Chapter 4 Differentiation

Section 4.1 The derivative of a function

Practice Problems

1 **(a)** $\frac{11-3}{3-(-1)}=\frac{8}{4}=2$

(b) $\frac{-2-3}{4-(-1)}=\frac{-5}{5}=-1$

(c) $\frac{3-3}{49-(-1)}=\frac{0}{50}=0$

2 Using a calculator, the values of the cube function, correct to 2 decimal places, are

x	–1.50	–1.25	–1.00	–0.75	
$f(x)$	–3.38	–1.95	–1.00	–0.42	
x	–0.50	–0.25	0.00	0.25	0.50
$f(x)$	–0.13	–0.02	0.00	0.02	0.13
x	0.75	1.00	1.25	1.50	
$f(x)$	0.42	1.00	1.95	3.38	

The graph of the cube function is sketched in Figure S4.1.

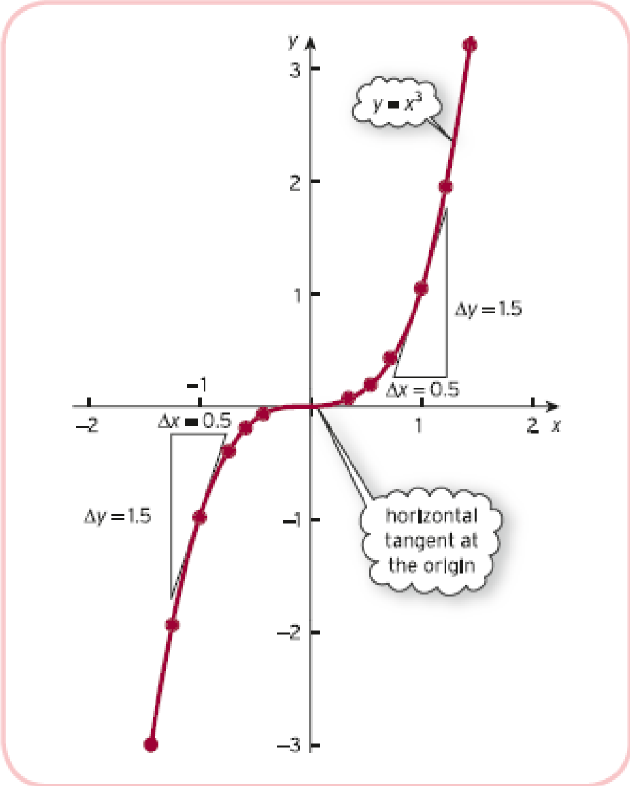

Figure S4.1

$$f'(-1) = \frac{1.5}{0.5} = 3.0$$

$f'(0) = 0$ (because the tangent is horizontal at $x = 0$)

$$f'(1) = \frac{1.5}{0.5} = 3.0$$

[Note: $f'(-1) = f'(1)$ because of the symmetry of the graph.]

3 If $n = 3$ then the general formula gives

$$f'(x) = 3x^{3-1} = 3x^2$$

Hence

$$f'(-1) = 3(-1)^2 = 3$$

$$f'(0) = 3(0)^2 = 0$$

$$f'(1) = 3(1)^2 = 3$$

4 **(a)** $5x^4$; **(b)** $6x^5$; **(c)** $100x^{99}$;

(d) $-x^{-2}$ (that is, $-1/x^2$);

(e) $-2x^{-3}$ (that is, $-2/x^3$).

Exercise 4.1 (p. 249)

1 **(a)** $\frac{9-5}{4-2}=2$; **(b)** $\frac{-5-(-1)}{7-3}=-1$; **(c)** $\frac{19-19}{4-7}=0$.

2 $2\times 0+3\times 2=6$ and $2\times 3+3\times 0=6$

$\frac{0-2}{3-0}=-\frac{2}{3}$ which is negative so the line is downhill.

3 The graph of $f(x) = 5$ is sketched in Figure S4.2. The graph is horizontal, so has zero slope at all values of x.

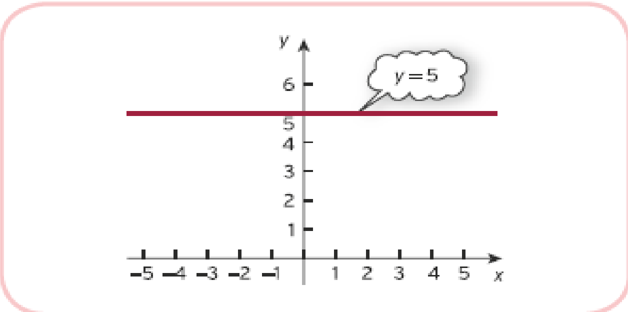

Figure S4.2

4 $7x^6$; $f'(2)=7\times 2^6=448$

5 **(a)** $8x^7$; **(b)** $50x^{49}$; **(c)** $19x^{18}$; **(d)** $999x^{998}$.

6 **(a)** $f(x)=x^{-3}\Rightarrow f'(x)=-3x^{-4}=\frac{-3}{x^4}$

(b) $f(x)=x^{\frac{1}{2}}\Rightarrow f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

(c) $f(x) = x^{-\frac{1}{2}} \Rightarrow -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}} = -\frac{1}{2x\sqrt{x}}$

(d) $f(x) = x^1 x^{\frac{1}{2}} = x^{\frac{3}{2}} \Rightarrow f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$

7 3, 1.25, 0, –0.75, –1, –0.75, 0, 1.25;

(a) –3; **(b)** 0; **(c)** 1.

Section 4.2 Rules of differentiation

Practice Problems

1 **(a)** $4(3x^2) = 12x^2$.

(b) $-2/x^3$ because $1/x = x^{-1}$, which differentiates to $-x^{-2}$.

2 **(a)** $5x^4 + 1$; (b) $2x + 0 = 2x$.

3 **(a)** $2x - 3x^2$; (b) $0 - (-3x^{-4}) = \frac{3}{x^4}$.

4 **(a)** $9(5x^4) + 2(2x) = 45x^4 + 4x$

(b) $5(8x^7) - 3(-1)x^{-2} = 40x^7 + 3/x^2$

(c) $2x + 6(1) + 0 = 2x + 6$

(d) $2(4x^3) + 12(3x^2) - 4(2x) + 7(1) - 0$

$= 8x^3 + 36x^2 - 8x + 7$

5 $f'(x) = 4(3x^2) - 5(2x) = 12x^2 - 10x$

$f''(x) = 12(2x) - 10(1) = 24x - 10$

$f''(6) = 24(6) - 10 = 134$

Exercise 4.2 (p. 258)

1 **(a)** $10x$; **(b)** $-3/x^2$; **(c)** 2; **(d)** $2x + 1$;

(e) $2x - 3$; **(f)** $3 + 7/x^2$; **(g)** $6x^2 - 12x + 49$;

(h) a; **(i)** $2ax + b$;

(j) $f(x) = 4x^{\frac{1}{2}} - 3x^{-1} + 7x^{-2} \Rightarrow f'(x) = 2x^{-\frac{1}{2}} + 3x^{-2} - 14x^{-3} = \dfrac{2}{\sqrt{x}} + \dfrac{3}{x^2} - \dfrac{14}{x^3}$

2 **(a)** $f'(x) = 27x^8 \Rightarrow f'(1) = 27$

(b) $f'(x) = 2x - 2 \Rightarrow f'(3) = 4$

(c) $f'(x) = 3x^2 - 8x + 2 \Rightarrow f'(0) = 2$

(d) $f(x) = 5x^4 - 4x^{-4} \Rightarrow f'(x) = 20x^3 + 16x^{-5} \Rightarrow f'(-1) = -36$

(e) $f(x) = x^{\frac{1}{2}} - 2x^{-1} \Rightarrow f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} + 2x^{-2} \Rightarrow f'(x) = \dfrac{3}{8}$

3 $4x^3 + 6x$;

(a) $f(x) = 3x^3 - 4x^2 \Rightarrow f'(x) = 9x^2 - 8x$

(b) $f(x) = 3x^4 - 2x^3 + 6x^2 - 7x \Rightarrow f'(x) = 12x^3 - 6x^2 + 12x - 7$

(c) $f(x) = x^2 - 5x - 6 \Rightarrow f'(x) = 2x - 5$

(d) $f(x) = x - 3x^{-1} \Rightarrow f'(x) = 1 + 3x^{-2} = 1 + \dfrac{3}{x^2}$

(e) $f(x) = x^{-2} - 4x^{-1} \Rightarrow f'(x) = -2x^{-3} + 4x^{-2} = -\dfrac{2}{x^3} + \dfrac{4}{x^2}$

(f) $f(x) = 1 - 3x^{-1} + 5x^{-2} \Rightarrow f'(x) = 3x^{-2} - 10x^{-3} = \dfrac{3}{x^2} - \dfrac{10}{x^3}$

4 **(a)** $\dfrac{dy}{dx} = 14x - 1 \Rightarrow \dfrac{d^2y}{dx^2} = 14$

(b) $y = x^{-2} \Rightarrow \dfrac{dy}{dx} = -2x^{-3} \Rightarrow \dfrac{d^2y}{dx^2} = 6x^{-4} = \dfrac{6}{x^4}$

(c) $\frac{dy}{dx} = a \Rightarrow \frac{d^2y}{dx^2} = 0$

5 $f'(x) = 3x^2 - 8x + 10 \Rightarrow f''(x) = 6x - 8 \Rightarrow f''(2) = 4$

6 $f'(x) = 2x - 6 \Rightarrow f'(3) = 0$; horizontal tangent, i.e. vertex of parabola must be at $x = 3$.

7 $f(x) = 2x^{\frac{1}{2}} \Rightarrow f'(x) = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

(a) $f(x) = \sqrt{25x} = \sqrt{25} \times \sqrt{x} = 5x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{5}{2}x^{-\frac{1}{2}} = \frac{5}{2\sqrt{x}}$

(b) $f(x) = \sqrt[3]{27x} = \sqrt[3]{27} \times \sqrt[3]{x} = 3x^{\frac{1}{3}} \Rightarrow f'(x) = x^{-\frac{2}{3}}$

(c) $f(x) = \sqrt[4]{16x^3} = \sqrt[4]{16} \times \sqrt[4]{x^3} = 2x^{\frac{3}{4}} \Rightarrow f'(x) = \frac{3}{2}x^{-\frac{1}{4}} = \frac{3}{2\sqrt[4]{x}}$

(d) $f(x) = \sqrt{\frac{25}{x}} = \frac{\sqrt{25}}{\sqrt{x}} = 5x^{-\frac{1}{2}} \Rightarrow f'(x) = -\frac{5}{2}x^{-\frac{3}{2}}$

8 **(a)** $2P + 1$

(b) $50 - 6Q$;

(c) $AC = 30Q^{-1} + 10 \Rightarrow \frac{d(AC)}{dQ} = -30Q^{-2} = -\frac{30}{Q^2}$

(d) 3;

(e) $Q = 10L^{\frac{1}{2}} \Rightarrow \frac{dQ}{dL} = 5L^{-\frac{1}{2}} = \frac{5}{\sqrt{L}}$

(f) $-6Q^2 + 30Q - 24$.

Section 4.3 Marginal functions

Practice Problems

1 TR = $PQ = (60 - Q)Q = 60Q - Q^2$

(1) MR = $60 - 2Q$

When $Q = 50$, MR = $60 - 2(50) = -40$

(2) **(a)** TR = $60(50) - (50)^2 = 500$

(b) TR = $60(51) - (51)^2 = 459$

so TR changes by –41, which is approximately the same as the exact value obtained in part (1).

2 MR = $1000 - 8Q$, so when $Q = 30$

MR = $1000 - 8(30) = 760$

(a) $\Delta(\text{TR}) \cong \text{MR} \times \Delta Q = 760 \times 3 = 2280$, so total revenue rises by about 2280.

(b) $\Delta(\text{TR}) \cong \text{MR} \times \Delta Q = 760 \times (-2)$

$= -1520$

so total revenue falls by about 1520.

3 $\text{TC} = (\text{AC})Q = \left(\frac{100}{Q} + 2\right)Q = 100 + 2Q$

This function differentiates to give MC = 2, so a 1 unit increase in Q always leads to a 2 unit increase in TC irrespective of the level of output.

4 If $K = 100$ then

$Q = 5L^{1/2}(100)^{1/2} = 50L^{1/2}$ because $\sqrt{100} = 10$.

Differentiating gives

$$MP_L = 50(\tfrac{1}{2}L^{-1/2}) = \frac{25}{\sqrt{L}}$$

(a) $\frac{25}{\sqrt{1}} = 25$ **(b)** $\frac{25}{\sqrt{9}} = \frac{25}{3} = 8.3$;

(c) $\frac{25}{\sqrt{10000}} = \frac{25}{100} = 0.25$.

The fact that these values decrease as L increases suggests that the law of diminishing marginal productivity holds for this function. This can be confirmed by differentiating a second time to get

$$\frac{d^2Q}{dL^2} = 25(-1/2L^{-3/2}) \quad \frac{-25}{2L^{3/2}}$$

which is negative for all values of L.

5 The savings function is given, so we begin by finding MPS. Differentiating S with respect to Y gives

MPS = $0.04Y - 1$

so when $Y = 40$,

MPS = $0.04(40) - 1 = 0.6$

To find MPC we use the formula

MPC + MPS = 1

that is,

MPC = 1 – MPS = 1 – 0.6 = 0.4

This indicates that, at the current level of income, a 1 unit increase in national income causes a rise of about 0.6 units in savings and 0.4 units in consumption.

Exercise 4.3 (p. 273)

1 $TR = PQ = (100 - 4Q)Q = 100Q - 4Q^2$

$$MR = \frac{d(TR)}{dQ} = 100 - 8Q$$

When $Q = 12$, MR = 4 so $\Delta(TR) \approx 4 \times 0.3 = 1.2$

2 TR = $PQ = (80 - 3Q)Q = 80Q - 3Q^2$,

so MR = $\dfrac{d(TR)}{dQ} = 80 - 6Q = 80 - 6(80 - P)/3 = 2P - 80$.

3 $TR = PQ = (100 - Q)Q = 100Q - Q^2$

$$MR = \frac{d(TR)}{dQ} = 100 - 2Q$$

Graphs of TR and MR are sketched in Figures S4.3 and S4.4 respectively.

MR = 0 when Q = 50. This is the value of Q at which TR is a maximum.

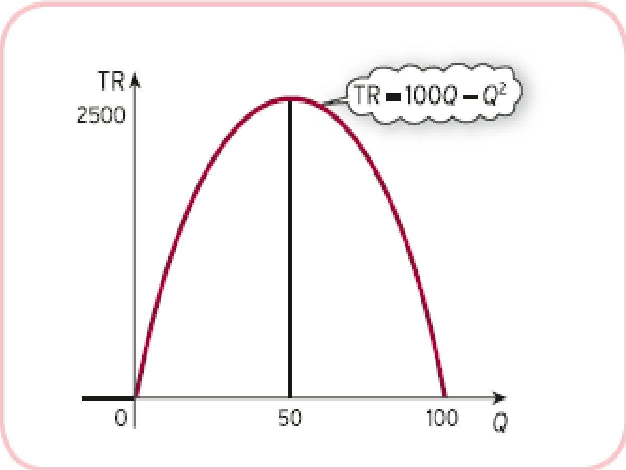

Figure S4.3

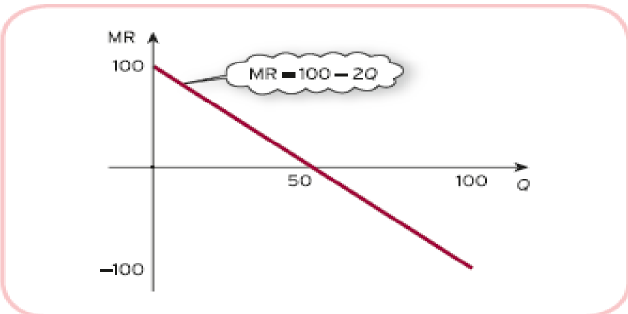

Figure S4.4

4 $TC = AC \times Q = \left(\frac{15}{Q} + 2Q + 9\right)Q = 15 + 2Q^2 + 9Q$

Fixed costs are 15 since this is the constant term in the expression for TC

$$MC = \frac{d(TC)}{dQ} = 4Q + 9$$

5 $MP_L = \frac{dQ}{dL} = 50 - 0.02L$

(a) 49.98; **(b)** 49.8;

(c) 48; **(d)** 30.

Yes, because $d^2Q/dL^2 = -0.02 < 0$.

6 $C = 50 + 2\sqrt{Y} = 50 + 2Y^{\frac{1}{2}} \Rightarrow MPC = \frac{dC}{dY} = Y^{-\frac{1}{2}}$

When $Y = 36$, MPC = $\frac{1}{6}$ so $MPS = 1 - MPC = \frac{5}{6}$

If national income rises by 1 unit, the approximate increase in consumption and savings is 1/6 and 5/6 respectively.

7 $MPC = \frac{dC}{dY} = 0.04Y + 0.1$

$MPS = 1 - MPC = 1 - (0.04Y + 0.1) = 0.9 - 0.04Y$

$$\Rightarrow 0.9 - 0.04Y = 0.38$$
$$\Rightarrow -0.04Y = -0.52$$
$$\Rightarrow Y = 13$$

Section 4.4 Further rules of differentiation

Practice Problems

1 **(a)** The outer power function differentiates to get $5(3x - 4)^4$ and the derivative of the inner function, $3x - 4$, is 3, so

$$\frac{dy}{dx} = 5(3x - 4)^4(3) = 15(3x - 4)^4$$

(b) The outer power function differentiates to get $3(x^2 + 3x + 5)^2$ and the derivative of the inner function, $x^2 + 3x + 5$, is $2x + 3$, so

$$\frac{dy}{dx} = 3(x^2 + 3x + 5)^2(2x + 3)$$

(c) Note that $y = (2x - 3)^{-1}$. The outer power function differentiates to get $-(2x - 3)^{-2}$ and the derivative of the inner function, $2x - 3$, is 2, so

$$\frac{dy}{dx} = -(2x - 3)^{-2}(2) = \frac{-2}{(2x-3)^2}$$

(d) Note that $y = (4x - 3)^{1/2}$. The outer power function differentiates to get $\tfrac{1}{2}(4x - 3)^{-1/2}$ and the derivative of the inner function, $4x - 3$, is 4, so

$$\frac{dy}{dx} = \tfrac{1}{2}(4x - 3)^{-1/2}(4) = \frac{2}{\sqrt{(4x-3)}}$$

2 **(a)** $u = x \qquad v = (3x - 1)^6$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 6(3x - 1)^5$$

So

$$\frac{dy}{dx} = 18x(3x - 1)^5 + (3x - 1)^6$$

$$= (3x - 1)^5\,[18x + (3x - 1)]$$

$$= (3x - 1)^5(21x - 1)$$

(b) $u = x^3 \qquad v = (2x + 3)^{1/2}$

$$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = \tfrac{1}{2}(2x + 3)^{-1/2}$$

$$= \frac{1}{\sqrt{(2x+3)}}$$

So

$$\frac{dy}{dx} = \frac{x^3}{\sqrt{(2x+3)}} + 3x^2\sqrt{(2x+3)}$$

(c) $u = x$ $v = (x-2)^{-1}$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -(x-2)^{-2}$$

So

$$\begin{aligned}\frac{dy}{dx} &= \frac{-x}{(x-2)^2} + \frac{1}{x-2} \\ &= \frac{-x+(x-2)}{(x-2)^2} \\ &= \frac{-2}{(x-2)^2}\end{aligned}$$

3 **(a)** $u = x$ $v = x - 2$

$$\frac{dy}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

So

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x-2)-x}{(x-2)^2} \\ &= \frac{-2}{(x-2)^2}\end{aligned}$$

(b) $u = x - 1$ $v = x + 2$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

So

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+1)-(x-1)}{(x+1)^2} \\ &= \frac{2}{(x+1)^2}\end{aligned}$$

Exercise 4.4 (p. 283)

1 **(a)** $\frac{dy}{dx}=3(5x+1)^2\times 5=15(5x+1)^2$

(b) $\frac{dy}{dx}=8(2x-7)^7\times 2=16(2x-7)^7$

(c) $\frac{dy}{dx}=5(x+9)^4\times 1=5(x+9)^4$;

(d) $\frac{dy}{dx}=3(4x^2-7)\times 8x=24x(4x^2-7)^2$;

(e) $\frac{dy}{dx}=4(x^2+4x-3)\times(2x+4)=8(x+2)(x^2+4x-3)$;

(f) $y=(2x+1)^{\frac{1}{2}}\Rightarrow\frac{dy}{dx}=\frac{1}{2}(2x+1)^{-\frac{1}{2}}\times 2=\frac{1}{\sqrt{2x+1}}$;

(g) $y=(3x+1)^{-1}\Rightarrow\frac{dy}{dx}=(-1)(3x+1)^{-2}\times 3=\frac{-3}{(3x+1)^2}$

(h) $y=(4x-3)^{-2}\Rightarrow\frac{dy}{dx}=(-2)(4x-3)^{-3}\times 4=\frac{-8}{(4x-3)^3}$;

(i) $y=(2x+5)^{-\frac{1}{2}}\Rightarrow\frac{dy}{dx}=-\frac{1}{2}(2x+5)^{-\frac{3}{2}}\times 2=-(2x+3)^{-\frac{3}{2}}=\frac{-1}{(2x+3)\sqrt{(2x+3)}}$

2 **(a)** $1\times(3x+4)^2+x\times 2(3x+4)^1(3)=(3x+4)^2+6x(3x+4)=(9x+4)(3x+4)$;

(b) $2x\times(x-2)^3+x^2\times 3(x-2)^2(1)=2x(x-2)^3+3x^2(x-2)^2=x(5x-4)(x-2)^2$;

(c) $y=x(x+2)^{\frac{1}{2}}\Rightarrow\frac{dy}{dx}=1\times(x+2)^{\frac{1}{2}}+x\times\frac{1}{2}(x+2)^{-\frac{1}{2}}(1)=\sqrt{x+2}+\frac{x}{2\sqrt{x+2}}=\frac{3x+4}{2\sqrt{x+2}}$

(d) $1\times(x+6)^3+(x-1)\times 3(x+6)^2(1)=(x+6)^3+3(x-1)(x+6)^2=(4x+3)(x+6)^2$;

(e) $2\times(x+5)^3+(2x+1)\times 3(x+5)^2(1)=2(x+5)^3+3(2x+1)(x+5)^2=(8x+13)(x+5)^2$;

(f)

$$3x^2 \times (2x-5)^4 + x^3 \times 4(2x-5)^3(2) = 3x^2(2x-5)^4 + 8x^3(2x-5)^3 = x^2(14x-15)(2x-5)^3.$$

3 **(a)** $\dfrac{(x-5)\times 1 - x\times 1}{(x-5)^2} = \dfrac{x-5-x}{(x-5)^2} = \dfrac{-5}{(x-5)^2}$;

(b) $\dfrac{(x+7)\times 1 - x\times 1}{(x+7)^2} = \dfrac{x+7-x}{(x+7)^2} = \dfrac{7}{(x+7)^2}$

(c) $\dfrac{(x-2)\times 1-(x+3)\times 1}{(x-2)^2} = \dfrac{x-2-x-3}{(x-2)^2} = \dfrac{-5}{(x-2)^2}$;

(d) $\dfrac{(3x+1)\times 2-(2x+9)\times 3}{(3x+1)^2} = \dfrac{6x+2-6x-27}{(3x+1)^2} = \dfrac{-25}{(3x+1)^2}$

(e) $\dfrac{(5x+6)\times 1 - x\times 5}{(5x+6)^2} = \dfrac{5x+6-5x}{(5x+6)^2} = \dfrac{6}{(5x+6)^2}$;

(f) $\dfrac{(3x-7)\times 1-(x+4)\times 3}{(3x-7)^2} = \dfrac{3x-7-3x-12}{(3x-7)^2} = \dfrac{-19}{(3x-7)^2}$

4 **(a)** $2(5x+7)(5) = 10(5x+7)$

(b) $y = 25x^2 + 70x + 49 \Rightarrow \dfrac{dy}{dx} = 50x + 70 = 10(5x+7)$

5 **(a)**
$$5x^4 \times (x+2)^2 + x^5 \times 2(x+2)(1) = 5x^4(x+2)^2 + 2x^5(x+2)$$
$$= 5x^4(x^2+4x+4) + 2x^6 + 4x^5 = 7x^6 + 24x^5 + 20x^4$$

(b) $y = x^5(x^2+4x+4) = x^7 + 4x^6 + 4x^5 \Rightarrow \dfrac{dy}{dx} = 7x^6 + 24x^5 + 20x^4$

6 **(a)**

$$TR = PQ = (100-Q)^3 Q$$
$$MR = \frac{d(TR)}{dQ} = 3(100-Q)^2(-1)\times Q + (100-Q)^3 \times 1 = -3Q(100-Q)^2 + (100-Q)^3$$

$$= (100-4Q)(100-Q)^2;$$

(b) $TR = PQ = \dfrac{1000Q}{Q+4}$

$$MR = \frac{d(TR)}{dQ} = \frac{(Q+4)\times 1000 - 1000Q\times 1}{(Q+4)^2} = \frac{4000}{(Q+4)^2}$$

7 $MPC = \dfrac{dC}{dY} = \dfrac{(1+Y)\times 4Y - (300+2Y^2)\times 1}{(1+Y)^2} = \dfrac{2Y^2+4Y-300}{(1+Y)^2}$

When Y = 36, MPC = 1.78 and MPS = $1 - MPC$ = –0.78. If national income rises by 1 unit, consumption rises by 1.78 units, whereas savings actually fall by 0.78 units.

Section 4.5 Elasticity

Practice Problems

1 We are given that P_1 = 210 and P_2 = 200. Substituting P = 210 into the demand equation gives

$$\begin{aligned} 1000 - 2Q_1 &= 210 \\ -2Q_1 &= -790 \\ Q_1 &= 395 \end{aligned}$$

Similarly, putting P = 200 gives Q_2 = 400. Hence

$$\Delta P = 200 - 210 = -10$$
$$\Delta Q = 400 - 395 = 5$$

Averaging the P values gives

$$P = \tfrac{1}{2}(210 + 200) = 205$$

Averaging the Q values gives

$$Q = \tfrac{1}{2}(395 + 400) = 397.5$$

Hence, arc elasticity is

$$-\left(\frac{205}{397.5}\right) \times \left(\frac{5}{-10}\right) = 0.26$$

2 The quickest way of solving this problem is to find a general expression for E in terms of P and then just to replace P by 10, 50 and 90 in turn. The equation

$$P = 100 - Q$$

rearranges as

$$Q = 100 - P$$

so

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = -1$$

Hence

$$E = -\frac{P}{Q} \times \frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{-P}{100 - P} \times (-1)$$
$$= \frac{P}{100 - P}$$

(a) If $P = 10$ then $E = 1/9 < 1$ so inelastic.

(b) If $P = 50$ then $E = 1$ so unit elastic.

(c) If $P = 90$ then $E = 9$ so elastic.

At the end of Section 4.5 it is shown quite generally that the price elasticity of demand for a linear function

$$P = aQ + b$$

is given by

$$E = \frac{P}{b - P}$$

The above is a special case of this with $b = 100$.

3 Substituting $Q = 4$ into the demand equation gives

$$P = -(4)^2 - 10(4) + 150 = 94$$

Differentiating the demand equation with respect to Q gives

$$\frac{dP}{dQ} = -2Q - 10$$

so

$$\frac{dQ}{dP} = \frac{1}{-2Q - 10}$$

When $Q = 4$

$$\frac{dQ}{dP} = -\frac{1}{18}$$

The price elasticity of demand is then

$$-\left(\frac{94}{4}\right) \times \left(-\frac{1}{18}\right) = \frac{47}{36}$$

From the definition

$$E = -\frac{\text{percentage change in demand}}{\text{percentage change in price}}$$

we have

$$\frac{47}{36} = -\frac{10}{\text{percentage change in price}}$$

Hence the percentage change in price is $-10 \times 36/47 = -7.7\%$: that is, the firm must reduce prices by 7.7% to achieve a 10% increase in demand.

4 **(a)** Putting $P = 9$ and 11 directly into the supply equation gives $Q = 203.1$ and 217.1 respectively, so

$$\Delta P = 11 - 9 = 2$$
$$\Delta Q = 217.1 - 203.1 = 14$$

Averaging the P values gives

$$P = \tfrac{1}{2}(9+11) = 10$$

Averaging the Q values gives

$$Q = \tfrac{1}{2}(203.1+217.1) = 210.1$$

Arc elasticity is

$$\frac{10}{210.1} \times \frac{14}{2} = 0.333175$$

(b) Putting P = 10 directly into the supply equation, we get Q = 210. Differentiating the supply equation immediately gives

$$\frac{dQ}{dP} = 5 + 0.2P$$

so when P = 10, dQ/dP = 7. Hence

$$E = \frac{10}{210} \times 7 = \frac{1}{3}$$

Note that, as expected, the results in parts (a) and (b) are similar. They are not identical, because in part (a) the elasticity is 'averaged' over the arc from P = 9 to P = 11, whereas in part (b) the elasticity is evaluated exactly at the midpoint, P = 10.

Exercise 4.5 (p. 298)

1 When Q = 8, $P = 500 - 4 \times 8^2 = 244$

When Q = 10, $P = 500 - 4 \times 10^2 = 100$

$$\Delta P = 100 - 244 = -144$$
$$\Delta Q = 10 - 8 = 2$$
$$P = \frac{1}{2}(244 + 100) = 172$$
$$Q = \frac{1}{2}(8 + 10) = 9$$

Hence $E = -\dfrac{P}{Q} \times \dfrac{\Delta Q}{\Delta P} = -\dfrac{172}{9} \times \dfrac{2}{-144} = \dfrac{43}{162} = 0.27$ (to 2 decimal places)

2 Putting Q = 9 into the demand function gives $P = 500 - 4 \times 9^2 = 176$

$\frac{dP}{dQ} = -8Q$ so when Q = 9, $\frac{dP}{dQ} = -72$

Hence $\frac{dQ}{dP} = -\frac{1}{72}$

$E = -\frac{P}{Q} \times \frac{dQ}{dP} = -\frac{176}{9} \times \frac{-1}{72} = \frac{22}{81} = 0.27$ (to 2 decimal places)

The values agree to 2 decimal places.

3 **(a)**

$$\begin{aligned} &30 - 2Q = 6 \\ &-2Q = -24 \\ &Q = 12 \end{aligned}$$

$\frac{dP}{dQ} = -2$ so $\frac{dP}{dQ} = \frac{-1}{2}$

Hence

$$E = -\frac{P}{Q} \times \frac{dQ}{dP} = -\frac{6}{12} \times \frac{-1}{2} = \frac{1}{4}$$

(b)

$$\begin{aligned} &30 - 12Q = 6 \\ &-12Q = -24 \\ &Q = 2 \end{aligned}$$

$\frac{dP}{dQ} = -12$ so $\frac{dP}{dQ} = \frac{-1}{12}$

Hence

$$E = -\frac{P}{Q} \times \frac{dQ}{dP} = -\frac{6}{2} \times \frac{-1}{12} = \frac{1}{4}$$

(c)

$$\sqrt{100-2Q}=6$$
$$100-2Q=36$$
$$-2Q=-64$$
$$Q=32$$

$$P=\left(100-2Q\right)^{\frac{1}{2}} \Rightarrow \frac{dP}{dQ}=\frac{1}{2}\left(100-2Q\right)^{-\frac{1}{2}}\times(-2)=-\left(100-2Q\right)^{-\frac{1}{2}}$$

so $\frac{dP}{dQ}=\frac{-1}{6}$ giving $\frac{dQ}{dP}=-6$

Hence

$$E=-\frac{P}{Q}\times\frac{dQ}{dP}=-\frac{6}{32}\times(-6)=\frac{9}{8}$$

4 **(a)** $0.2P$.

(b) $0.1P^2=Q-4$

$$P^2=10(Q-4)=10Q-40$$

$P=\sqrt{(10Q-40)}$

$$P=(10Q-40)^{\frac{1}{2}} \Rightarrow \frac{dP}{dQ}=\frac{1}{2}(10Q-40)^{-\frac{1}{2}}\times(10)=\frac{5}{\sqrt{(10Q-40)}}$$

(c) $\frac{1}{\mathrm{d}P/\mathrm{d}Q}=\frac{\sqrt{(10Q-40)}}{5}$

$$=\frac{P}{5}=0.2P=\frac{\mathrm{d}Q}{\mathrm{d}P}$$

(d) $Q=14 \Rightarrow P=10$ and $\frac{dQ}{dP}=2$

$$E=\frac{P}{Q}\times\frac{dQ}{dP}=\frac{10}{14}\times2=\frac{10}{7}$$

5 $\frac{dQ}{dP} = 0.1 + 0.008P$

At P = 80, Q = 40.6 and $\frac{dQ}{dP} = 0.74$

Hence $E = \frac{P}{Q} \times \frac{dQ}{dP} = \frac{80}{40.6} \times 0.74 = 1.46$

(a) elastic; **(b)** % change in supply = 5×1.46 = 7.3 %.

Section 4.6 Optimization of economic functions

Practice Problems

1 **(a)** ***Step 1***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x + 12 = 0$$

has solution x = –2.

Step 2

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6 > 0$$

so minimum.

Finally, note that when x = –2, y = –47, so the minimum point has coordinates (–2, –47).

A graph is sketched in Figure S4.5.

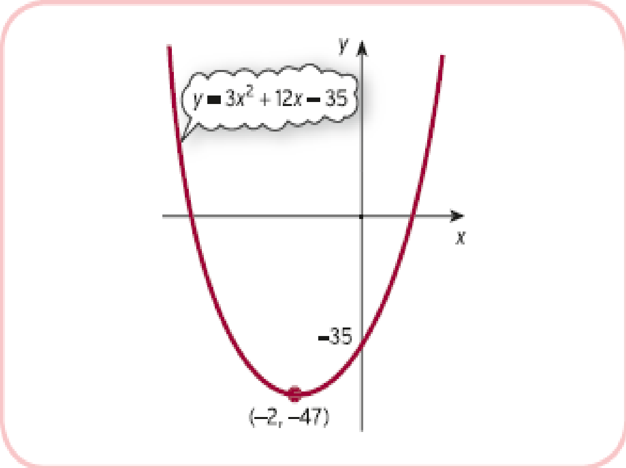

Figure S4.5

(b) ***Step 1***

$$\frac{dy}{dx} = -6x^2 + 30x - 36 = 0$$

has solutions $x = 2$ and $x = 3$.

Step 2

$$\frac{d^2y}{dx^2} = -12x + 30$$

which takes the values 6 and –6 at $x = 2$ and $x = 3$ respectively. Hence minimum at $x = 2$ and maximum at $x = 3$.

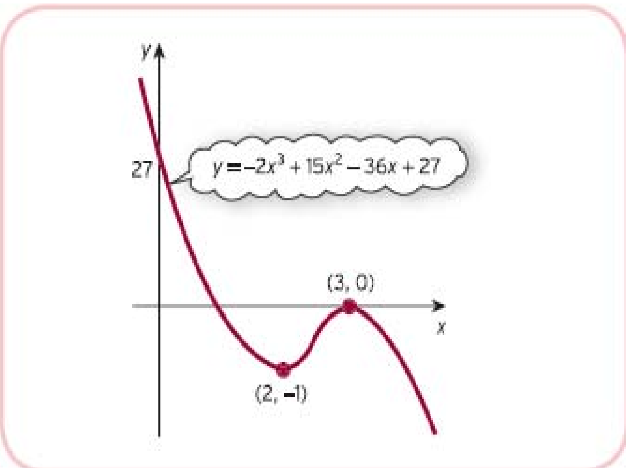

Figure S4.6

A graph is sketched in Figure S4.6 based on the following table of function values:

X	–10	0	2	3	10
$f(x)$	3887	27	–1	0	–833

2 $\text{AP}_L = \dfrac{Q}{L} = \dfrac{300L^2 - L^4}{L} = 300L - L^3$

Step 1

$$\frac{\text{d}(\text{AP}_L)}{\text{d}L} = 300 - 3L^2 = 0$$

has solution $L = \pm 10$. We can ignore –10 because it does not make sense to employ a negative number of workers.

Step 2

$$\frac{\text{d}^2(\text{AP}_L)}{\text{d}L^2} = -6L$$

which takes the value $-60 < 0$ at $L = 10$. Hence $L = 10$ is a maximum.

Now

$$\text{MP}_L = \frac{\text{d}Q}{\text{d}L} = 600L - 4L^3$$

so at $L = 10$

$$\text{MP}_L = 600(10) - 4(10)^3 = 2000$$

$$\text{AP}_L = 300(10) - (10)^3 = 2000$$

that is, $\text{MP}_L = \text{AP}_L$.

3 **(a)** $\text{TR} = PQ = (20 - 2Q)Q = 20Q - 2Q^2$

Step 1

$$\frac{\text{d}(\text{TR})}{\text{d}Q} = 20 - 4Q = 0$$

has solution $Q = 5$.

Step 2

$$\frac{\mathrm{d}^2(\mathrm{TR})}{\mathrm{d}Q^2} = -2 < 0$$

so maximum.

(b) $\pi = \mathrm{TR} - \mathrm{TC}$

$$= (20Q - 2Q^2) - (Q^3 - 8Q^2 + 20Q + 2)$$
$$= -Q^2 + 6Q^2 - 2$$

Step 1

$$\frac{\mathrm{d}\pi}{\mathrm{d}Q} = -3Q^2 + 12Q = 0$$

has solutions $Q = 0$ and $Q = 4$.

Step 2

$$\frac{\mathrm{d}^2\pi}{\mathrm{d}Q^2} = -6Q + 12$$

which takes the values 12 and –12 when $Q = 0$ and $Q = 4$, respectively. Hence minimum at $Q = 0$ and maximum at $Q = 4$.

Finally, evaluating π at $Q = 4$ gives the maximum profit, $\pi = 30$. Now

$$\mathrm{MR} = \frac{\mathrm{d}(\mathrm{TR})}{\mathrm{d}Q} = 20 - 4Q$$

so at $Q = 4$, MR = 4;

$$\mathrm{MC} = \frac{\mathrm{d}(\mathrm{TC})}{\mathrm{d}Q} = 3Q^2 - 16Q + 20$$

so at Q = 4, MC = 4.

4 $\text{AC} = Q + 3 + \dfrac{36}{Q}$

Step 1

$$\frac{\text{d(AC)}}{\text{d}Q} = 1 - \frac{36}{Q^2} = 0$$

has solution $Q = \pm 6$. A negative value of Q does not make sense, so we just take $Q = 6$.

Step 2

$$\frac{\text{d}^2\text{(AC)}}{\text{d}Q^2} = \frac{72}{Q^3}$$

is positive when $Q = 6$, so it is a minimum.

Now when $Q = 6$, AC = 15. Also

$$\text{MC} = \frac{\text{d(TC)}}{\text{d}Q} = 2Q + 3$$

which takes the value 15 at $Q = 6$. We observe that the values of AC and MC are the same: that is, at the point of minimum average cost

$$\boxed{\text{average cost}} = \boxed{\text{marginal cost}}$$

There is nothing special about this example and in the next section we show that this result is true for any average cost function.

5 After tax the supply equation becomes

$$P = \tfrac{1}{2}Q_S + 25 + t$$

In equilibrium, $Q_S = Q_D = Q$, so

$$P = \tfrac{1}{2}Q + 25 + t$$
$$P = -2Q + 50$$

Hence

$$\tfrac{1}{2}Q + 25 + t = -2Q + 50$$

which rearranges to give

$$Q = 10 - \frac{2}{5}t$$

Hence the tax revenue, T, is

$$T = tQ = 10t - \frac{2}{5}t^2$$

Step 1

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 10 - \frac{4}{5}t^2 = 0$$

has solution $t = 12.5$.

Step 2

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = \frac{-4}{5} < 0$$

so maximum. Government should therefore impose a tax of $12.50 per good.

Exercise 4.6 (p. 316)

1 **(a)** Step 1 $\quad \frac{dy}{dx} = -2x + 1$

At a stationary point,

$$-2x + 1 = 0$$
$$-2x = -1$$
$$x = \frac{1}{2}$$

The coordinates of the stationary point are $\left(\frac{1}{2}, \frac{5}{4}\right)$

Step 2 $\quad \frac{d^2y}{dx^2} = -2 < 0$ so the point is a maximum.

The graph is sketched in Figure S4.7.

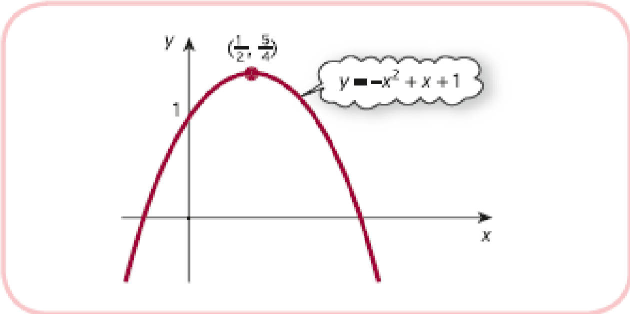

Figure S4.7

(b) Step 1 $\frac{dy}{dx} = 2x - 4$

At a stationary point,

$$\begin{aligned}2x - 4 &= 0\\ 2x &= 4\\ x &= 2\end{aligned}$$

The coordinates of the stationary point are $(2, 0)$

Step 2 $\frac{d^2y}{dx^2} = 2 > 0$ so the point is a minimum.

The graph is sketched in Figure S4.8.

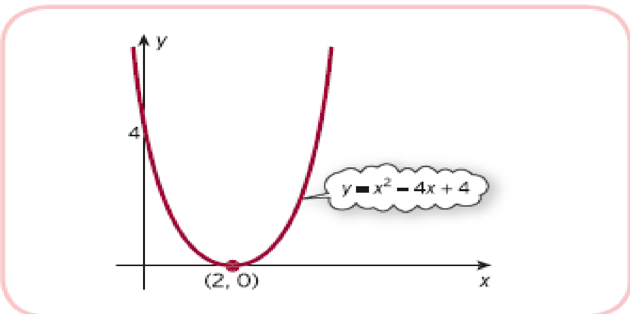

Figure S4.8

(c) Step 1 $\frac{dy}{dx} = 2x - 20$

At a stationary point,

$$2x - 20 = 0$$
$$2x = 20$$
$$x = 10$$

The coordinates of the stationary point are $(10, 5)$

Step 2 $\quad \frac{d^2y}{dx^2} = 2 > 0$ so the point is a minimum.

The graph is sketched in Figure S4.9.

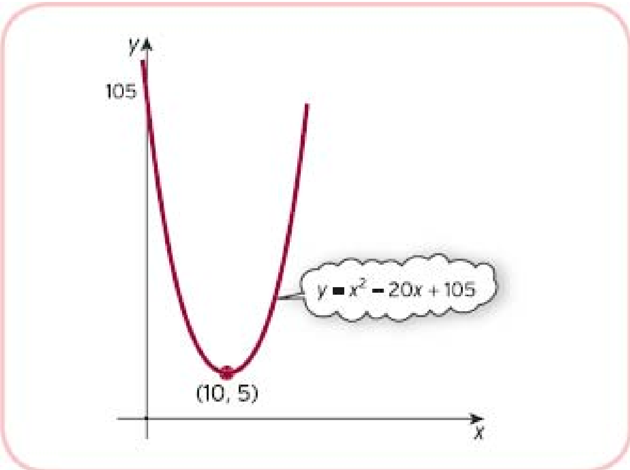

Figure S4.9

(d) Step 1 $\quad \frac{dy}{dx} = -3x^2 + 3$

At a stationary point,

$$-3x^2 + 3 = 0$$
$$-3x^2 = -3$$
$$x^2 = 1$$
$$x = \pm 1$$

The coordinates of the stationary points are $(-1, -2)$ and $(1,2)$

Step 2 $\frac{d^2y}{dx^2} = -6x$

At $x = -1$, $\frac{d^2y}{dx^2} = 6 > 0$ so this point is a minimum

At $x = -1$, $\frac{d^2y}{dx^2} = -6 < 0$ so this point is a maximum

Maximum at (1, 2), minimum at (–1, –2)

The graph is sketched in Figure S4.10.

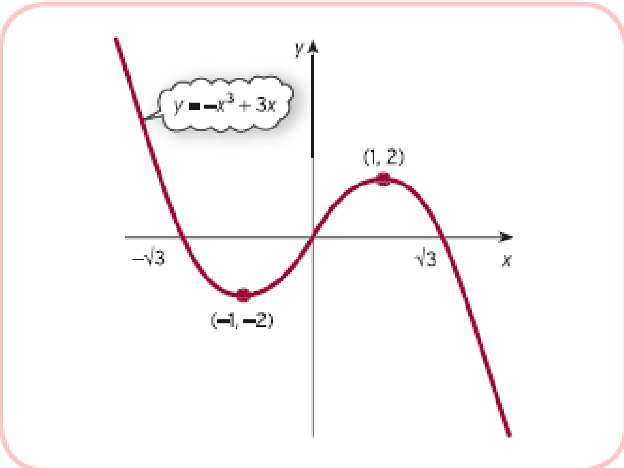

Figure S4.10

2 $TR = PQ = (40 - 2Q)Q = 40Q - 2Q^2$

Step 1 $\frac{d(TR)}{dQ} = 40 - 4Q$

At a stationary point,

$$40 - 4Q = 0$$
$$4Q = 40$$
$$Q = 10$$

Step 2 $\frac{d^2(TR)}{dQ^2} = -4 < 0$ so the stationary point is a maximum.

3 $AP_L = \frac{Q}{L} = 30L - 0.5L^2$

Step 1 $\frac{d(AP_L)}{dL} = 30 - L$ so there is one stationary point at $L = 30$.

Step 2 $\frac{d^2(AP_L)}{dL} = -1 < 0$ so it is a maximum.

At $L = 30$. $AP_L = 30 \times 30 - 0.5 \times 30^2 = 450$

$$MP_L = \frac{dQ}{dL} = 60L - 1.5L^2$$

At $L = 30$, $MP_L = 60 \times 30 - 1.5 \times 30^2 = 450$

4 **(a)** $\text{TC} = 13 + (Q + 2)Q$

$$= 13 + Q^2 + 2Q$$

$$\text{AC} = \frac{\text{TC}}{Q} = \frac{13}{Q} + Q + 2$$

Q	1	2	3	4	5	6
AC	16	10.5	9.3	9.3	9.6	10.2

The graph of AC is sketched in Figure S4.11.

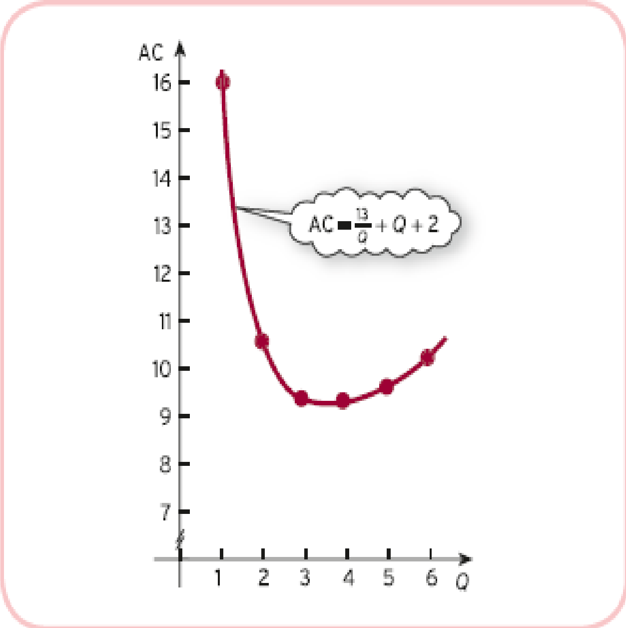

Figure S4.11

(b) From Figure S4.11 minimum average cost is 9.2.

(c) Step 1

$$AC = 13Q^{-1} + Q \Rightarrow \frac{d(AC)}{dQ} = -13Q^{-2} + 1 = \frac{-13}{Q^2} + 1$$

At a stationary point

$$\frac{-13}{Q^2} + 1 = 0$$

$$Q^2 = 13$$

$$Q = \sqrt{13}$$

Step 2

$$\frac{d^2(AC)}{dL^2} = 26Q^{-3} > 0$$ when $Q = \sqrt{13}$ so the stationary point is a minimum.

At $Q = \sqrt{13}$, AC = 9.21.

5 **(a)** $\text{TR} = PQ = \left(4 - \frac{Q}{4}\right)Q = 4Q - \frac{Q^2}{4}$

$$\pi = TR - TC = \left(4Q - \frac{Q^2}{4}\right) - \left(4 + 2Q - \frac{3Q^2}{10} + \frac{Q^3}{20}\right) = \frac{-Q^3}{20} + \frac{Q^2}{20} + 2Q - 4$$

$$\text{MR=}\frac{\text{d(TR)}}{\text{dQ}}\text{=}4 - \frac{Q}{2}$$

$$\text{MC=}\frac{\text{d(TC)}}{\text{dQ}} = 2 - \frac{3Q}{5} + \frac{3Q^2}{20}$$

(b) Step 1

$$\frac{d\pi}{dQ} = \frac{-3Q^2}{20} + \frac{Q}{10} + 2 = 0 \Rightarrow 3Q^2 - 2Q - 40 = 0$$

$$Q = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-40)}}{2(3)} = \frac{2 \pm \sqrt{484}}{6}$$

$$Q = \frac{2 \pm 22}{6} = -\frac{10}{3}, \quad 4$$

Ignoring the negative solution gives, $Q = 4$.

Step 2

$$\frac{d^2\pi}{dQ^2} = \frac{-3Q}{10} + \frac{1}{10}$$

At $Q = 4$, $\frac{d^2\pi}{dQ^2} = -\frac{11}{10} < 0$ so maximum.

(c) $MR = \frac{d(TR)}{dQ} = 4 - \frac{Q}{2}$ which takes the value 2 at Q = 4

$MC = \frac{d(TC)}{dQ} = 2 - \frac{6Q}{10} + \frac{3Q^2}{20}$ also takes the value 2 at $Q = 4$.

6 The new supply equation is

$$3(P - t) - Q_S = 3$$
$$3P - 3t - Q_S = 3$$

which can be reaaranged as $Q_S = 3P - 3t - 3$

The demand equation can be rearranged as $Q_D = 14 - 2P$.

In equilibrium, $Q_S = Q_D = Q$ so

$$\Rightarrow 3P - 3t - 3 = 14 - 2P$$
$$\Rightarrow 5P - 3t - 3 = 14$$
$$\Rightarrow 5P = 3t + 17$$
$$\Rightarrow P = \frac{1}{5}(3t + 17)$$

Hence $Q = 14 - \frac{2}{5}(3t + 17) = \frac{6}{5}(6 - t)$

$$TR = PQ = \frac{6}{25}(3t + 17)(6 - t) = \frac{6}{25}(-3t^2 + t + 102)$$

Step 1 $\frac{d(TR)}{dt} = \frac{6}{25}(-6t + 1) = 0 \Rightarrow t = \frac{1}{6}$

Step 2 $\frac{d^2(TR)}{dt^2} = -\frac{36}{25} < 0$ so the stationary point is a maximum.

7 (a) $TC = (2Q - 36)Q + 200 = 2Q^2 - 36Q + 200$

$$AC = \frac{TC}{Q} = 2Q - 36 + \frac{200}{Q}$$

(b) Step 1 $AC = 200Q^{-1} + 2Q - 36 \Rightarrow \frac{d(AC)}{dQ} = -200Q^{-2} + 2$

At a stationary point,

$$-\frac{200}{Q^2} + 2 = 0 \Rightarrow Q^2 = 100 \Rightarrow Q = 10 \quad \text{(ignoring the negative solution)}$$

Step 2 $\frac{d^2(AC)}{dQ^2} = 400Q^{-3}$

When $Q = 10$, $\frac{d^2(AC)}{dQ^2} = 0.4 > 0$ so the stationary point is a minimum.

(c) $Q = 10 \quad \Rightarrow \quad AC = 4$

$$MC = \frac{d(TC)}{dQ} = 4Q - 36$$

$Q = 10 \Rightarrow MC = 4 = AC$

Section 4.7 Further optimization of economic functions

Practice Problems

1 **(a)** $\text{TR} = (25 - 0.5Q)Q = 25Q - 0.5Q^2$

$\text{TC} = 7 + (Q+1)Q = Q^2 + Q + 7$
$\text{MR} = 25 - Q$
$\text{MC} = 2Q + 1$

(b) From Figure S4.13 the point of intersection of the MR and MC curves occurs at $Q = 8$.

The MC curve cuts the MR curve from below, so this must be a maximum point.

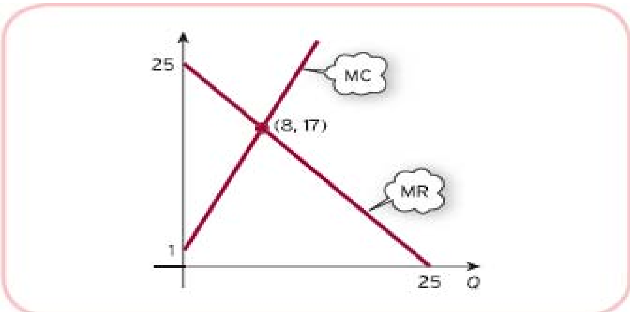

Figure S4.13

2 MC = 100.

(a) *Domestic market* $P_1 = 300 - Q_1$

$\text{TR}_1 = 300Q_1 - Q_1^2$

so

$\text{MR}_1 = 200 - 2Q_1$

To maximize profit, $MR_1 = MC$: that is,

$$300 - 2Q_1 = 100$$

which has solution $Q_1 = 100$.

Corresponding price is

$$P_1 = 300 - 100 = \$200$$

Foreign market $P_2 = 200 - \frac{1}{2}Q_2$

$$TR_2 = 200Q_2 - \frac{1}{2}Q_2^2$$

so

$$MR_2 = 200 - Q_2$$

To maximize profit, $MR_2 = MC$: that is,

$$200 - Q_2 = 100$$

which has solution $Q_2 = 100$.

Corresponding price is

$$P_2 = 200 - \frac{1}{2}(100) = \$150$$

(b) Without discrimination, $P_1 = P_2 = P$, say, so individual demand equations become

$$Q_1 = 300 - P$$
$$Q_2 = 400 - 2P$$

Adding shows that the demand equation for combined market is

$$Q = 700 - 3P$$

where $Q = Q_1 + Q_2$.

$$TR = \frac{700}{3}Q - \frac{Q^2}{3}$$

so

$$\text{MR}=\frac{700}{3}-\frac{2Q}{3}$$

To maximize profit, MR = MC: that is,

$$\frac{700}{3}-\frac{2Q}{3}=100$$

which has solution $Q = 200$.

Corresponding price is

$$P = 700/3 - 200/3 = \$500/3$$

Total cost of producing 200 goods is

$$5000 + 100(200) = \$25000$$

With discrimination, total revenue is

$$100 \times 200 + 100 \times 150 \ = \ \$35000$$

so profit is \$35000 – \$25000 = \$10000.

Without discrimination, total revenue is

$$200 \times \frac{500}{3} = \$33333$$

so profit is \$33333 – \$25000 = \$8333.

3 *Domestic market* From Practice Problem 2, profit is maximum when $P_1 = 200$, $Q_1 = 100$.

Also, since $Q_1 = 300 - P_1$ we have $\mathrm{d}Q_1/\mathrm{d}P_1 = -1$. Hence

$$\begin{aligned} E_1 &= -\frac{P_1}{Q_1}\times\frac{\mathrm{d}Q_1}{\mathrm{d}P_1} \\ &= -\frac{200}{100}\times(-1) = 2 \end{aligned}$$

Foreign market From Practice Problem 2, profit is maximum when $P_2 = 150$, $Q_2 = 100$.

Also, since $Q_2 = 400 - 2P_2$ we have $\mathrm{d}Q_2/\mathrm{d}P_2 = -2$. Hence

$$E_2 = \frac{P_2}{Q_2} \times \frac{dQ_2}{dP_2}$$
$$= -\frac{150}{100} \times (-2) = 3$$

We see that the firm charges the higher price in the domestic market, which has the lower elasticity of demand.

Section 4.8 The derivative of the exponential and natural logarithm functions

Practice Problems

1

x	0.50	1.00	1.50	2.00
$f(x)$	–0.69	0.00	0.41	0.69

x	2.50	3.00	3.50	4.00
$f(x)$	0.92	1.10	1.25	1.39

The graph of the natural logarithm function is sketched in Figure S4.14.

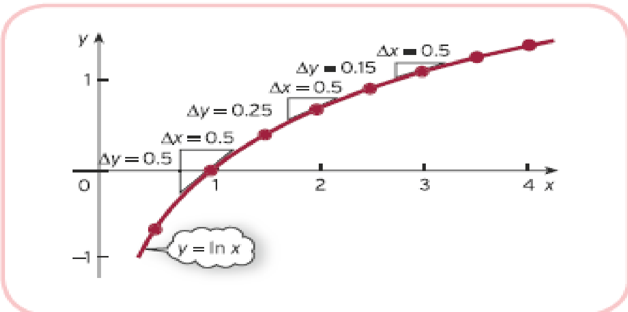

Figure S4.14

$$f'(1) = \frac{0.50}{0.50} = 1.00 = 1$$
$$f'(2) = \frac{0.25}{0.50} = 0.50 = \frac{1}{2}$$
$$f'(3) = \frac{0.15}{0.50} = 0.30 = \frac{1}{3}$$

These results suggest that $f'(x) = 1/x$.

2 **(a)** $3e^{3x}$; **(b)** $-e^{-x}$; **(c)** $1/x$; **(d)** $1/x$.

3 **(a)** For the product rule we put

$u = x^4$ and $v = \ln x$

for which

$$\frac{du}{dx} = 4x^3 \qquad \text{and} \qquad \frac{dv}{dx} = \frac{1}{x}$$

By the product rule

$$\frac{dy}{dx} = x^4 \times \frac{1}{x} + \ln x \times 4x^3$$
$$= x^3 + 4x^3 \ln x$$
$$= x^3(1 + 4\ln x)$$

(b) By the chain rule

$$\frac{dy}{dx} = e^{x^2} \times 2x = 2xe^{x^2}$$

(c) If

$$u = \ln x \qquad \text{and} \qquad v = x + 2$$

then

$$\frac{du}{dx} = \frac{1}{x} \text{ and } \qquad \frac{dv}{dx} = 1$$

By the quotient rule

$$\frac{dy}{dx} = \frac{(x+2) \times \frac{1}{x} - (\ln x) \times 1}{(x+2)^2}$$
$$\frac{x + 2 - x\ln x}{x(x+2)^2} \quad \text{(multiply top and bottom by } x\text{)}$$

4 **(a)** $y = \ln x^3 + \ln(x+2)^4$ (rule 1)

$= 3 \ \ln x + 4\ln(x+2)$ (rule 3)

Hence

$$\frac{dy}{dx} = \frac{3}{x} + \frac{4}{x+2}$$
$$\frac{3(x+2)+4x}{x(x+2)} = \frac{7x+6}{x(x+2)}$$

(b) $y = \ln x^2 - \ln(2x+3)$ (rule 2)

$= 2 \ln x - \ln(2x+3)$ (rule 3)

Hence

$$\frac{dy}{dx} = \frac{2}{x} - \frac{2}{2x+3}$$ (chain rule)

$$= \frac{2(2x+3)-2x}{x(2x+3)}$$
$$= \frac{2x+6}{x(2x+3)}$$

5 In terms of P the total revenue function is given by

$$\text{TR} = PQ = 1000Pe^{-0.2P}$$

and the total cost function is

$$\text{TC} = 100 + 2Q = 100 + 2000e^{-0.2P}$$

Hence

$$\pi = \text{TR} - \text{TC}$$
$$= 1000Pe^{-0.2P} - 2000e^{-0.2P} - 100$$

Step 1

At a stationary point

$$\frac{d\pi}{dP} = 0$$

To differentiate the first term, $1000Pe^{-0.2P}$, we use the product rule with

$u = 1000P$ and $v = e^{-0.2P}$

for which

$$\frac{\mathrm{d}u}{\mathrm{d}P} = 1000 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}P} = -0.2\mathrm{e}^{-0.2P}$$

Hence the derivative of $1000P\mathrm{e}^{-0.2P}$ is

$$u\frac{\mathrm{d}v}{\mathrm{d}P} + v\frac{\mathrm{d}u}{\mathrm{d}P}$$
$$= 1000P(-0.2\mathrm{e}^{-0.2P}) + \mathrm{e}^{-0.2P}(1000)$$
$$= \mathrm{e}^{-0.2P}(1000 - 200P)$$

Now

$$\pi = 1000P\mathrm{e}^{-0.2P} - 2000\mathrm{e}^{-0.2P} - 100$$

so

$$\frac{\mathrm{d}\pi}{\mathrm{d}P} = \mathrm{e}^{-0.2P}(1000 - 200P) - 2000(-0.2\mathrm{e}^{-0.2P})$$
$$= \mathrm{e}^{-0.2P}(1400 - 200P)$$

This is zero when

$$1400 - 200P = 0$$

because $\mathrm{e}^{-0.2P} \neq 0$.

Hence $P = 7$.

Step 2

To find $\frac{\mathrm{d}^2\pi}{\mathrm{d}P^2}$ we differentiate

$$\frac{\mathrm{d}\pi}{\mathrm{d}P} = \mathrm{e}^{-0.2P}(1400 - 200P)$$

using the product rule. Taking

$$u = \mathrm{e}^{-0.2P} \quad \text{and} \quad v = 1400 - 200P$$

gives

$$\frac{\mathrm{d}u}{\mathrm{d}P} = -0.2\mathrm{e}^{-0.2P} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}P} = -200$$

Hence

$$\frac{d^2\pi}{dP^2} = u\frac{dv}{dP} + v\frac{du}{dP}$$
$$= e^{-02.P}(-200) + (1400 - 200P)(-0.2e^{-02.P})$$
$$= e^{-0.2P}(10P - 480)$$

Putting $P = 7$ gives

$$\frac{d^2\pi}{dP^2} = -200e^{-1.4}$$

This is negative, so the stationary point is a maximum.

6 To find the price elasticity of demand we need to calculate the values of P, Q and dQ/dP. We are given that $Q = 20$ and the demand equation gives

$P = 200 - 40 \ln(20 + 1) = 78.22$

The demand equation expresses P in terms of Q, so we first evaluate dP/dQ and then use the result

$$\frac{dQ}{dP} = \frac{1}{dP/dQ}$$

To differentiate ln(Q + 1) by the chain rule we differentiate the outer log function to get

$$\frac{1}{Q+1}$$

and then multiply by the derivative of the inner function, Q + 1, to get 1. Hence the derivative of ln(Q + 1) is

$$\frac{1}{Q+1}$$

and so

$$\frac{dP}{dQ} = \frac{-40}{Q+1}$$

Putting $Q = 20$ gives dP/dQ = –40/21, so that dQ/dP = –21/40. Finally, we use the formula

$$E = -\frac{P}{Q} \times \frac{\mathrm{d}Q}{\mathrm{d}P}$$

to calculate the price elasticity of demand as

$$E = -\frac{78.22}{20} \times \left(\frac{-21}{40}\right) = 2.05$$

Exercise 4.8 (p. 339)

1 **(a)** $6e^{6x}$; **(b)** $-342e^{-342x}$; **(c)** $-2e^{-x} + 4e^{x}$;

(d) $40e^{4x} - 4x$.

2 **(1)** **(a)** \$4885.61; **(b)** \$4887.57;

Rate of growth is approximately $\dfrac{4887.57 - 4885.61}{0.01} = 196$

(2) $160e^{0.04t}$; 195.42

3 **(a)** $\dfrac{1}{x}$; **(b)** $\dfrac{1}{x}$.

4 **(a)** $3x^2e^{x^3}$; (b) $\dfrac{4x^3 + 6x}{x^4 + 3x^2}$

5 **(a)** $4x^3 \times e^{2x} + x^4 \times 2e^{2x} = \left(4x^3 + 2x^4\right)e^{2x}$

(b) $1 \times \ln x + x \times \dfrac{1}{x} = \ln x + 1$

6 **(a)** $\dfrac{(x^2+2)\times 4e^{4x} - e^{4x} \times 2x}{(x^2+2)^2} = \dfrac{2e^{4x}(2x^2 - x + 4)}{(x^2+2)^2}$ **(b)** $\dfrac{(\ln x)\times e^{x} - e^{x} \times \dfrac{1}{x}}{(\ln x)^2} = \dfrac{e^{x}(x\ln x - 1)}{x(\ln x)^2}$

7

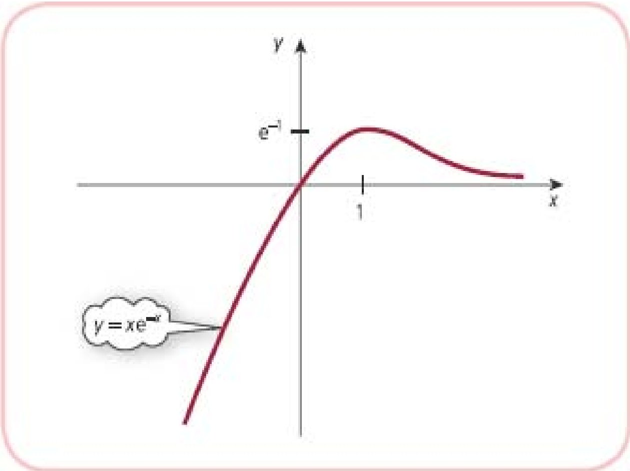

Figure S4.15

(a) Step 1 $\frac{dy}{dx} = 1 \times e^{-x} + x \times (-e^{-x}) = e^{-x}(1 - x)$

At a stationary point $e^{-x}(1 - x) = 0 \Rightarrow x = 1 \Rightarrow y = e^{-1}$

Step 2 $\frac{d^2y}{dx^2} = e^{-x} \times (-1) + (-e^{-x}) \times (1 - x) = e^{-x}(x - 2)$

When $x = 1$, $\frac{d^2y}{dx^2} = -e^{-1} < 0$

Maximum at (1, e^{-1}); the graph is sketched in Figure S4.15.

(b) Step 1 $\frac{dy}{dx} = \frac{1}{x} - 1$

At a stationary point, $\frac{1}{x} - 1 = 0 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$

Step 2 $\frac{dy}{dx} = x^{-1} - 1 \Rightarrow \frac{d^2y}{dx^2} = -x^{-2}$

When $x = 1$, $\frac{d^2y}{dx^2} = -1 < 0$

Maximum at (1, –1); the graph is sketched in Figure S4.16.

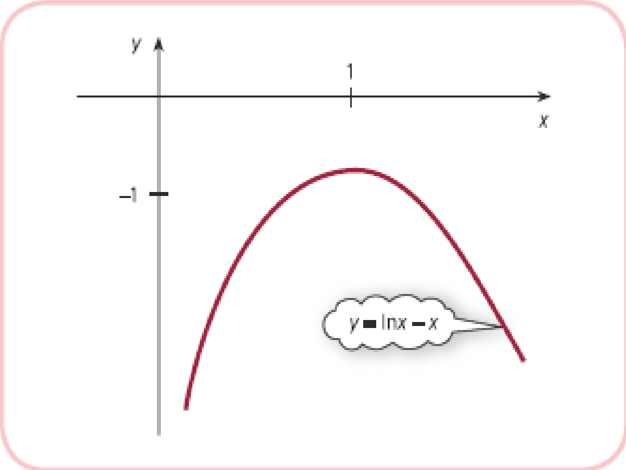

Figure S4.16

8 $\pi = TR - TC = 100\ln(Q+1) - 2Q$

Step 1 $\frac{d\pi}{dQ} = \frac{100}{Q+1} - 2 = 0 \Rightarrow Q+1 = 50 \Rightarrow Q = 49$

Step 2 $\frac{d\pi}{dQ} = 100(Q+1)^{-1} - 2 \Rightarrow \frac{d^2\pi}{dQ^2} = -100(Q+1)^{-2}$

When $Q = 49$, $\frac{d^2\pi}{dQ^2} = -\frac{100}{2401} < 0$

so the point is a maximum.

9 Step 1 $\frac{dQ}{dL} = 700L \times (-0.02e^{-0.02L}) + 700 \times e^{-0.02L} = e^{-0.02L}(700 - 14L)$

At a stationary point, $\frac{dQ}{dL} = 0 \Rightarrow 700 - 14L = 0 \Rightarrow L = 50$

Step 2 $\frac{d^2Q}{dL^2} = -0.02e^{-0.02L}(700 - 14L) + e^{-0.02L} \times (-14) = e^{-0.02L}(0.28L - 28)$

When $L = 50$, $\frac{d^2Q}{dL^2} = -14e^{-1} < 0$

so the stationary point is a maximum .

10 $Q = 10 \Rightarrow P = 100e^{-1}$

$$\frac{dP}{dQ} = -10e^{-0.1Q} \text{ so when } Q = 10,\ \frac{dP}{dQ} = -10e^{-1} \Rightarrow \frac{dQ}{dP} = -0.1e$$

$$E = -\frac{P}{Q} \times \frac{dQ}{dP} = -\frac{100e^{-1}}{10} \times (-0.1e) = 1$$

Chapter 5 Partial Differentiation

Section 5.1 Functions of several variables

Practice Problems

1 **(a)** –10; **(b)** –1; **(c)** 2; **(d)** 21; **(e)** 0;

(f) 21. The value of g is independent of the ordering of the variables. Such a function is said to be *symmetric*.

2 **(a)** Differentiating $5x^4$ with respect to x gives $20x^3$ and, since y is held constant, y^2 differentiates to zero.

Hence

$$\frac{\partial f}{\partial x} = 20x^3 - 0 = 20x^3$$

Differentiating $5x^4$ with respect to y gives zero because x is held fixed. Also differentiating y^2 with respect to y gives $2y$, so

$$\frac{\partial f}{\partial y} = 0 - 2y = -2y$$

(b) To differentiate the first term, x^2y^3, with respect to x we regard it as a constant multiple of x^2 (where the constant is y^3), so we get $2xy^3$. The second term obviously gives –10, so

$$\frac{\partial f}{\partial x} = 2xy^3 - 10$$

To differentiate the first term, x^2y^3, with respect to y we regard it as a constant multiple of y^3 (where the constant is x^2), so we get $3x^2y^2$. The second term is a constant and goes to zero, so

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 0 = 3x^2y^2$$

3 **(a)** $f_{xx} = 60x^2$

$$f_{yy} = 2$$
$$f_{yx} = f_{xy} = 0$$

(b) $f_{xx} = 2y^3$

$$f_{yy} = 6x^2y$$
$$f_{yx} = f_{xy} = 6xy^2$$

4 $f_1 = \frac{\partial f}{\partial x_1} = x_2 + 5x_1^4$

$$f_{11} = \frac{\partial^2 f}{\partial x_1^2} = 20x_1^3$$
$$f_{21} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$$

5 $\frac{\partial z}{\partial x} = y - 5, \frac{\partial z}{\partial y} = x + 2,$ so, at $(2, 6)$.

$$\frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = 4$$

(a) $\Delta x = -0.1$, $\Delta y = 0.1$;

$z \cong 1(-0.1) + 4(0.1) = 0.3$, so z increases by approximately 0.3.

(b) At (2, 6), $z = 14$, and at (1.9, 6.1), $z = 14.29$, so the exact increase is 0.29.

6 **(a)** $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y}{x - 3y^2 + 1}$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{5y^4 - 2xy}$

Exercise 5.1 (p. 360)

1 324; 75; 0.

2 **(a)** $f_x = 2x$, $f_y = 20y^4$;

(b) $f_x = 9x^2$, $f_y = -2e^y$;

(c) $f_x = y$, $f_y = x + 6$;

(d) $f_x = 6x^5y^2$, $f_y = 2x^6y + 15y^2$.

3 $f_x = 4x^3y^5 - 2x$

$f_y = 5x^4y^4 + 2y$

$f_x(1, 0) = -2$

$f_y(1, 1) = 7$

4 **(a)** $\frac{\partial z}{\partial x} = 2xy^4 - 6x^5$ so at (1,0), $\frac{\partial z}{\partial x} = -6$

Hence $\Delta z \approx -6 \times 0.1 = -0.6$

(b) $\frac{\partial z}{\partial y} = 4x^2y^3 + 4$ so at (1,0), $\frac{\partial z}{\partial y} = 4$

Hence $\Delta z \approx 4 \times (-0.5) = -2$

(c) $\Delta z \approx -6 \times 0.1 + 4 \times (-0.5) = -2.6$

5 **(a)** $f_x = -3x^2 + 2$, $f_y = 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(-3x^2 + 2)}{1} = 3x^2 - 2$$

(b) $y = x^3 - 2x + 1$, so

$$\frac{dy}{dx} = 3x^2 - 2 \quad ✓$$

Section 5.2 Partial elasticity and marginal functions

Practice Problems

1 Substituting the given values of P, P_A and Y into the demand equation gives

$$Q = 500 - 3(20) - 2(30) + 0.01(5000)$$
$$= 430$$

(a) $\frac{\partial Q}{\partial P} = -3$, so

$$E_P = -\frac{20}{430} \times (-3) = 0.14$$

(b) $\frac{\partial Q}{\partial P_A} = -2$, so

$$E_{P_A} = \frac{30}{430} \times (-2) = -0.14$$

(c) $\frac{\partial Q}{\partial Y} = 0.01$, so

$$E_Y = \frac{5000}{430} \times 0.01 = 0.12$$

By definition,

$$E_Y = \frac{\text{percentage change in } Q}{\text{percentage change in } Y}$$

so demand rises by 0.12 × 5 = 0.6%. A rise in income causes a rise in demand, so good is superior.

2 $\dfrac{\partial U}{\partial x_1} = 1000 + 5x_2 - 4x_1$

$\dfrac{\partial U}{\partial x_2} = 450 + 5x_1 - 2x_2$, so at (138, 500)

$\dfrac{\partial U}{\partial x_1} = 2948$ and $\dfrac{\partial U}{\partial x_2} = 140$

If working time increases by 1 hour then leisure time decreases by 1 hour, so $\Delta x_1 = -1$. Also $\Delta x_2 = 15$. By the small increments formula

$$\Delta U = 2948(-1) + 140(15) = -848$$

The law of diminishing marginal utility holds for both x_1 and x_2 because

$$\frac{\partial^2 U}{\partial x_1^2} = -4 < 0$$

and

$$\frac{\partial^2 U}{\partial x_2^2} = -2 < 0$$

3 Using the numerical results in Practice Problem 2,

$$\text{MRCS} = \frac{2948}{140} = 21.06$$

This represents the increase in x_2 required to maintain the current level of utility when x_1 falls by 1 unit. Hence if x_1 falls by 2 units, the increase in x_2 is approximately

$$21.06 \times 2 = \$42.12$$

4 $\text{MP}_K = 2K$ and $\text{MP}_L = 4L$

(a) $\text{MRTS} = \dfrac{\text{MP}_L}{\text{MP}_K} = \dfrac{4L}{2K} = \dfrac{2L}{K}$

(b) $K\dfrac{\partial Q}{\partial K} + L\dfrac{\partial Q}{\partial L} = K(2K) + L(4L)$

$$= 2(K^2 + 2L^2) = 2Q \quad ✓$$

Exercise 5.2 (p. 374)

1 $\frac{\partial Q}{\partial Y} = 0.015Y^2$

When $P = 15$, $P_A = 20$ and $Y = 100$ we have $Q = 5525$ and $\frac{\partial Q}{\partial Y} = 150$

Hence $E_Y = \frac{Y}{Q} \times \frac{\partial Q}{\partial Y} = \frac{100}{5525} \times 150 = 2.71$

2 **(a)** $\frac{\partial Q}{\partial P} = -2$

When $P = 10$, $P_A = 20$ and $Y = 100$ we have $Q = 1165$

Hence $E_P = -\frac{P}{Q} \times \frac{\partial Q}{\partial P} = \frac{-10}{1165} \times (-2) = \frac{20}{1165}$

(b) $\frac{\partial Q}{\partial P_A} = -1$

Hence $E_{P_A} = \frac{P_A}{Q} \times \frac{\partial Q}{\partial P_A} = \frac{15}{1165} \times (-1) = -\frac{3}{233}$

(c) $\frac{\partial Q}{\partial Y} = 0.2Y$ so when $Y = 100$ we have $\frac{\partial Q}{\partial Y} = 20$

Hence $E_Y = \frac{Y}{Q} \times \frac{\partial Q}{\partial Y} = \frac{100}{1165} \times 20 = \frac{40}{233}$

% change in demand = $-\frac{3}{233} \times 3 = -0.04\%$;

Complementary since E_{PA} is negative so that an increase in the price of the alternative good causes a decrease in demand.

3 $\frac{\partial Q}{\partial Y} = \frac{2P_AY}{P}$

When $P_A = 10$, $Y = 2$ and $P = 4$ we have $Q = 10$ and $\frac{\partial Q}{\partial Y} = \frac{2 \times 10 \times 2}{4} = 10$

Hence $E_Y = \frac{Y}{Q} \times \frac{\partial Q}{\partial Y} = \frac{2}{10} \times 10 = 2$;

% change in Y = 2/2 = 1%.

4 $\frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{3}}$ and $\frac{\partial U}{\partial x_2} = \frac{1}{3} x_1^{\frac{1}{2}} x_2^{-\frac{2}{3}}$ so at the point (25,8) we have

$\frac{\partial U}{\partial x_1} = \frac{1}{5}$ and $\frac{\partial U}{\partial x_2} = \frac{5}{12}$

(a) $\Delta U \approx \frac{1}{5} \times 1 + \frac{5}{12} \times 1 = \frac{37}{60}$ **(b)** $MRCS = \frac{1/5}{5/12} = \frac{12}{25}$

5 $Q = 2LK + L^{\frac{1}{2}} \Rightarrow MP_K = \frac{\partial Q}{\partial K} = 2L;$ $MP_L = \frac{\partial Q}{\partial L} = 2K + \frac{1}{2} L^{-\frac{1}{2}}$ so when K = 7 and L = 4

MP_K = 8, MP_L = 14¼;

(a) $MRTS = \frac{14\frac{1}{4}}{8} = \frac{57}{32} = 1\frac{25}{32}$

(b) This is the value of $MRTS$ so is $1\frac{25}{32}$.

6 $MP_K = \frac{\partial Q}{\partial K} = 6K^2 + 3L^2$ and $MP_L = \frac{\partial Q}{\partial L} = 6LK$

$K(MP_K) + L(MP_L) = K(6K^2 + 3L^2) + L(6LK) = 6K^3 + 9L^2K = 3(K^3 + 3L^2K)$

Section 5.3 Comparative statics

Practice Problems

1 $C = a\left(\frac{b + I^*}{I - a}\right) + b$

$\frac{\partial S}{\partial I^*} = \frac{a}{1 - a} > 0$

because $0 < a < 1$

Hence an increase in I^* leads to an increase in C. If $a = \frac{1}{2}$ then

$$\frac{\partial C}{\partial I^*} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Change in C is

$$1 \times 2 = 2$$

2 **(a)** Substitute C, I, G, X and M into the Y equation to get

$$Y = aY + b + I^* + G^* + X^* - (mY + M^*)$$

Collecting like terms gives

$$(1 - a + m)Y = b + I^* + G^* + X^* - M^*$$

so

$$Y = \frac{b + I^* + G^* + X^* - M^*}{1 - a + m}$$

(b) $$\frac{\partial Y}{\partial X^*} = \frac{1}{1-a+m}$$

$$\frac{\partial Y}{\partial m} = \frac{b + I^* + G^* + X^* - M^*}{(1-a+m)^2}$$

Now $a < 1$ and $m > 0$, so $1 - a + m > 0$. The autonomous export multiplier is positive, so an increase in X^* leads to an increase in Y. The marginal propensity to import multiplier is also positive. To see this note from part (a) that $\partial Y/\partial m$ can be written as

$$\frac{Y}{1-a+m}$$

and $Y > 0$ and $1 - a + m > 0$.

(c) $$Y = \frac{120 + 100 + 300 + 150 - 40}{1 - 0.8 + 0.1}$$

$$= 2100$$

$$\frac{\partial Y}{\partial X^*} = \frac{1}{1 - 0.8 + 0.1} = \frac{10}{3}$$

and

$$\Delta X^* = 10$$

so

$$\Delta Y = \frac{10}{3} \times 10 = \frac{100}{3}$$

3 If d increases by a small amount then the intercept increases and the demand curve shifts upwards slightly. Figure S5.2 shows that the effect is to increase the equilibrium quantity from Q_1 to Q_2, confirming that $\partial Q/\partial d > 0$.

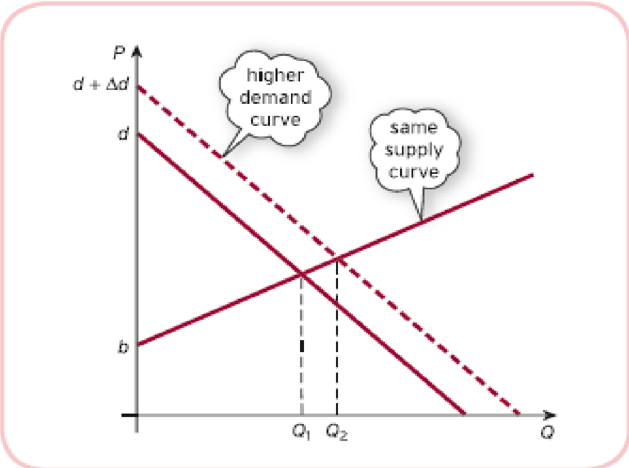

Figure S5.2

Section 5.4 Unconstrained optimization

Practice Problems

1 $f_x = 2x$, $f_y = 6 - 6y$, $f_{xx} = 2$, $f_{yy} = -6$, $f_{xy} = 0$.

Step 1

At a stationary point

$$2x = 0$$
$$6 - 6y = 0$$

which shows that there is just one stationary point at (0, 1).

Step 2

$$f_{xx}f_{yy} - f^2_{xy} = 2(-6) - 0^2 = -12 < 0$$

so it is a saddle point.

2 Total revenue from the sale of G1 is

$$\mathrm{TR}_1 = P_1Q_1 = (50 - Q_1)Q_1 = 50Q_1 - Q_1^2$$

Total revenue from the sale of G2 is

$$\begin{aligned}\mathrm{TR}_2 &= P_2Q_2 = (95 - 3Q_2)Q_2\\ &= 95Q_2 - 3Q_2^2\end{aligned}$$

Total revenue from the sale of both goods is

$$\begin{aligned}\mathrm{TR} &= \mathrm{TR}_1 + \mathrm{TR}_2\\ &= 50Q_1 - Q_1^2 + 95Q_2 - 3Q_2^2\end{aligned}$$

Profit is

$$\begin{aligned}\pi &= \mathrm{TR} - \mathrm{TC}\\ &= (50Q_1 - Q_1^2 + 95Q_2 - 3Q_2^2) - (Q_1^2 + 3Q_1Q_2 + Q_2^2)\\ &= 50Q_1 - 2Q_1^2 + 95Q_2 - 4Q_2^2 - 3Q_1Q_2\end{aligned}$$

Now

$$\frac{\partial \pi}{\partial Q_1} = 50 - 4Q_1 - 3Q_2,$$

$$\frac{\partial \pi}{\partial Q_2} = 95 - 8Q_2 - 3Q_1$$

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -4, \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = -3,$$

$$\frac{\partial^2 \pi}{\partial Q_2^2} = -8$$

Step 1

At a stationary point

$$\begin{aligned}50 - 4Q_1 - 3Q_2 &= 0\\ 95 - 3Q_1 - 8Q_2 &= 0\end{aligned}$$

that is,

$$4Q_1 + 3Q_2 = 50 \quad (1)$$

$$3Q_1 + 8Q_2 = 95 \quad (2)$$

Multiply equation (1) by 3, and equation (2) by 4 and subtract to get

$$23Q_2 = 230$$

so Q_2 = 10. Substituting this into either equation (1) or equation (2) gives Q_1 = 5.

Step 2

This is a maximum because

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -4 < 0, \ \frac{\partial^2 \pi}{\partial Q_2^2} = -8 < 0$$

and

$$\left(\frac{\partial^2 \pi}{\partial Q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial Q_2^2}\right) - \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2$$
$$= (-4)(-8) - (-3)^2 = 23 > 0$$

Corresponding prices are found by substituting Q_1 = 5 and Q_2 = 10 into the original demand equations to obtain P_1 = 45 and P_2 = 65.

3 For the domestic market, $P_1 = 300 - Q_1$, so

$$\text{TR}_1 = P_1Q_1 = 300Q_1 - Q_1^2$$

For the foreign market, $P_2 = 200 - \frac{1}{2}Q_2$, so

$$\text{TR}_2 = P_2Q_2 = 200Q_2 - \tfrac{1}{2}Q_2^2$$

Hence

$$\begin{aligned}\text{TR} &= \text{TR}_1 + \text{TR}_2 \\ &= 300Q_1 - Q_1^2 + 200Q_2 - \tfrac{1}{2}Q_2^2\end{aligned}$$

We are given that

$$\begin{aligned}\mathrm{TC} &= 5000 + 100(Q_1 + Q_2)\\ &= 5000 + 100Q_1 + 100Q_2\end{aligned}$$

so

$$\begin{aligned}\pi &= \mathrm{TR} - \mathrm{TC}\\ &= (300Q_1 - Q_1^2 + 200Q_2 - \tfrac{1}{2}Q_2^2) - (5000 + 100Q_1 + 100Q_2)\\ &= 200Q_1 - Q_1^2 + 100Q_2 - \tfrac{1}{2}Q_2^2 - 5000\end{aligned}$$

Now

$$\frac{\partial \pi}{\partial Q_1} = 200 - 2Q_1, \frac{\partial \pi}{\partial Q_2} = 100 - Q_2$$

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -2, \frac{\partial^2 \pi}{\partial Q_1^2 \partial Q_2} = 0, \frac{\partial^2 \pi}{\partial Q_2^2} = -1$$

Step 1

At a stationary point

$$\begin{aligned}200 - 2Q_1 &= 0\\ 100 - Q_2 &= 0\end{aligned}$$

which have solution $Q_1 = 100$, $Q_2 = 100$.

Step 2

This is a maximum because

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -2 < 0,$$

$$\frac{\partial^2 \pi}{\partial Q_2^2} = -1 < 0$$

and

$$\left(\frac{\partial^2 \pi}{\partial Q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial Q_2^2}\right) - \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2$$

$$= (-2)(-1) - 0^2 = 2 > 0$$

Substitute $Q_1 = 100$, $Q_2 = 100$, into the demand and profit functions to get $P_1 = 200$, $P_2 = 150$

and $\pi = 10000$.

Exercise 5.4 (p. 401)

1 (a)

$$f_x = 3x^2 - 3$$
$$f_y = 3y^2 - 3$$
$$f_{xx} = 6x$$
$$f_{yy} = 6y$$
$$f_{xy} = 0$$

Step 1 At a stationary point,

$$f_x = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$
$$f_y = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

so there are four stationary points at (1, 1), (–1, –1), (1, –1) and (–1, 1).

Step 2 At (1, 1), $f_{xx} = f_{yy} = 6 > 0$ and $f_{xx} f_{yy} - f_{xy}^2 = 36 > 0$ so the point is a minimum.

At (–1, –1), $f_{xx} = f_{yy} = -6 < 0$ and $f_{xx} f_{yy} - f_{xy}^2 = 36 > 0$ so the point is a maximum.

At (1, –1) and (–1, 1), $f_{xx} f_{yy} - f_{xy}^2 = -36 < 0$ so these are both saddle points.

(b)

$$f_x = 3x^2 + 3y^2 - 6x$$
$$f_y = 6xy - 6y$$
$$f_{xx} = 6x - 6$$
$$f_{yy} = 6x - 6$$
$$f_{xy} = 6y$$

Step 1 At a stationary point, $f_y = (x-1)y = 0$ so either $x = 1$ or $y = 0$

When $x = 1$ the equation $f_x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

When $y = 0$, the equation, $f_x = 0 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or 2

so there are four stationary points at (2, 0), (0, 0), (1, 1) and (1, –1).

Step 2 At (2,0), $f_{xx} = f_{yy} = 6 > 0$ and $f_{xx} f_{yy} - f_{xy}^2 = 36 > 0$ so the point is a maximum.

At (0,0), $f_{xx} = f_{yy} = -6 < 0$ and $f_{xx} f_{yy} - f_{xy}^2 = 36 > 0$ so the point is a minimum.

At $(1, 1)$ and $(1, -1)$, $f_{xx} f_{yy} - f_{xy}^2 = -36 < 0$ so these are both saddle points.

2 At a stationary point,

$$\frac{\partial \pi}{\partial Q_1} = 24 - 2Q_1 - Q_2 = 0 \Rightarrow 2Q_1 + Q_2 = 24$$
$$\frac{\partial \pi}{\partial Q_2} = -Q_1 - 4Q_2 + 33 = 0 \Rightarrow Q_1 + 4Q_2 = 33$$

Subtracting twice the second equation from the first gives: $-7Q_2 = 42 \Rightarrow Q_2 = 6$

Substituting this into the second equation gives $Q_1 + 24 = 33 \Rightarrow Q_1 = 9$.

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -2 < 0, \quad \frac{\partial^2 \pi}{\partial Q_2^2} = -4 < 0$$

$$\left(\frac{\partial^2 \pi}{\partial Q_1^2}\right)\left(\frac{\partial^2 \pi}{\partial Q_2^2}\right) - \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2 = (-2)(-4) - (-1)^2 = 7 > 0 \quad \Rightarrow \quad \text{max}$$

3 $\pi = TR - TC = 70Q_1 + 50Q_2 - (Q_1^2 + Q_1Q_2 + Q_2^2) = 70Q_1 + 50Q_2 - Q_1^2 - Q_1Q_2 - Q_2^2$

$$\frac{\partial \pi}{\partial Q_1} = 70 - 2Q_1 - Q_2$$
$$\frac{\partial \pi}{\partial Q_2} = 50 - Q_1 - 2Q_2$$
$$\frac{\partial^2 \pi}{\partial Q_1^2} = -2$$
$$\frac{\partial^2 \pi}{\partial Q_2^2} = -2$$
$$\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} = -1$$

<u>Step 1</u> At a stationary point

$$2Q_1 + Q_2 = 70$$
$$Q_1 + 2Q_2 = 50$$

Doubling the second equation and subtracting it from the first gives $-3Q_2 = -30 \Rightarrow Q_2 = 10$

Substituting this into the second equation gives $Q_1 + 20 = 50 \Rightarrow Q_1 = 30$

Step 2 $\dfrac{\partial^2 \pi}{\partial Q_1^2} = -2 < 0,\ \dfrac{\partial^2 \pi}{\partial Q_2^2} = -2 < 0\ \ \dfrac{\partial \pi}{\partial Q_1^2}\dfrac{\partial \pi}{\partial Q_2^2} - \left(\dfrac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2 = 3 > 0$ so the point is a maximum

Finally substituting $Q_1 = 30$ and $Q_2 = 10$ into the formula for profit gives \$1300.

4

$$\frac{\partial U}{\partial x_1} = 260 + 5x_2 - 20x_1$$
$$\frac{\partial U}{\partial x_2} = 310 + 5x_1 - 2x_2$$
$$\frac{\partial^2 U}{\partial x_1^2} = -20$$
$$\frac{\partial^2 U}{\partial x_2^2} = -2$$
$$\frac{\partial^2 U}{\partial x_1 \partial x_2} = 5$$

Step 1 At a stationary point

$$\begin{aligned} 20x_1 - 5x_2 &= 260 \\ -5x_1 + 2x_2 &= 310 \end{aligned}$$

Adding four times the second equation to the first gives

$$3x_2 = 1500 \Rightarrow x_2 = 500$$

Substiuting this into the first equation gives

$$20x_1 - 2500 = 260 \Rightarrow 20x_1 = 2760 \Rightarrow x_1 = 138$$

Step 2

$$\frac{\partial^2 U}{\partial x_1^2} = -20 < 0;\ \frac{\partial^2 U}{\partial x_2^2} = -2 < 0;\ \frac{\partial^2 U}{\partial x_1^2}\frac{\partial^2 U}{\partial x_2^2} - \left(\frac{\partial^2 U}{\partial x_1 \partial x_2}\right)^2 = 3 > 0$$

so the stationary point is a maximum.

The individual works for 30 hours a week and earns \$500 so the hourly rate of pay is

$$\frac{500}{30} = \$16.67$$

5

$$TR = PQ = (100 - 2Q)Q = 100Q - 2Q^2 = 100(Q_1 + Q_2) - 2\left(Q_1 + Q_2\right)^2$$
$$= 100Q_1 + 100Q_2 - 2Q_1^2 - 2Q_2^2 - 4Q_1Q_2$$

Hence

$$\pi = TR - TC = 100Q_1 + 100Q_2 - 2Q_1^2 - 2Q_2^2 - 4Q_1Q_2 - (8Q_1 + Q_2^2)$$
$$= 92Q_1 + 100Q_2 - 2Q_1^2 - 3Q_2^2 - 4Q_1Q_2$$

$$\frac{\partial \pi}{\partial Q_1} = 92 - 4Q_1 - 4Q_2$$
$$\frac{\partial \pi}{\partial Q_2} = 100 - 6Q_2 - 4Q_1$$
$$\frac{\partial^2 \pi}{\partial Q_1^2} = -4$$
$$\frac{\partial^2 \pi}{\partial Q_2^2} = -6$$
$$\frac{\partial \pi}{\partial Q_1 \partial Q_2} = -4$$

<u>Step 1</u> At a stationary point

$$4Q_1 + 4Q_2 = 92$$
$$4Q_1 + 6Q_2 = 100$$

Subtract the second equation from the first:

$$-2Q_2 = -8 \Rightarrow Q_2 = 4$$

Subsituting this into the first equation gives

$$4Q_1 + 16 = 92 \Rightarrow 4Q_1 = 76 \Rightarrow Q_1 = 19$$

<u>Step 2</u>

$$\frac{\partial^2 \pi}{\partial Q_1^2} = -4 < 0; \quad \frac{\partial^2 \pi}{\partial Q_2^2} = -6 < 0; \quad \frac{\partial^2 \pi}{\partial Q_1^2}\frac{\partial^2 \pi}{\partial Q_2^2} - \left(\frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}\right)^2 = 8 > 0$$

so the stationary point is a maximum.

Section 5.5 Constrained optimization

Practice Problems

1 ***Step 1***

We are given that $y = x$, so no rearrangement is necessary.

Step 2

Substituting $y = x$ into the objective function

$$z = 2x^2 - 3xy + 2y + 10$$

gives

$$z = 2x^2 - 3x^2 + 2x + 10$$
$$= -x^2 + 2x + 10$$

Step 3

At a stationary point

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 0$$

that is,

$$-2x + 2 = 0$$

which has solution $x = 1$. Differentiating a second time gives

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -2$$

confirming that the stationary point is a maximum.

The value of z can be found by substituting $x = 1$ into

$$z = -2x^2 + 2x + 10$$

to get $z = 11$. Finally, putting $x = 1$ into the constraint $y = x$ gives $y = 1$. The constrained function therefore has a maximum value of 11 at the point (1, 1).

2 We want to maximize the objective function

$$U = x_1x_2$$

subject to the budgetary constraint

$$2x_1 + 10x_2 = 400$$

Step 1

$$x_1 = 200 - 5x_2$$

Step 2

$$U = 200x_2 - 5x_2^2$$

Step 3

$$\frac{\mathrm{d}U}{\mathrm{d}x_1} = 200 - 10x_2 = 0$$

has solution $x_2 = 20$.

$$\frac{\mathrm{d}^2U}{\mathrm{d}x_2^2} = -10 < 0$$

so maximum.

Putting $x_2 = 20$ into constraint gives $x_1 = 100$.

$$U_1 = \frac{\partial U}{\partial x_1} = x_2 = 20$$

and

$$U_2 = \frac{\partial U}{\partial x_2} = x_2 = 100$$

so the ratios of marginal utilities to prices are

$$\frac{U_1}{P_1} = \frac{20}{2} = 10$$

and

$$\frac{U_2}{P_2} = \frac{100}{10} = 10$$

which are the same.

3 We want to minimize the objective function

$$\text{TC} = 3x_1^2 + 2x_1x_2 + 7x_2^2$$

subject to the production constraint

$$x_1 + x_2 = 40$$

Step 1

$$x_1 = 40 - x_2$$

Step 2

$$\begin{aligned}\text{TC} &= 3(40 - x_2)^2 + 2(40 - x_2)x_2 + 7x_2^2 \\ &= 4800 - 160x_2 + 8x_2^2\end{aligned}$$

Step 3

$$\frac{\text{d(TC)}}{\text{d}x_2} = -160 + 16x_2 = 0$$

has solution $x_2 = 10$.

$$\frac{\text{d}^2\text{(TC)}}{\text{d}x_2^2} = 16 > 0$$

so minimum.

Finally, putting $x_2 = 10$ into constraint gives $x_1 = 30$.

Exercise 5.5 (p. 413)

1 **(a)** $9x + 3y = 2$

$3y = 2 - 9x$ (subtract $9x$ from both sides)

$y = \frac{2}{3} - 3x$ (divide both sides by 3)

(b) $z = 3xy = 3x\left(\frac{2}{3} - 3x\right) = 2x - 9x^2$

At a stationary point, $\frac{dz}{dx} = 0 \Rightarrow 2 - 18x = 0 \Rightarrow x = \frac{1}{9}$

Hence $y = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ and $z = 3 \times \frac{1}{9} \times \frac{1}{3} = \frac{1}{9}$

$\frac{d^2z}{dx^2} = -18 < 0$ so the stationary point is a maximum

2 $y - x^2 = 2 \Rightarrow y = x^2 + 2 \Rightarrow z = 6x - 3x^2 + 2(x^2 + 2) = 6x - x^2 + 4$

At a stationary point, $\frac{dz}{dx} = 0 \Rightarrow 6 - 2x = 0 \Rightarrow x = 3$

Hence y = 11 and z = 13.

$\frac{d^2z}{dx^2} = -2 < 0$ so the stationary point is a maximum

3 $x + y = 500 \Rightarrow y = 500 - x \Rightarrow z = 80x - 0.1x^2 + 100(500 - x) - 0.2(500 - x)^2$

$z = 80x - 0.1x^2 + 50000 - 100x - 50000 + 200x - 0.2x^2 = 180x - 0.3x^2$

At a stationary point, $\frac{dz}{dx} = 0 \Rightarrow 180 - 0.6x = 0 \Rightarrow x = 300$

Hence y = 200 and z = 27000.

$\frac{d^2z}{dx^2} = -0.6 < 0$ so the stationary point is a maximum

4 $50KL = 1200 \Rightarrow K = 24L^{-1}$

$TC = 2K + 3L = 48L^{-1} + 3L$

At a stationary point, $\dfrac{d(TC)}{dL} = 0 \Rightarrow -48L^{-2} + 3 = 0 \Rightarrow L^2 = 16 \Rightarrow L = 4$

Hence $K = 6$

$\dfrac{d^2(TC)}{dL^2} = 96L^{-3}$ so at L=4 the value of $\dfrac{d^2(TC)}{dL^2} = \dfrac{3}{2} > 0$ so the stationary point is a minimum.

5

$$x + y = 20 \Rightarrow y = 20 - x \Rightarrow TC = 22x^2 + 8(20-x)^2 - 5x(20-x)$$
$$TC = 22x^2 + 3200 - 320x + 8x^2 - 100x + 5x^2 = 35x^2 - 420x + 3200$$

At a stationary point $\dfrac{d(TC)}{dx} = 0 \Rightarrow 70x - 420 = 0 \Rightarrow x = 6 \Rightarrow y = 14$

$\dfrac{d^2(TC)}{dx^2} = 70 > 0 \Rightarrow \text{min}$

6

$$x_1 + 4x_2 = 360 \Rightarrow x_1 = 360 - 4x_2$$
$$U = x_1x_2 = (360 - 4x_2)x_2 = 360x_2 - 4x_2^2$$

At a stationary point, $\dfrac{dU}{dx_2} = 0 \Rightarrow 360 - 8x_2 = 0 \Rightarrow x_2 = 45 \Rightarrow U = 8100$

$\dfrac{d^2U}{dx_2^2} = -8 < 0$ so the stationary point is a maximum.

Section 5.6 Lagrange multipliers

Practice Problems

1 ***Step 1***

$g(x, y, \lambda) = 2x^2 - xy + \lambda(12 - x - y)$

Step 2

$$\frac{\partial g}{\partial x} = 4x - y - \lambda = 0$$
$$\frac{\partial g}{\partial y} = -x - \lambda = 0$$
$$\frac{\partial g}{\partial \lambda} = 12 - x - y = 0$$

that is,

$$4x - y - \lambda = 0 \qquad (1)$$

$$-x - \lambda = 0 \qquad (2)$$

$$x + y = 12 \qquad (3)$$

Multiply equation (2) by 4 and add equation (1), multiply equation (3) by 4 and subtract from equation (1) to get

$$-y - 5\lambda = 0 \qquad (4)$$

$$-5y - \lambda = -48 \qquad (5)$$

Multiply equation (4) by 5 and subtract equation (5) to get

$$-24\lambda = 48 \qquad (6)$$

Equations (6), (5) and (1) can be solved in turn to get

$\lambda = -2, y = 10, x = 2$

so the optimal point has coordinates (2, 10). The corresponding value of the objective function is

$$2(2)^2 - 2(10) = -12$$

2 Maximize

$$U = 2x_1x_2 + 3x_1$$

subject to

$$x_1 + 2x_2 = 83$$

Step 1

$$g(x_1, x_2, \lambda) = 2x_1x_2 + 3x_1 + \lambda(83 - x_1 - 2x_2)$$

Step 2

$$\frac{\partial g}{\partial x_1} = 2x_2 + 3 - \lambda = 0$$

$$\frac{\partial g}{\partial x_2} = 2x_1 - 2\lambda = 0$$

$$\frac{\partial g}{\partial \lambda} = 83 - x_1 - 2x_2 = 0$$

that is,

$$2x_2 - \lambda = -3 \quad (1)$$

$$2x_1 - 2\lambda = 0 \quad (2)$$

$$x_1 + 2x_2 = 83 \quad (3)$$

The easiest way of solving this system is to use equations (1) and (2) to get

$$\lambda = 2x_2 + 3 \quad \text{and} \quad \lambda = x_1$$

respectively. Hence

$$x_1 = 2x_2 + 3$$

Substituting this into equation (3) gives

$$4x_2 + 3 = 83$$

which has solution $x_2 = 20$ and so $x_1 = \lambda = 43$.

The corresponding value of U is

2(43)(20) + 3(43) = 1849

The value of λ is 43, so when income rises by 1 unit, utility increases by approximately 43 to 1892.

3 ***Step 1***

$$g(x_1,\ x_2,\lambda) = x_1^{1/2} + x_2^{1/2} + \lambda(M - P_1x_1 - P_2x_2)$$

Step 2

$$\frac{\partial g}{\partial x_1} = \frac{1}{2}x_1^{-1/2} - \lambda P_1 = 0$$
$$\frac{\partial g}{\partial x_2} = \frac{1}{2}x_2^{-1/2} - \lambda P_2 = 0$$
$$\frac{\partial g}{\partial \lambda} = M - P_1x_1 - P_2x_2 = 0$$

From equations (1) and (2)

$$\lambda = \frac{1}{2x_1^{1/2}P_1} \quad \text{and} \quad \lambda = \frac{1}{2x_2^{1/2}P_2}$$

respectively. Hence

$$\frac{1}{2x_1^{1/2}P_1} = \frac{1}{2x_2^{1/2}P_2}$$

that is,

$$x_1P_1^2 = x_2P_2^2$$

so

$$x_1 = \frac{x_2P_2^2}{P_1^2} \qquad (4)$$

Substituting this into equation (3) gives

$$M - \frac{x_2P_2^2}{P_1} - P_2x_2 = 0$$

which rearranges as

$$x_2 = \frac{P_1 M}{P_2(P_1 + P_2)}$$

Substitute this into equation (4) to get

$$x_1 = \frac{P_2 M}{P_1(P_1 + P_2)}$$

Exercise 5.6 (p. 424)

1 Step 1

$$g(x, y, \lambda) = x + 2xy + \lambda(5 - x - 2y)$$

Step 2

$$\frac{\partial g}{\partial x} = 1 + 2y - \lambda = 0 \Rightarrow \lambda - 2y = 1$$
$$\frac{\partial g}{\partial y} = 2x - 2\lambda = 0 \Rightarrow \lambda = x$$
$$\frac{\partial g}{\partial \lambda} = 5 - x - 2y = 0 \Rightarrow x + 2y = 5$$

From the first two equations we have $x - 2y = 1$

Adding this to the third equation we get $2x = 6 \Rightarrow x = 3$

The constraint gives $3 + 2y = 5 \Rightarrow 2y = 2 \Rightarrow y = 1$

Hence the maximum value of z is 9.

2 **(a)** Step 1

$$g(x, y, \lambda) = 4xy + \lambda(40 - x - 2y)$$

Step 2

$$\frac{\partial g}{\partial x} = 4y - \lambda = 0 \Rightarrow y = \frac{\lambda}{4}$$
$$\frac{\partial g}{\partial y} = 4x - 2\lambda = 0 \Rightarrow x = \frac{\lambda}{2}$$
$$\frac{\partial g}{\partial \lambda} = 40 - x - 2y = 0 \Rightarrow x + 2y = 40$$

Substituting the first two equations into the third gives

$$\frac{1}{2}\lambda + \frac{1}{2}\lambda = 40 \Rightarrow \lambda = 40 \text{ and so } x = 20 \text{ and } y = 10$$

The minimum value of z is 800

(b) Replacing 40 by 40.5 in part (a) gives $\lambda = 41$ and so $x = 20.5$, $y = 10.25$ and

$z = 840.5$

(c) The change is 40.5 compared to a multiplier of 40

3 Want to maximize $Q = KL$ subject to $2K + L = 6$

Step 1 $g(K, L, \lambda) = KL + \lambda(6 - 2K - L)$

Step 2

$$\frac{\partial g}{\partial K} = L - 2\lambda = 0 \Rightarrow L = 2\lambda$$
$$\frac{\partial g}{\partial L} = K - \lambda = 0 \Rightarrow K = \lambda$$
$$\frac{\partial g}{\partial \lambda} = 6 - 2K - L = 0 \Rightarrow 2K + L = 6$$

Substituting the first two equations into the third:

$4\lambda = 6 \Rightarrow \lambda = 1.5$ and so $K = 1.5$ and $L = 3$

Hence the maximum level of output is 4.5

4

$$\begin{aligned} TR &= P_1Q_1 + P_2Q_2 = (50 - Q_1 - Q_2)Q_1 + (100 - Q_1 - 4Q_2)Q_2 \\ &= 50Q_1 - Q_1^2 - Q_2Q_1 + 100Q_2 - Q_1Q_2 - 4Q_2^2 = 50Q_1 + 100Q_2 - 2Q_1Q_2 - Q_1^2 - 4Q_2^2 \end{aligned}$$

Hence

$$\pi = TR - TC = 50Q_1 + 100Q_2 - 2Q_1Q_2 - Q_1^2 - 4Q_2^2 - (5Q_1 + 10Q_2) = 45Q_1 + 90Q_2 - 2Q_1Q_2 - Q_1^2 - 4Q_2^2$$

Step 1

$$g(Q_1, Q_2, \lambda) = 45Q_1 + 90Q_2 - 2Q_1Q_2 - Q_1^2 - 4Q_2^2 + \lambda(100 - 5Q_1 - 10Q_2)$$

Step 2

$$\frac{\partial g}{\partial Q_1} = 45 - 2Q_2 - 2Q_1 - 5\lambda = 0 \Rightarrow 2Q_1 + 2Q_2 + 5\lambda = 45$$
$$\frac{\partial g}{\partial Q_2} = 90 - 2Q_1 - 8Q_2 - 10\lambda = 0 \Rightarrow 2Q_1 + 8Q_2 + 10\lambda = 90$$
$$\frac{\partial g}{\partial \lambda} = 100 - 5Q_1 - 10Q_2 = 0 \Rightarrow 5Q_1 + 10Q_2 = 100$$

The Lagrange multiplier can be eliminated from the first two equations by doubling the first and subtracting the second:

$$2Q_1 - 4Q_2 = 0 \Rightarrow Q_1 = 2Q_2$$

Substituting this into the third equation gives $20Q_2 = 100 \Rightarrow Q_2 = 5$

Hence $Q_1 = 10$.

Substituting these values into the formula for π shows that the maximum profit is $600.

Substituting these values into the first equation gives $30 + 5\lambda = 45 \Rightarrow 5\lambda = 15 \Rightarrow \lambda = 3$

Lagrange multiplier is 3, so profit rises to $603 when total cost increases by 1 unit.

Chapter 6 Integration

Section 6.1 Indefinite integration

Practice Problems

1 **(a)** x^2; **(b)** x^4; **(c)** x^{100}; **(d)** $\frac{1}{4}x^4$; **(e)** $\frac{1}{19}x^{19}$.

1st year = \$1600; 2nd year = \$2925; 3rd year = \$3788; 4th year = \$4486 so the first year it happens is the fourth

8 $P = \int_0^8 12000e^{-0.075t}dt = \left[-160000e^{-0.075t}\right]_0^8 = \$72190.14$

Chapter 7 Matrices

Section 7.1 Basic matrix operations

Practice Problems

1 **(a)** 2 × 2, 1 × 5, 3 × 5, 1 × 1.

(b) 1, 4, 6, 2, 6, ?, 6; the value of c_{43} does not exist, because C has only three rows.

2 $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 4 & 7 & 1 & -5 \\ 0 & 6 & 3 & 1 \\ 1 & 1 & 5 & 8 \\ 2 & 4 & -1 & 0 \end{bmatrix}$

$\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 5 \\ 7 \\ 9 \end{bmatrix}$

$\mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \mathbf{C}$

Matrices with the property that $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$ are called symmetric. Elements in the top right-hand corner are a mirror image of those in the bottom left-hand corner.

3 **(a)** $\begin{bmatrix} 1 & 7 \\ 3 & -8 \end{bmatrix}$; **(c)** $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$**;** **(d)** $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$; **(e)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Part (b) is impossible because **A** and **C** have different orders.

4 **(1)** **(a)** $\begin{bmatrix} 2 & -4 \\ 6 & 10 \\ 0 & 8 \end{bmatrix}$; **(b)** $\begin{bmatrix} 2 & -2 \\ 4 & 14 \\ 2 & 12 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & -3 \\ 5 & 12 \\ 1 & 10 \end{bmatrix}$; **(d)** $\begin{bmatrix} 2 & -6 \\ 10 & 24 \\ 2 & 20 \end{bmatrix}$.

From (a) and (b)

$$2A + 2B = \begin{bmatrix} 2 & -4 \\ 6 & 10 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 4 & 14 \\ 2 & 12 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 10 & 24 \\ 2 & 20 \end{bmatrix}$$

which is the same as (d), so

$2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$

(2) **(a)** $\begin{bmatrix} 3 & -6 \\ 9 & 15 \\ 0 & 12 \end{bmatrix}$; **(b)** $\begin{bmatrix} -6 & 12 \\ -18 & -30 \\ 0 & -24 \end{bmatrix}$.

From (a),

$$-2(3A) = -2\begin{bmatrix} 3 & -6 \\ 9 & 15 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ -18 & -30 \\ 0 & -24 \end{bmatrix}$$

which is the same as (b), so

$-2(3\mathbf{A}) = -6\mathbf{A}$

5 **(a)** [8] because

$$1(0) + (-1)(-1) \\ + 0(1) + 3(1) + 2(2) = 8$$

(b) [0] because $1(-2) + 2(1) + 9(0) = 0$.

(c) This is impossible, because **a** and **d** have different numbers of elements.

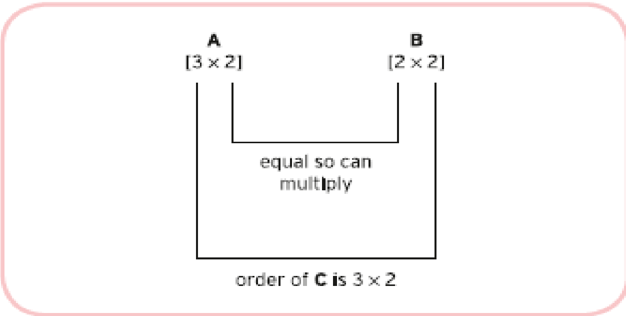

Figure S7.1

6

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} \boxed{1} & \boxed{2} \\ 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 \\ \boxed{3} & 4 \end{bmatrix} = \begin{bmatrix} 7 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} \boxed{1} & \boxed{2} \\ 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \boxed{2} \\ 3 & \boxed{4} \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ \boxed{0} & \boxed{1} \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 \\ \boxed{3} & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ \boxed{3} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ \boxed{0} & \boxed{1} \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \boxed{2} \\ 3 & \boxed{4} \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & \boxed{4} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ \boxed{3} & \boxed{1} \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 \\ \boxed{3} & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \\ \boxed{6} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ \boxed{3} & \boxed{1} \end{bmatrix} \begin{bmatrix} 1 & \boxed{2} \\ 3 & \boxed{4} \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \\ 6 & \boxed{10} \end{bmatrix}$$

7 **(a)** $\begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$; **(d)** $\begin{bmatrix} 4 & 3 \\ 2 & -1 \\ 5 & 5 \end{bmatrix}$; **(f)** $\begin{bmatrix} 9 & 6 & 13 \\ 27 & 15 & 28 \end{bmatrix}$;

(g) $\begin{bmatrix} 5 & 7 & 9 \\ 3 & 3 & 3 \\ 6 & 9 & 12 \end{bmatrix}$; **(h)** $\begin{bmatrix} 5 & 6 \\ 11 & 15 \end{bmatrix}$.

Parts (b), (c) and (e) are impossible because, in each case, the number of columns in the first matrix is not equal to the number of rows in the second.

8 **Ax** is the 3×1 matrix

$$\begin{bmatrix} x + 4y + 7z \\ 2z + 6y + 5z \\ 8x + 9y + 5z \end{bmatrix}$$

However, $x + 4y + 7z = -3$, $2x + 6y + 5z = 10$ and $8x + 9y + 5z = 1$, so this matrix is just

$$\begin{bmatrix} -3 \\ 10 \\ 1 \end{bmatrix}$$

which is **b**. Hence **Ax** = **b**.

Exercise 7.1 (p. 478)

1 **(a)** $\mathbf{J} = \begin{bmatrix} 35 & 27 & 13 \\ 42 & 39 & 24 \end{bmatrix}$; $\mathbf{F} = \begin{bmatrix} 31 & 17 & 3 \\ 25 & 29 & 16 \end{bmatrix}$.

(b) $\begin{bmatrix} 66 & 44 & 16 \\ 67 & 68 & 40 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 10 & 10 \\ 17 & 10 & 8 \end{bmatrix}$

2 **(a)** $\begin{bmatrix} 4 & 6 & 2 & 18 \\ 2 & 0 & 10 & 0 \\ 12 & 14 & 16 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 14 & 18 & 12 \\ 4 & 2 & 0 & 10 \\ 12 & 8 & 10 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 20 & 20 & 30 \\ 6 & 2 & 10 & 10 \\ 24 & 22 & 26 & 14 \end{bmatrix}$

(d) Same answer as (c).

3 4**B**, (**CB**)^T, **CBA** are possible with order 2×3, 3×4, 2×4 respectively.

4 **(a)** $\begin{bmatrix} 5900 \\ 1100 \end{bmatrix}$

Total cost charged to each customer.

(b) $\begin{bmatrix} 13 & 7 & 23 & 22 \\ 3 & 1 & 4 & 5 \end{bmatrix}$

Amount of raw materials used to manufacture each customer's goods.

(c) $\begin{bmatrix} 35 \\ 75 \\ 30 \end{bmatrix}$

Total raw material costs to manufacture one item of each good.

(d) $\begin{bmatrix} 1005 \\ 205 \end{bmatrix}$

Total raw material costs to manufacture requisite number of goods for each customer.

(e) [7000]

Total revenue received from customers.

(f) [1210]

Total cost of raw materials.

(g) [5790]

Profit before deduction of labour, capital and overheads.

5 **(1)** **(a)** $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 \\ 5 & 5 \\ 2 & 10 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 5 & 2 \\ 1 & 5 & 10 \end{bmatrix}$

$(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$: that is, 'transpose of the sum is the sum of the transposes'.

(2) **(a)** $\begin{bmatrix} 1 & 5 \\ 4 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 1 \\ 1 & 5 \\ 4 & 9 \end{bmatrix}$

$(\mathbf{CD})^{\mathrm{T}} = \mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}$: that is 'transpose of a product is the product of the transposes multiplied in reverse order'.

6 **(a)** $\mathbf{B} + \mathbf{C} = \begin{bmatrix} 0 & 6 \\ 5 & 2 \end{bmatrix}$

so $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -15 & 24 \\ 5 & 14 \end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix} -7 & 25 \\ 6 & 10 \end{bmatrix}$ and

$\mathbf{AC} = \begin{bmatrix} -8 & -1 \\ -1 & 4 \end{bmatrix}$, so

$\mathbf{AB} + \mathbf{AC} = \begin{bmatrix} -15 & 24 \\ 5 & 14 \end{bmatrix}$

(b) $\mathbf{AB} = \begin{bmatrix} -7 & 25 \\ 6 & 10 \end{bmatrix}$, so

$(\mathbf{AB})\mathbf{C} = \begin{bmatrix} 32 & 43 \\ 4 & 26 \end{bmatrix}$

$\mathbf{BC} = \begin{bmatrix} 4 & 11 \\ -4 & 4 \end{bmatrix}$, so

$\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 32 & 43 \\ 4 & 26 \end{bmatrix}$

7 $\mathbf{AB} = [-3]$; $\mathbf{BA} = \begin{bmatrix} 1 & 2 & -4 & 0 \\ 7 & 14 & -28 & 21 \\ 3 & 6 & -12 & 9 \\ -2 & -4 & 8 & -6 \end{bmatrix}$

8 **(a)** $\begin{bmatrix} 7x + 5y \\ x + 3y \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$.

Section 7.2 Matrix inversion

Practice Problems

1 $|\mathbf{A}| = 6(2) - 4(1) = 8 \neq 0$

so **A** is non-singular and its inverse is given by

$$\frac{1}{8}\begin{bmatrix} 2 & -4 \\ -1 & -6 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/2 \\ -1/8 & 3/4 \end{bmatrix}$$

$$|\mathbf{B}| = 6(2) - 4(3) = 0$$

so **B** is singular and its inverse does not exist.

2 We need to solve **Ax** = **b**, where

$$\mathbf{A} = \begin{bmatrix} 9 & 1 \\ 2 & 7 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 43 \\ 57 \end{bmatrix}$$

Now

$$\mathbf{A}^{-1} = \frac{1}{61}\begin{bmatrix} 7 & -1 \\ -2 & 9 \end{bmatrix}$$

so

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{61}\begin{bmatrix} 7 & -1 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} 43 \\ 57 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

3 In equilibrium, $Q_S = Q_D = Q$, say, so the supply equation becomes

$$P = aQ + b$$

Subtracting aQ from both sides gives

$$P - aQ = b \qquad (1)$$

Similarly, the demand equation leads to

$$P + cQ = d \tag{2}$$

In matrix notation equations (1) and (2) become

$$\begin{bmatrix} 1 & -a \\ 1 & c \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

The coefficient matrix has an inverse,

$$\frac{1}{c+a} \begin{bmatrix} c & a \\ -1 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \frac{1}{c+a} \begin{bmatrix} c & a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix}$$

that is,

$$P = \frac{cb + ad}{c+a} \quad \text{and} \quad Q = \frac{-b + d}{c+a}$$

The multiplier for Q due to changes in b is given by the element in row 2, column 1, of the inverse matrix so is

$$\frac{-1}{c+a}$$

Given that c and a are both positive it follows that the multiplier is negative. Consequently, an increase in b leads to a decrease in Q.

4 $A_{11} = +\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$A_{13} = +\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$$

$$A_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$$

$$A_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$A_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$A_{31} = +\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$$

$$A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$A_{33} = +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

5 Expanding along the top row of **A** gives

$$\begin{aligned} |\mathbf{A}| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= 1(7) + 3(-1) + 3(-1) = 1 \end{aligned}$$

using the values of A_{11}, A_{12} and A_{13} from Practice Problem 4. Other rows and columns are treated similarly. Expanding down the last column of **B** gives

$$\begin{aligned} |\mathbf{B}| &= b_{13}B_{13} + b_{23}B_{23} + b_{33}B_{33} \\ &= 0(B_{13}) + 0(B_{23}) + 0(B_{33}) = 0 \end{aligned}$$

6 The cofactors of **A** have already been found in Practice Problem 4. Stacking them in their natural positions gives the adjugate matrix

$$\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Transposing gives the adjoint matrix

$$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The determinant of **A** has already been found in Practice Problem 5 to be 1, so the inverse matrix is the same as the adjoint matrix.

The determinant of **B** has already been found in Practice Problem 5 to be 0, so **B** is singular and does not have an inverse.

7 Using the inverse matrix in Practice Problem 6,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 32 \\ 37 \\ 35 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}$$

Exercise 7.2 (p. 497)

1 **(1)** **(a)** $|\mathbf{A}| = -3$;

(b) $|\mathbf{B}| = 4$;

(c) $\mathbf{AB} = \begin{bmatrix} 4 & 4 \\ 7 & 4 \end{bmatrix}$

so $|\mathbf{AB}| = -12$. These results give $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$: that is, 'determinant of a product is the product of the determinants'.

(2) **(a)** $\mathbf{A}^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 5/3 & -2/3 \end{bmatrix}$

(b) $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -1/2 & -1/4 \end{bmatrix}$

(c) $(\mathbf{AB})^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 7/12 & -1/3 \end{bmatrix}$

These results give $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$: that is, 'inverse of a product is the product of the inverses multiplied in reverse order'.

2 $2a + 3 = 0 \Rightarrow a = -3/2$

$-8 - 3b = 0 \Rightarrow b = -8/3$

3 **(a)** $\begin{bmatrix} x \\ y \end{bmatrix} = -\dfrac{1}{23}\begin{bmatrix} -1 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Hence $x = 1, y = -1$;

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = -\dfrac{1}{13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ Hence $x = 2, y = 2$.

4 $\dfrac{1}{25}\begin{bmatrix} -9 & -1 \\ -2 & -3 \end{bmatrix}$;

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} -9 & -1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -110 \\ -10 \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \end{bmatrix};$$

$P_1 = 40;\quad P_2 = 10$

5 **(a)** $50 - 2P_1 + P_2 = -20 + P_1 \quad \Rightarrow \quad 3P_1 - P_2 = 70$

$10 + P_1 - 4P_2 = -10 + 5P_2 \quad \Rightarrow \quad -P_1 + 9P_2 = 20$

(b) Inverse $= \dfrac{1}{26}\begin{bmatrix} 9 & 1 \\ 1 & 3 \end{bmatrix}$

$$\frac{1}{26}\begin{bmatrix} 9 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 70 \\ 20 \end{bmatrix} = \begin{bmatrix} 25 \\ 5 \end{bmatrix}$$

$P_1 = 25,\ P_2 = 5$

Section 7.3 Cramer's rule

Practice Problems

1 **(a)** By Cramer's rule

$$x_2 = \frac{\det(A_2)}{\det(A)}$$

where

$$\det(\mathbf{A}_2) = \begin{vmatrix} 2 & 16 \\ 3 & -9 \end{vmatrix} = -66$$

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix} = -22$$

Hence

$$x_2 = \frac{-66}{-22} = 3$$

(b) By Cramer's rule

$$x_3 = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})}$$

where

$$\begin{aligned} \det(\mathbf{A}_3) &= \begin{vmatrix} 4 & 1 & 8 \\ -2 & 5 & 4 \\ 3 & 2 & 9 \end{vmatrix} \\ &= 4\begin{vmatrix} 5 & 4 \\ 2 & 9 \end{vmatrix} - 1\begin{vmatrix} -2 & 4 \\ 3 & 9 \end{vmatrix} + 8\begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix} \\ &= 4(37) - 1(-30) + 8(-19) \\ &= 26 \end{aligned}$$

and

$$\det(\mathbf{A}) = \begin{vmatrix} 4 & 1 & 3 \\ -2 & 5 & 1 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= 4\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix}$$

$$= 4(18) - 1(-11) + 3(-19)$$

$$= 26$$

Hence

$$x_3 = \frac{26}{26} = 1$$

1. The variable Y_d is the third, so Cramer's rule gives

$$Y_d = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})}$$

where

$$\mathbf{A}_3 = \begin{bmatrix} 1 & -1 & I^* + G^* & 0 \\ 0 & 1 & b & 0 \\ -1 & 0 & 0 & 1 \\ -t & 0 & T^* & 1 \end{bmatrix}$$

Expanding along the second row gives

$$\det(\mathbf{A}_3) = 1\begin{vmatrix} 1 & I^* + G^* & 0 \\ -1 & 0 & 1 \\ -t & T^* & 1 \end{vmatrix} - \mathrm{b}\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ -t & 0 & 0 \end{vmatrix}$$

since along the second row the pattern is '$- + - +$'. Now

$$\begin{vmatrix} 1 & I^* + G^* & 0 \\ -1 & 0 & 1 \\ -t & T^* & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 4 \\ T^* & 1 \end{vmatrix} - (I^* + G^*)\begin{vmatrix} -1 & 1 \\ -t & 1 \end{vmatrix}$$

$$= T^* - (I^* + G^*)(-1 + t)$$

(expanding along the first row) and

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ -t & 0 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} -1 & 1 \\ -t & 1 \end{vmatrix} = -1+t$$

(expanding down the second column).

Hence

$$\det(\mathbf{A}_3) = -T^* - (I^* + G^*)(-1 + t) - b(-1 + t)$$

From the worked example given in the text,

$$\det(\mathbf{A}) = 1 - a + at$$

Hence

$$\mathrm{y_d} = \frac{-T^* - (I^* + G^*)(-1+t) - b(-1+t)}{1-a+at}$$

3 Substituting C_1, M_1 and I^*_1 into the equation for Y_1 gives

$$Y_1 = 0.7Y_1 + 50 + 200 + X_1 - 0.3Y_1$$

Also, since $X_1 = M_2 = 0.1Y_2$, we get

$$Y_1 = 0.7Y_1 + 50 + 200 + 0.1Y_2 - 0.3Y_1$$

which rearranges as

$$0.6Y_1 - 0.1Y_2 = 250$$

In the same way, the second set of equations leads to

$$-0.3Y_1 + 0.3Y_2 = 400$$

Hence

$$\begin{bmatrix} 0.6 & -0.1 \\ -0.3 & 0.6 \end{bmatrix}\begin{bmatrix} \mathrm{Y_1} \\ \mathrm{Y_2} \end{bmatrix} = \begin{bmatrix} 250 \\ 400 \end{bmatrix}$$

In this question both Y_1 and Y_2 are required, so it is easier to solve using matrix inverses rather than Cramer's rule, which gives

$$Y_1 = \frac{1}{0.15}\begin{bmatrix}0.3 & 0.1\\ 0.1 & 0.6\end{bmatrix}\begin{bmatrix}250\\ 400\end{bmatrix}$$
$$= \frac{1}{0.15}\begin{bmatrix}115\\ 315\end{bmatrix}$$

Hence Y_1 = 766.67 and Y_2 = 2100. The balance of payments for country 1 is

$$\begin{aligned} X_1 - M_1 &= M_2 - M_1 \\ &= 0.1Y_2 - 0.3Y_1 \\ &= 0.1(2100) - 0.3(766.67) \\ &= -20 \end{aligned}$$

Moreover, since only two countries are involved, it follows that country 2 will have a surplus of 20

Exercise 7.3 (p. 508)

1 **(a)** $x = \dfrac{\begin{vmatrix}4 & -3\\ 7 & 5\end{vmatrix}}{\begin{vmatrix}7 & -3\\ 2 & 5\end{vmatrix}} = \dfrac{41}{41} = 1$

(b) $x = \dfrac{\begin{vmatrix}5 & 4\\ 12 & 5\end{vmatrix}}{\begin{vmatrix}-3 & 4\\ 2 & 5\end{vmatrix}} = \dfrac{-23}{-23} = 1$;

(c) $x = \dfrac{\begin{vmatrix}9 & 4\\ 3 & -7\end{vmatrix}}{\begin{vmatrix}1 & 4\\ 2 & -7\end{vmatrix}} = \dfrac{-75}{-15} = 5$

2 **(a)** $y = \dfrac{\begin{vmatrix}1 & 9\\ 2 & -2\end{vmatrix}}{\begin{vmatrix}1 & 3\\ 2 & -4\end{vmatrix}} = \dfrac{-20}{-10} = 2$

(b) $y = \dfrac{\begin{vmatrix} 5 & 7 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix}} = \dfrac{-19}{19} = -1$

(c) $y = \dfrac{\begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix}} = \dfrac{-19}{-19} = 1$

3 **(a)** $x = \dfrac{\begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{14}{14} = 1;$ $y = \dfrac{\begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{-14}{14} = -1$

(b) $x = \dfrac{\begin{vmatrix} 1 & 3 \\ 11 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{-28}{14} = -2;$ $y = \dfrac{\begin{vmatrix} 4 & 1 \\ 2 & 11 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{42}{14} = 3$

(c) $x = \dfrac{\begin{vmatrix} -2 & 3 \\ -36 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{98}{14} = 7;$ $y = \dfrac{\begin{vmatrix} 4 & -2 \\ 2 & -36 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{-140}{14} = -10$

4 **(a)**

$$400 - 5P_1 - 3P_2 = -60 + 3P_1 \Rightarrow 8P_1 + 3P_2 = 460$$
$$300 - 2P_1 - 3P_2 = -100 + 2P_2 \Rightarrow 2P_1 + 5P_2 = 400$$

(b) $\dfrac{\begin{vmatrix} 460 & 3 \\ 400 & 5 \end{vmatrix}}{\begin{vmatrix} 8 & 3 \\ 2 & 5 \end{vmatrix}} = \dfrac{1100}{34} = \dfrac{550}{17} = 32\dfrac{6}{17}$

5 **(a)**

$$Y - C = I^*$$
$$-aY + C = b$$

$$\begin{bmatrix} 1 & -1 \\ -a & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I^* \\ b \end{bmatrix}$$

(b) $$C = \frac{\begin{vmatrix} 1 & I^* \\ -a & b \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -a & 1 \end{vmatrix}} = \frac{b + aI^*}{1-a}$$

Chapter 8 Linear Programming

Section 8.1 Graphical solution of linear programming problems

Practice Problems

1 The line $-x + 3y = 6$ passes through (0, 2) and (–6, 0). Substituting $x = 1$, $y = 4$ into the equation gives

$$-1 + 3(4) = 11$$

This is greater than 6, so the test point satisfies the inequality. The corresponding region is shown in Figure S8.1.

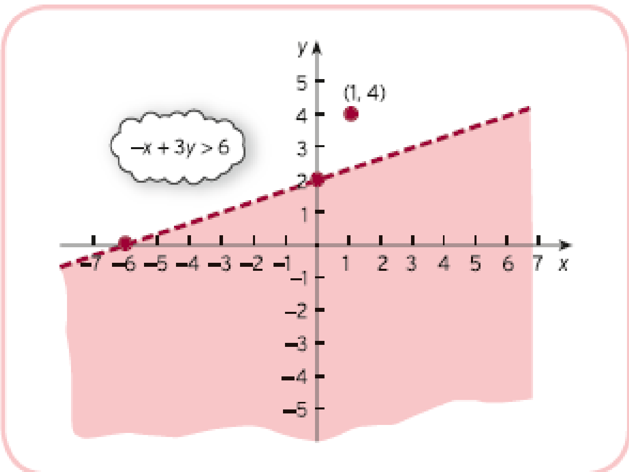

Figure S8.1