag{15}$$

Solving the system of equations for x and y gives us

$$\begin{aligned} x^H &= \left(\tfrac{P_y U_0}{P_x}\right)^{\frac{1}{2}} = \tfrac{P_y^{1/2} U_0^{1/2}}{P_x^{1/2}} \\ y^H &= \left(\tfrac{P_x U_0}{P_y}\right)^{\frac{1}{2}} = \tfrac{P_x^{1/2} U_0^{1/2}}{P_y^{1/2}} \end{aligned} \tag{16}$$

which denoted by the superscript H for "Hicksian" demands.

If we substitute x^H and y^H into the objective function,

$$\begin{aligned} E &= P_x \frac{P_y^{1/2} U_0^{1/2}}{P_x^{1/2}} + P_y \frac{P_x^{1/2} U_0^{1/2}}{P_y^{1/2}} \\ E &= \frac{P_x^{1/2} P_y^{1/2} U_0^{1/2}}{1} + \frac{P_y^{1/2} P_x^{1/2} U_0^{1/2}}{1} \\ E &= 2 P_x^{1/2} P_y^{1/2} U_0^{1/2} \end{aligned} \tag{17}$$

Which is the same expenditure we derived earlier from the indirect utility function.