Lecture notes Exponentials and Logarithms (Chapter 10)

Kevin Wainwright

September 20, 2012

1 Simple Compound Interest

Suppose you have x dollars to invest at an interest rate of r percent per year. In one year you will have y dollars, where

$$y = x + rx = x(1+r)$$

in two years

$$y = [x(1+r)](1+r) = x(1+r)^2$$

in three years

$$y = \left[x(1+r)^2\right](1+r) = x(1+r)^3$$

The present value (PV) of y 3 years from now is

$$x = \frac{y}{(1+r)^3} = y(1+r)^{-3}$$

PV: Tells you "how much to invest now" in order to have y dollars in 3 years.

1.1 Compounding Within a Year

1. (a) Semi-annual compounding
   at six months
   $$y = x\left(1 + \frac{r}{2}\right) = x + \frac{xr}{2}$$
   at one year
   $$y = \left[x\left(1 + \frac{r}{2}\right)\right]\left(1 + \frac{r}{2}\right) = x\left(1 + \frac{r}{2}\right)^2$$
   (b) Monthly compounding
   $$\begin{aligned} y &= x\left(1 + \frac{r}{12}\right)^{12} \quad \text{for one year} \\ y &= \left[x\left(1 + \frac{r}{12}\right)^{12}\right]\left(1 + \frac{r}{2}\right) = x\left(1 + \frac{r}{12}\right)^{24} \quad \text{for two years} \\ y &= x\left(1 + \frac{r}{12}\right)^{12n} \quad \text{for n years} \end{aligned}$$

1.2 Converting Compound Interest into an Annual Yield

Suppose you are offered a choice:

1. 10% compounded semi-annually, or
2. 10.2% annually

Which would you choose?
We know for semi-annual

$$\begin{aligned} y &= x\left(1 + \frac{r}{2}\right)^2 = x\left(1 + \frac{.10}{2}\right)^2 = (1.05)^2 x \\ y &= 1.1025x \end{aligned}$$

Yield = y - principal = $y - x$

Yield = $1.1025x - x = 0.1025x$ or you can earn 10.25% annually since 10.25% > 10.20% $\Longrightarrow\Longrightarrow$ Pick option (1)

1.3 Continuous Compounding

1. (a) Daily interest for one year

$$y = x\left(1 + \frac{r}{365}\right)^{365}$$

Suppose x=$1 and r=100% (or r=1)

$$y = 1\left(1 + \frac{1}{365}\right)^{365} = \$2.71456$$

(b) Compound hourly (365 x 24=8760)

$$y = x\left(1 + \frac{1}{8760}\right)^{8760} = \$2.71812$$

or if

$$y = 1\left(1 + \frac{1}{m}\right)^{m}$$

if we let m $\Longrightarrow$ infinity (∞)

$$y = \left(1 + \frac{1}{m}\right)^{m} \Longrightarrow 2.71828... \doteq e$$

for any r as $m \longrightarrow \infty$ {and $x = \$1$}

$$y = \left(1 + \frac{1}{m}\right)^{m} \Longrightarrow e^{r}$$

2 The Number "e"

The number $e = 2.71828...$ is the value of \$1 compounded continuously for one year (or one period) at an interest rate of 100%.

Continuous compounding at r percent for t years of a principal equal to x

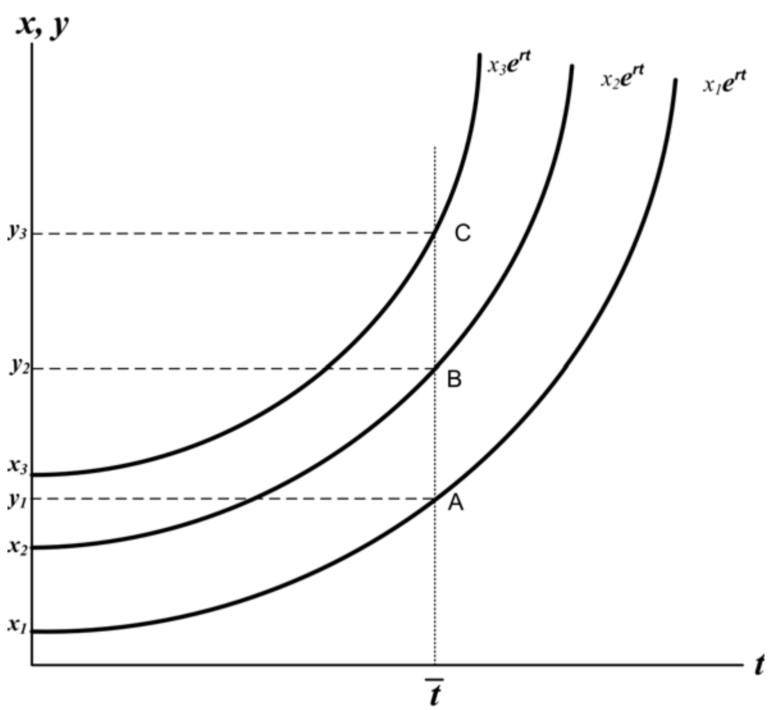
$$y = xe^{rt}$$

The present value of y is

$$x = \frac{y}{e^{rt}} = ye^{-rt}$$

which tells you the amount needed to invest today that will be worth y dollars in t years of continuous compounding

Present Value (xe^{rt}) Graphically



Slope $= \frac{dy}{dt} = rxe^{rt}$

3 Derivative rules of e

1. $$y = e^x \qquad \frac{dy}{dx} = e^x$$

2. $$y = e^{f(x)} \qquad \frac{dy}{dx} = f'(x)e^{f(x)}$$

3. Examples:

   (a) $y = e^{3x}$ $\qquad \frac{dy}{dx} = 3e^{3x}$

   (b) $y = e^{-rt}$ $\qquad \frac{dy}{dt} = -re^{-rt}$

   (c) $y = ae^{(t^2-t)}$ $\qquad \frac{dy}{dt} = a\,(2t-1)\,e^{\left(t^2-t\right)}$

4. $$e^{-\infty} = \frac{1}{e^{\infty}} \approx 0 \qquad e^0 = 1$$

3.1 Growth Rates

Given

$$y = xe^{rt}$$

The change in y is

$$\frac{dy}{dt} = rxe^{rt} = ry$$

However, the percentage change in y, or the "growth rate" is

$$\text{Growth Rate} = \frac{\Delta\ in\ y}{y} \approx \frac{dy}{y}$$

Therefore

$$\text{Growth Rate } = \frac{\frac{dy}{dt}}{y} = \frac{rxe^{rt}}{xe^{rt}} = r$$

Where r is the continuous rate of growth of y over time. NOTE: the growth rate is constant, however, the slope of $y = xe^{rt}$ is <u>not</u> constant.

4 Logarithms

4.1 Common Log (or log base 10)

Given

$$10^2 = 100$$

The exponent 2 is defined as the logarithm of 100 to the base 10.

eg.

$$\begin{array}{ccc} \log 1000 = 3 & \text{because} & \left\{10^3 = 1000\right\} \\ \log 10 = 1 & \text{because} & 10^1 = 10 \\ \log 1 = 0 & \text{because} & 10^0 = 1 \\ \log 0.1 = -1 & \text{because} & 10^{-1} = .1 \\ \log 0.01 = -2 & \text{because} & 10^{-2} = .001 \end{array}$$

4.2 Natural Logarithm

If $y = e^x$ $\ln y = \ln e^x = x$ where ln is the logarithm to base e

4.3 Rules of Logarithms

1. $\ln(AB) = \ln A + \ln B$
2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3. $\ln(A^b) = b \ln A$

4.3.1 Example:

$$\ln(x^3y^2) = 3\ln x + 2\ln y$$

4.3.2 Other Properties

if $x = y$ then $\ln x = \ln y$
if $x > y$ then $\ln x > \ln y$

**$\ln(-3)$ does NOT exist!! You cannot take a logarithm of a negative number.
**$ln(A+B) \neq \ln A + \ln B$!!!

5 Derivatives of the Natural Logarithm

1. $y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x}$ or $dy = \frac{dx}{x}$

2. $y = \ln ax \qquad \frac{dy}{dx} = \frac{a}{ax} = \frac{1}{x}$

   OR $y = \ln ax = \ln x + \ln a$

   $\frac{dy}{dx} = \frac{1}{x}\left\{\text{since } \frac{d(\ln a)}{dx} = 0\right\}$

3. $y = ln(x^2 + 2x)$

   $\frac{dy}{dx} = \frac{1}{(x^2+2x)}(2x+2) = \frac{2x+2}{x^2+2x} = \frac{1}{x+2} + \frac{1}{x}$

   OR $y = ln(x^2+2x) = \ln\left[(x+2)\,x\right] = \ln(x+2) + \ln x$
   $\frac{dy}{dx} = \underline{\underline{\frac{1}{x+2} + \frac{1}{x}}}$

6 Optimal Timing Problems

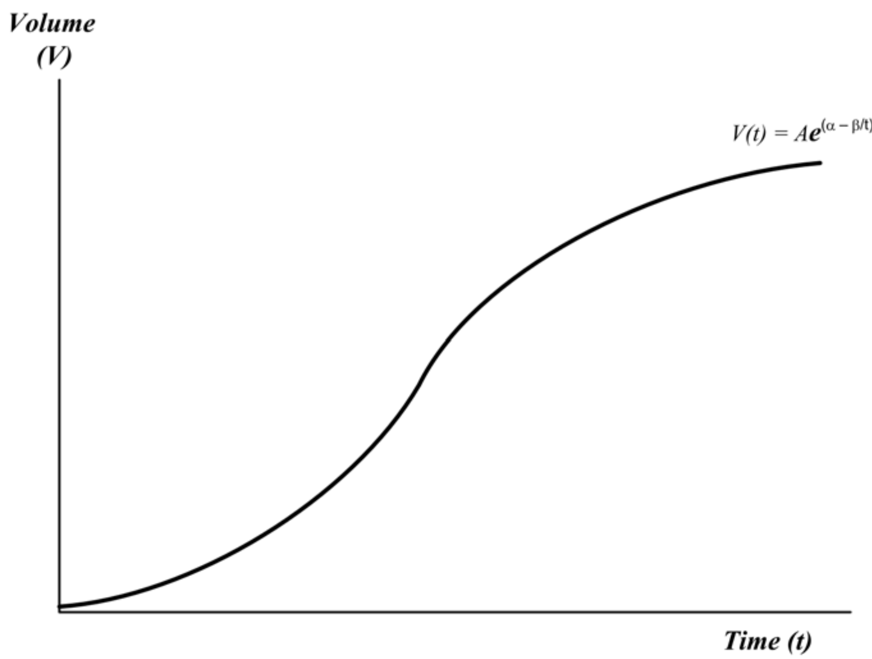


Figure 1:

6.1 The Forest Harvesting Problem

Assume a stand of trees grows according to the following function

$$\text{Volume} = V(t) = Ae^{\alpha - \frac{\beta}{t}} \quad \left\{ \text{measured in } (\text{ft})^3 \right\}$$

Question: When is the best time to harvest the stand of trees?

· For simplicity, assume that the price per ft^3 for lumber is \$1 and it remains constant over time

· if the market rate of interest is r gthen the problem is to choose a time to harvest the trees that maximizes the present value of the asset

· at any time, t_0 the present value is:

$$\begin{aligned} PV &= V(t_0)\, e^{-rt_0} \\ &= \left(Ae^{\alpha - \frac{\beta}{t_0}}\right) e^{-rt} \\ PV &= Ae^{\alpha - \frac{\beta}{t} - rt} \end{aligned}$$

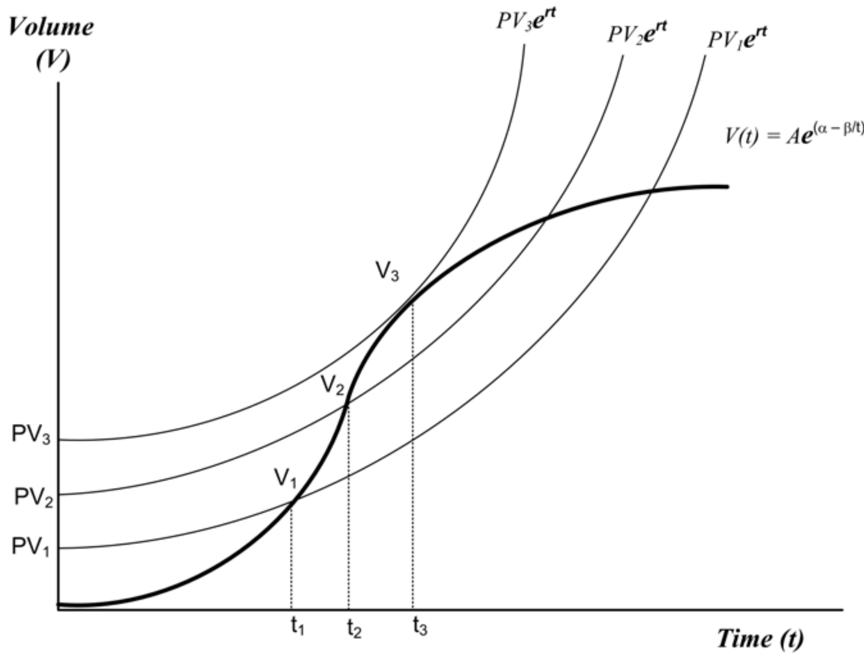


Figure 2:

Optimal harvest time: t_3

Maximum present value: $PV_3 \left\{ \frac{V_3'}{V_3} = r \right\}$

At V_1 the growth rate of trees exceeds the growth rate of a financial asset since $\left(\frac{V_1(t)'}{V_1} > r \right)$

Present Value is

$$
\begin{aligned}
PV(t) &= V(t)e^{-rt} \\
PV(t) &= Ae^{\alpha - \frac{\beta}{t} - rt} \qquad \left\{ V(t) = Ae^{\alpha - \frac{\beta}{t}} \right\}
\end{aligned}
$$

Max PV with respect to t $\left\{ \frac{dPV}{dt} = 0 \right\}$

$$
\begin{aligned}
\frac{dPV}{dt} &= Ae^{\alpha-\frac{\beta}{t}-rt}\left(\frac{\beta}{t^2}-r\right)=0 \\
\frac{dPV}{dt} &= 0 \;\; \text{If} \;\; \frac{\beta}{t^2}-r=0
\end{aligned}
$$

Therefore

$$
\frac{\beta}{t^2}=r \text{ or } t=\sqrt{\frac{\beta}{r}}
$$

Logarithmic Approach

$$
\begin{aligned}
\ln PV &= \ln(e^{\alpha-\frac{\beta}{t}-rt})=\alpha-\frac{\beta}{t}-rt \\
\frac{d(\ln PV)}{dt} &= \frac{dPV}{dt}=\frac{\beta}{t^2}-r=0 \\
&= \frac{\beta}{t^2}=r \quad \left\{t=\sqrt{\frac{\beta}{r}}\right\}
\end{aligned}
$$

$\frac{\beta}{t^2}$ = growth rate of the value of uncut trees

r = growth rate of the optimally invested money

$\frac{\beta}{t^2}$ = the growth in your wealth from leaving trees uncut

r = Growth in your wealth if you cut down the trees, sell them, and put the money into a savings account paying e^{rt}

Comparative Statistics: if interest rate rises: $r \to r'$ then the optimal cutting time falls: $t^* \to t'$

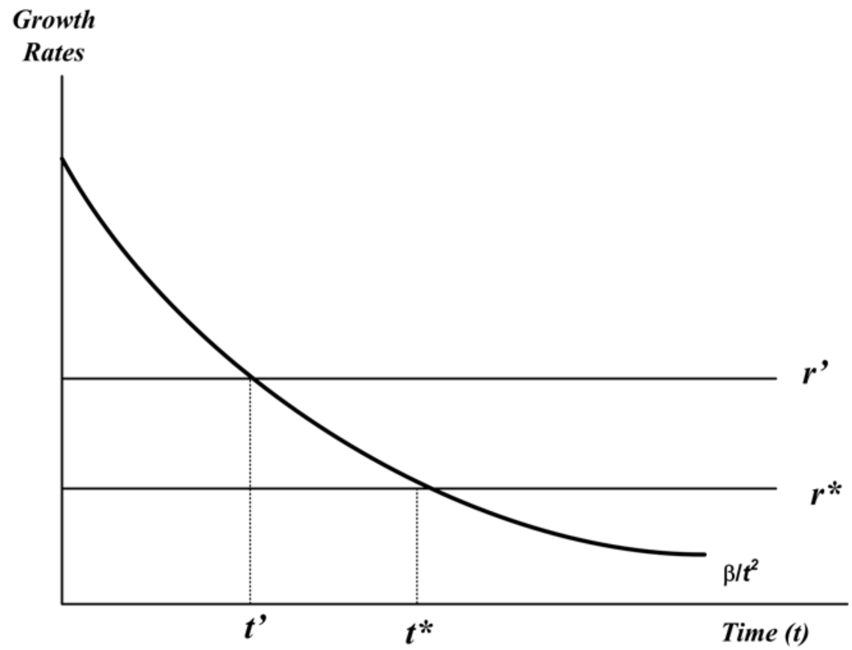


Figure 3:

6.2 Review: When to use the Implicit Function Theorem (Jacobian)??

6.2.1 General Form

Max

$$U(x, y) + \lambda(B - P_x x + P_y y)$$

F.O.C.

$$\begin{aligned} L_x &: U_x - \lambda P_x = 0 && \text{Eq. 1} \\ L_y &: U_y - \lambda P_y = 0 && \text{Eq. 2} \\ L_\lambda &: B - P_x x + P_y y = 0 && \text{Eq. 3} \end{aligned}$$

Equations 1, 2, and 3 IMPLICITY define

$$x^* = x^*(B, P_x, P_y)$$
$$y^* = y^*(B, P_x, P_y)$$
$$\lambda^* = \lambda^*(B, P_x, P_y)$$

S.O.C.

$$\left|\bar{H}\right| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & U_{xx} & U_{xy} \\ -P_y & U_{yx} & U_{yy} \end{vmatrix} \underset{\text{(By Assumption)}}{> 0}$$

Find $\frac{dx^*}{dP_x}$: use Implicit Function Theorem

6.2.2 Specific Form

Max

$$xy + \lambda(B - P_x x + P_y y)$$

F.O.C

$$L_x : y - \lambda P_x = 0 \qquad \text{Eq. 1}$$
$$L_y : x - \lambda P_y = 0 \qquad \text{Eq. 2}$$
$$L_\lambda : B - P_x x + P_y y = 0 \qquad \text{Eq. 3}$$

Equations 1, 2, and 3 EXPLICITLY define

$$x^* = \frac{B}{2P_x}$$
$$y^* = \frac{B}{2P_y}$$
$$\lambda^* = \frac{B}{2P_x P_y}$$

S.O.C.

$$\left|\bar{H}\right| = \begin{vmatrix} 0 & -P_x & -P_y \\ -P_x & 0 & 1 \\ -P_y & 1 & 0 \end{vmatrix} = 2P_x P_y > 0$$

Find $\frac{dx^*}{dP_x}$: Differentiate x^* directly

$\frac{dx^*}{dP_x} = -\frac{B}{2P_x^2} < 0$