SECTION 4.7

Further optimisation of economic functions

Objectives

At the end of this section you should be able to:

- Show that, at the point of maximum profit, marginal revenue equals marginal cost.
- Show that, at the point of maximum profit, the slope of the marginal revenue curve is less than that of marginal cost.
- Maximise profits of a firm with and without price discrimination in different markets.
- Show that, at the point of maximum average product of labour, average product of labour equals marginal product of labour.

The previous section demonstrated how mathematics can be used to optimise particular economic functions. Those examples suggested two important results:

1. If a firm maximises profit then MR = MC.
2. If a firm maximises average product of labour then $AP_L = MP_L$.

Although these results were found to hold for all of the examples considered in Section 4.6, it does not necessarily follow that the results are always true. The aim of this section is to prove these assertions without reference to specific functions and hence to demonstrate their generality.

Advice

You may prefer to skip these proofs at a first reading and just concentrate on the worked example (and Practice Problem 2 and Question 3 in Exercise 4.7*) on price discrimination.

Justification of the first result turns out to be really quite easy. Profit, π, is defined to be the difference between total revenue, TR, and total cost, TC: that is,

$$\pi = \text{TR} - \text{TC}$$

To find the stationary points of π we differentiate with respect to Q and equate to zero: that is,

$$\frac{d\pi}{dQ} = \frac{d(\text{TR})}{dQ} - \frac{(\text{TC})}{dQ} = 0$$

where we have used the difference rule to differentiate the right-hand side. In Section 4.3 we defined

$$\text{MR} = \frac{\text{d(TR)}}{\text{d}Q} \text{ and } \text{MC} = \frac{\text{d(TR)}}{\text{d}Q}$$

so the previous equation is equivalent to

$$\text{MR} - \text{MC} = 0$$

and so MR = MC as required.

The stationary points of the profit function can therefore be found by sketching the MR and MC curves on the same diagram and inspecting the points of intersection. Figure 4.26 shows typical marginal revenue and marginal cost curves. The result

$$\text{MR} = \text{MC}$$

holds for any stationary point. Consequently, if this equation has more than one solution then we need some further information before we can decide on the profit-maximising level of output. In Figure 4.26 there are two points of intersection, Q_1 and Q_2, and it turns out (as you discovered in Practice Problem 3 and Question 5 of Exercise 4.6 in the previous section) that one of these is a maximum while the other is a minimum. Obviously, in any actual example, we can classify these points by evaluating second-order derivatives. However, it would be nice to make this decision just by inspecting the graphs of marginal revenue and marginal cost. To see how this can be done let us return to the equation

$$\frac{\text{d}\pi}{\text{d}Q} = \text{MR} - \text{MC}$$

and differentiate again with respect to Q to get

$$\frac{\text{d}^2\pi}{\text{d}Q^2} = \frac{\text{d(MR)}}{\text{d}Q} - \frac{\text{d(MC)}}{\text{d}Q}$$

Now if $\text{d}^2\pi/\text{d}Q^2 < 0$ then the profit is a maximum. This will be so when

$$\frac{\text{d(MR)}}{\text{d}Q} < \frac{\text{d(MC)}}{\text{d}Q}$$

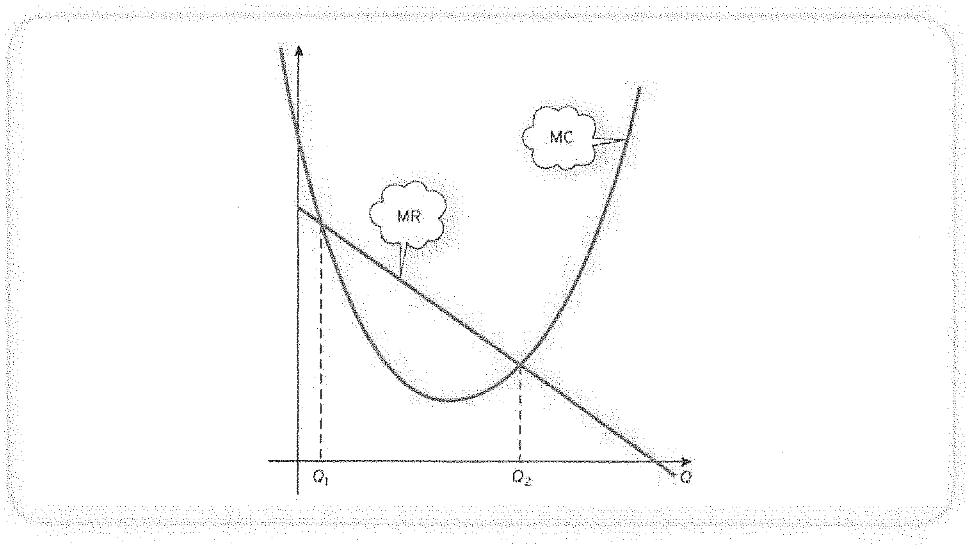

Figure 4.26

that is, when the slope of the marginal revenue curve is less than the slope of the marginal cost curve.

Looking at Figure 4.26, we deduce that this criterion is true at Q_2, so this must be the desired level of output needed to maximise profit. Note also from Figure 4.26 that the statement 'the slope of the marginal revenue curve is less than the slope of the marginal cost curve' is equivalent to saying that 'the marginal cost curve cuts the marginal revenue curve from below'. It is this latter form that is often quoted in economics textbooks. A similar argument shows that, at a minimum point, the marginal cost curve cuts the marginal revenue curve from above and so we can deduce that profit is a minimum at Q_1 in Figure 4.26. In practice, the task of sketching the graphs of MR and MC and reading off the coordinates of the points of intersection is not an attractive one, particularly if MR and MC are complicated functions. However, it might turn out that MR and MC are both linear, in which case a graphical approach is feasible.

Practice Problem

1. A monopolist's demand function is

$$P = 25 - 0.5Q$$

The fixed costs of production are 7 and the variable costs are $Q + 1$ per unit.

(a) Show that

$$TR = 25Q - 0.5Q^2 \text{ and } TC = Q^2 + Q + 7$$

and deduce the corresponding expressions for MR and MC.

(b) Sketch the graphs of MR and MC on the same diagram and hence find the value of Q which maximises profit.

Quite often a firm identifies more than one market in which it wishes to sell its goods. For example, a firm might decide to export goods to several countries and demand conditions are likely to be different in each one. The firm may be able to take advantage of this and increase overall profit by charging different prices in each country. The theoretical result 'marginal revenue equals marginal cost' can be applied in each market separately to find the optimal pricing policy.

Example

A firm is allowed to charge different prices for its domestic and industrial customers. If P_1 and Q_1 denote the price and demand for the domestic market then the demand equation is

$$P_1 + Q_1 = 500$$

If P_2 and Q_2 denote the price and demand for the industrial market then the demand equation is

$$2P_2 + 3Q_2 = 720$$

The total cost function is

$$TC = 50\,000 + 20Q$$

where $Q = Q_1 + Q_2$. Determine the prices (in dollars) that the firm should charge to maximise profits:

(a) with price discrimination

(b) without price discrimination.

Compare the profits obtained in parts (a) and (b).

Solution

(a) The important thing to notice is that the total cost function is independent of the market and so marginal costs are the same in each case. In fact, since

$$TC = 50\,000 + 20Q$$

we have MC = 20. All we have to do to maximise profits is to find an expression for the marginal revenue for each market and to equate this to the constant value of marginal cost.

Domestic market

The demand equation

$$P_1 + Q_1 = 500$$

rearranges to give

$$P_1 = 500 - Q_1$$

so the total revenue function for this market is

$$TR_1 = (500 - Q_1)Q_1 = 500Q_1 - Q_1^2$$

Hence

$$MR_1 = \frac{d(TR_1)}{dQ_1} = 500 - 2Q_1$$

For maximum profit

$$MR_1 = MC$$

so

$$500 - 2Q_1 = 20$$

which has solution $Q_1 = 240$. The corresponding price is found by substituting this value into the demand equation to get

$$P_1 = 500 - 240 = \$260$$

To maximise profit the firm should charge its domestic customers $260 per good.

Industrial market

The demand equation

$$2P_2 + 3Q_2 = 720$$

rearranges to give

$$P_2 = 360 - {}^3\!/_2Q_2$$

so the total revenue function for this market is

$$TR_2 = (360 - {}^3\!/_2Q_2)Q_2 = 360Q - {}^3\!/_2Q_2^2$$

Hence

$$MR_2 = \frac{d(TR_2)}{dQ_2} = 360 - 3Q_2$$

For maximum profit

$$MR_2 = MC$$

so

$$360 - 3Q_2 = 20$$

which has solution $Q_2 = 340/3$. The corresponding price is obtained by substituting this value into the demand equation to get

$$P_2 = 360 - \frac{3}{2}\left(\frac{340}{3}\right) = \$190$$

To maximise profits the firm should charge its industrial customers \$190 per good, which is lower than the price charged to its domestic customers.

(b) If there is no price discrimination then $P_1 = P_2 = P$, say, and the demand functions for the domestic and industrial markets become

$$P + Q_1 = 500$$

and

$$2P + 3Q_2 = 720$$

respectively. We can use these to deduce a single demand equation for the combined market. We need to relate the price, P, of each good to the total demand, $Q = Q_1 + Q_2$.

This can be done by rearranging the given demand equations for Q_1 and Q_2 and then adding. For the domestic market

$$Q_1 = 500 - P$$

and for the industrial market

$$Q_2 = 240 - {}^2\!/_3P$$

Hence

$$Q = Q_1 + Q_2 = 740 - {}^5\!/_3P$$

The demand equation for the combined market is therefore

$$Q + {}^5\!/_3P = 740$$

The usual procedure for profit maximisation can now be applied. This demand equation rearranges to give

$$P = 444 - {}^3\!/_5Q$$

enabling the total revenue function to be written down as

$$TR = \left(444 - \frac{3}{5}Q\right)Q = 444Q - \frac{3Q^2}{5}$$

Hence

$$MR = \frac{d(TR)}{dQ} = 444 - \frac{6}{5}Q$$

For maximum profit

$$MR = MC$$

so

$$444 - \frac{6}{5}Q = 20$$

which has solution $Q = 1060/3$. The corresponding price is found by substituting this value into the demand equation to get

$$P = 444 - \frac{3}{5}\left(\frac{1060}{3}\right) = \$232$$

To maximise profit without discrimination the firm needs to charge a uniform price of \$232 for each good. Notice that this price lies between the prices charged to its domestic and industrial customers with discrimination.

To evaluate the profit under each policy we need to work out the total revenue and subtract the total cost. In part (a) the firm sells 240 goods at \$260 each in the domestic market and sells 340/3 goods at \$190 each in the industrial market, so the total revenue received is

$$240 \times 260 + \frac{340}{3} \times 190 = \$83\,933.33$$

The total number of goods produced is

$$240 + \frac{340}{3} = \frac{1060}{3}$$

so the total cost is

$$50000 + 20 \times \frac{1060}{3} = \$57\,066.67$$

Therefore the profit with price discrimination is

$$83\,933.33 - 57\,066.67 = \$26\,866.67$$

In part (b) the firm sells 1060/3 goods at \$232 each, so total revenue is

$$\frac{1060}{3} \times 232 = \$81\,973.33$$

Now the total number of goods produced under both pricing policies is the same: that is, 1060/3. Consequently, the total cost of production in part (b) must be the same as part (a): that is,

$$TC = \$57\,066.67$$

The profit without price discrimination is

$$81\,973.33 - 57\,066.67 = \$24\,906.66$$

As expected, the profits are higher with discrimination than without.

Practice Problem

2. A firm has the possibility of charging different prices in its domestic and foreign markets. The corresponding demand equations are given by

$$Q_1 = 300 - P_1$$
$$Q_2 = 400 - 2P_2$$

The total cost function is

$$TC = 5000 + 100Q$$

where $Q = Q_1 + Q_2$.

Determine the prices (in dollars) that the firm should charge to maximise profits

(a) with price discrimination

(b) without price discrimination.

Compare the profits obtained in parts (a) and (b).

In the previous example and in Practice Problem 2 we assumed that the marginal costs were the same in each market. The level of output that maximises profit with price discrimination was found by equating marginal revenue to this common value of marginal cost. It follows that the marginal revenue must be the same in each market. In symbols

$$MR_1 = MC \text{ and } MR_2 = MC$$

so

$$MR_1 = MR_2$$

This fact is obvious on economic grounds. If it were not true then the firm's policy would be to increase sales in the market where marginal revenue is higher and to decrease sales by the same amount in the market where the marginal revenue is lower. The effect would be to increase revenue while keeping costs fixed, thereby raising profit. This property leads to an interesting result connecting price, P, with elasticity of demand, E. In Section 4.5 we derived the formula

$$MR = P\left(1 - \frac{1}{E}\right)$$

If we let the price elasticity of demand in two markets be denoted by E_1 and E_2 corresponding to prices P_1 and P_2 then the equation

$$MR_1 = MR_2$$

becomes

$$P_1\left(1 - \frac{1}{E_1}\right) = P_2\left(1 - \frac{1}{E_2}\right)$$

This equation holds whenever a firm chooses its prices P_1 and P_2 to maximise profits in each market. Note that if $E_1 < E_2$ then this equation can only be true if $P_1 > P_2$. In other words, the firm charges the higher price in the market with the lower elasticity of demand.

Practice Problem

3. Calculate the price elasticity of demand at the point of maximum profit for each of the demand functions given in Practice Problem 2 with price discrimination. Verify that the firm charges the higher price in the market with the lower elasticity of demand.

The previous discussion concentrated on profit. We now turn our attention to average product of labour and prove result (2) stated at the beginning of this section. This concept is defined by

$$\text{AP}_L = \frac{Q}{L}$$

where Q is output and L is labour. The maximisation of AP_L is a little more complicated than before, since it is necessary to use the quotient rule to differentiate this function. In the notation of Section 4.4 we write

$$u = Q \quad \text{and} \quad v = L$$

so

$$\frac{\text{d}u}{\text{d}L} = \frac{\text{d}Q}{\text{d}L} = \text{MP}_L \quad \text{and} \quad \frac{\text{d}v}{\text{d}L} = \frac{\text{d}L}{\text{d}L} = 1$$

where we have used the fact that the derivative of output with respect to labour is the marginal product of labour.

The quotient rule gives

$$\frac{\text{d}(\text{Ap}_L)}{\text{d}L} = \frac{v\text{d}u / \text{d}L - u\text{d}v / \text{d}L}{v^2}$$

$$= \frac{L(\text{MP}_L) - Q(1)}{L^2}$$

$$= \frac{\text{MP}_L - Q / L}{L}$$

divide top and bottom by L

$$= \frac{\text{MP}_L - \text{AP}_L}{L}$$

by definition, $AP_L = \frac{Q}{L}$

At a stationary point

$$\frac{\text{d}(\text{AP}_L)}{\text{d}L} = 0$$

Hence

$$\frac{\text{MP}_L = \text{AP}_L}{L} = 0$$

as required.

This analysis shows that, at a stationary point of the average product of labour function, the marginal product of labour equals the average product of labour. The above argument provides a formal proof that this result is true for any average product of labour function. Figure 4.27

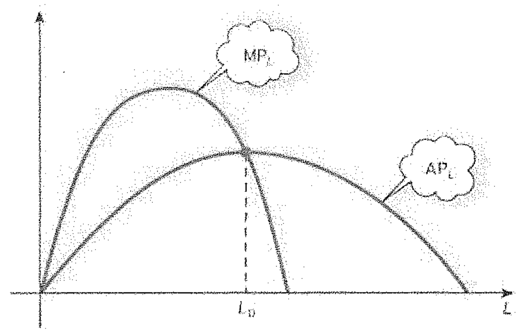

Figure 4.27

shows typical average and marginal product functions. Note that the two curves intersect at the peak of the AP_L curve. To the left of this point the AP_L function is increasing, so that

$$\frac{d(AP_L)}{dL} > 0$$

Now we have just seen that

$$\frac{d(AP_L)}{dL} = \frac{MP_L - AP_L}{L}$$

so we deduce that, to the left of the maximum, $MP_L > AP_L$. In other words, in this region the graph of marginal product of labour lies above that of average product of labour. Similarly, to the right of the maximum, AP_L is decreasing, so that

$$\frac{d(AP_L)}{dL} < 0$$

and hence $MP_L < AP_L$. The graph of marginal product of labour therefore lies below that of average product of labour in this region.

We deduce that if the stationary point is a maximum then the MP_L curve cuts the AP_L curve from above. A similar argument can be used for any average function. The particular case of the average cost function is investigated in Question 6 in Exercise 4.7*.

Exercise 4.7*

1. A firm's demand function is

$$P = aQ + b(a < 0, b > 0)$$

Fixed costs are c and variable costs per unit are d.

(a) Write down general expressions for TR and TC.

(b) By differentiating the expressions in part (a), deduce MR and MC.

(c) Use your answers to (b) to show that profit, π, is maximised when

$$Q = \frac{d-b}{2a}$$

2. (a) In Section 4.5 the following relationship between marginal revenue, MR, and price elasticity of demand, E, was derived:

$$\text{MR} = P\left(1 - \frac{1}{E}\right)$$

Use this result to show that at the point of maximum total revenue, $E = 1$.

(b) Verify the result of part (a) for the demand function

$$2P + 3Q = 60$$

3. The demand functions for a firm's domestic and foreign markets are

$$P_1 = 50 - 5Q_1$$
$$P_2 = 30 - 4Q_2$$

and the total cost function is

$$\text{TC} = 10 + 10Q$$

where $Q = Q_1 + Q_2$. Determine the prices needed to maximise profit

(a) with price discrimination

(b) without price discrimination.

Compare the profits obtained in parts (a) and (b).

4. Show that if the marginal cost curve cuts the marginal revenue curve from above then profit is a minimum.

5. The economic order quantity, EOQ, is used in cost accounting to minimise the total cost, TC, to order and carry a firm's stock over the period of a year.

The annual cost of placing orders, ACO, is given by

$$\text{ACO} = \frac{(\text{ARU})(\text{CO})}{\text{EOQ}}$$

where

ARU = annual required units

CO = cost per order

SECTION 4.8

The derivative of the exponential and natural logarithm functions

Objectives

At the end of this section you should be able to:

- Differentiate the exponential function.
- Differentiate the natural logarithm function.
- Use the chain, product and quotient rules to differentiate combinations of these functions.
- Appreciate the use of the exponential function in economic modelling.

In this section we investigate the derived functions associated with the exponential and natural logarithm functions, e^x and $\ln x$. The approach that we adopt is similar to that used in Section 4.1. The derivative of a function determines the slope of the graph of a function. Consequently, to discover how to differentiate an unfamiliar function we first produce an accurate sketch and then measure the slopes of the tangents at selected points.

Advice

The functions, e^x and $\ln x$ were first introduced in Section 2.4. You might find it useful to remind yourself how these functions are defined before working through the rest of the current section.

Example

Complete the following table of function values and hence sketch a graph of $f(x) = e^x$:

x	−2.0	−1.5	−1.0	0.0	0.5	1.0	1.5
$f(x)$							

Draw tangents to the graph at $x = -1$, 0 and 1. Hence estimate the values of $f'(-1)$, $f'(0)$ and $f'(1)$. Suggest a general formula for the derived function $f'(x)$.

Solution

Using a calculator we obtain

x	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5
$f(x)$	0.14	0.22	0.37	0.61	1.00	1.65	2.72	4.48

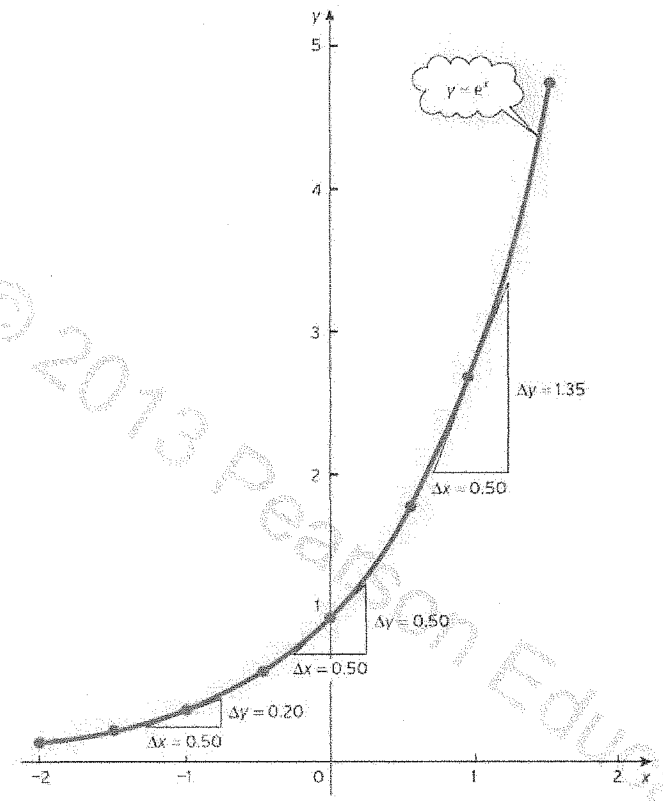

Figure 4.28

The corresponding graph of the exponential function is sketched in Figure 4.28. From the graph we see that the slopes of the tangents are

$$f'(-1) = \frac{0.20}{0.50} = 0.4$$

$$f'(0) = \frac{0.50}{0.50} = 1.0$$

$$f'(1) = \frac{1.35}{0.50} = 2.7$$

These results are obtained by measurement and so are quoted to only 1 decimal place. We cannot really expect to achieve any greater accuracy using this approach.

The values of x, $f(x)$ and $f'(x)$ are summarised in the following table. The values of $f(x)$ are rounded to 1 decimal place in order to compare with the graphical estimates of $f'(x)$.

x	-1	0	1
$f(x)$	0.4	1.0	2.7
$f'(x)$	0.4	1.0	2.7

Notice that the values of $f(x)$ and $f'(x)$ are identical to within the accuracy quoted.

These results suggest that the slope of the graph at each point is the same as the function value at that point: that is, e^x differentiates to itself. Symbolically,

if $f(x) = e^x$ then $f'(x) = e^x$

or, equivalently,

if $y = e^x$ then $\frac{dy}{dx} = e^x$

Practice Problem

1. Use your calculator to complete the following table of function values and hence sketch an accurate graph of $f(x) = \ln x$:

x	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
$f(x)$			0.41				1.25	

Draw the tangents to the graph at $x = 1$, 2 and 3. Hence estimate the values of $f'(1)$, $f'(2)$ and $f'(3)$. Suggest a general formula for the derived function $f'(x)$.
[Hint: for the last part you may find it helpful to rewrite your estimates of $f'(x)$ as simple fractions.]

In fact, it is possible to prove that, for any value of the constant m,

if $y = e^{mx}$ then $\frac{dy}{dx} = me^{mx}$

and

if $y = \ln mx$ then $\frac{dy}{dx} = \frac{1}{x}$

In particular, we see by setting $m = 1$ that

e^x differentiates to e^x

and that

$\ln x$ differentiates to $\frac{1}{x}$

which agree with our practical investigations.

Example

Differentiate

(a) $y = e^{2x}$

(b) $y = e^{-7x}$

(c) $y = \ln 5x$ $(x > 0)$

(d) $y = \ln 559x$ $(x > 0)$

Solution

(a) Setting $m = 2$ in the general formula shows that

$$\text{if} \quad y = e^{2x} \quad \text{then} \quad \frac{dy}{dx} = 2e^{2x}$$

Notice that when exponential functions are differentiated the power itself does not change. All that happens is that the coefficient of x comes down to the front.

(b) Setting $m = -7$ in the general formula shows that

$$\text{if} \quad y = e^{-7x} \quad \text{then} \quad \frac{dy}{dx} = -7e^{-7x}$$

(c) Setting $m = 5$ in the general formula shows that

$$\text{if} \quad y = \ln 5x \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{x}$$

Notice the restriction $x > 0$ stated in the question. This is needed to ensure that we do not attempt to take the logarithm of a negative number, which is impossible.

(d) Setting $m = 559$ in the general formula shows that

$$\text{if} \quad y = \ln 559x \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{x}$$

Notice that we get the same answer as part (c). The derivative of the natural logarithm function does not depend on the coefficient of x. This fact may seem rather strange but it is easily accounted for. The third rule of logarithms shows that $\ln 559x$ is the same as

$$\ln 559 + \ln x$$

The first term is merely a constant, so differentiates to zero, and the second term differentiates to $1/x$.

Practice Problem

2. Differentiate

(a) $y = e^{3x}$ **(b)** $y = e^{-x}$ **(c)** $y = \ln 3x \ (x > 0)$ **(d)** $y = \ln 51\,234x \ (x > 0)$

The chain rule can be used to explain what happens to the m when differentiating e^{mx}. The outer function is the exponential, which differentiates to itself, and the inner function is mx, which differentiates to m. Hence, by the chain rule,

$$\text{if} \quad y = e^{mx} \quad \text{then} \quad \frac{dy}{dx} = e^{mx} \times m = me^{mx}$$

Similarly, noting that the natural logarithm function differentiates to the reciprocal function,

$$\text{if} \quad y = \ln mx \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{mx} \times m = \frac{1}{x}$$

The chain, product and quotient rules can be used to differentiate more complicated functions involving e^x and $\ln x$.

Example

Differentiate

(a) $y = x^3e^{2x}$ **(b)** $y = \ln(x^2 + 2x + 1)$ **(c)** $y = \dfrac{e^{3x}}{x^2 + 2}$

Solution

(a) The function x^3e^{2x} involves the product of two simpler functions, x^3 and e^{2x}, so we need to use the product rule to differentiate it. Putting

$$u = x^3 \quad \text{and} \quad v = e^{2x}$$

gives

$$\frac{du}{dx} = 3x^2 \quad \text{and} \quad \frac{dv}{dx} = 2e^{2x}$$

By the product rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^3\left[2e^{2x}\right] + e^{2x}\left[3x^2\right] = 2x^3e^{2x} + 3x^2e^{2x}$$

There is a common factor of x^2e^{2x}, which goes into the first term $2x$ times and into the second term three times. Hence

$$\frac{dy}{dx} = x^2e^{2x}(2x + 3)$$

(b) The expression $\ln(x^2 + 2x + 1)$ can be regarded as a function of a function, so we can use the chain rule to differentiate it. We first differentiate the outer log function to get

$$\frac{1}{x^2 + 2x + 1}$$

and then multiply by the derivative of the inner function, $x^2 + 2x + 1$, which is $2x + 2$. Hence

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 1}$$

(c) The function

$$\frac{e^{3x}}{x^2 + 2}$$

is the quotient of the simpler functions

$$u = e^{3x} \quad \text{and} \quad v = x^2 + 2$$

for which

$$\frac{du}{dx} = 3e^{3x} \quad \text{and} \quad \frac{du}{dx} = 2x$$

By the quotient rule

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(x^2 + 2)(3e^{3x}) - e^{3x}(2x)}{(x^2 + 2)^2} = \frac{e^{3x}[3(x^2 + 2) - 2x]}{(x^2 + 2)^2} = \frac{e^{3x}(3x^2 - 2x + 6)}{(x^2 + 2)^2}$$

Practice Problem

3. Differentiate

(a) $y = x^4 \ln x$

(b) $y = e^{x^2}$

(c) $y = \dfrac{\ln x}{x+2}$

Advice

If you ever need to differentiate a function of the form:

ln(an inner function involving products, quotients or powers of *x*)

then it is usually quicker to use the rules of logs to expand the expression before you begin. The three rules are

Rule 1 $\ln(x \times y) = \ln x + \ln y$

Rule 2 $\ln(x \div y) = \ln x - \ln y$

Rule 3 $\ln x^m = m \ln x$

The following example shows how to apply this 'trick' in practice.

Example

Differentiate

(a) $y = \ln(x(x+1)^4)$ (b) $y = \ln\left(\dfrac{x}{\sqrt{(x+5)}}\right)$

Solution

(a) From rule 1

$$\ln(x(x+1)^4) = \ln x + \ln(x+1)^4$$

which can be simplified further using rule 3 to give

$$y = \ln x + 4\ln(x+1)$$

Differentiation of this new expression is simple. We see immediately that

$$\frac{dy}{dx} = \frac{1}{x} + \frac{4}{x+1}$$

If desired the final answer can be put over a common denominator

$$\frac{1}{x} + \frac{4}{x+1} = \frac{(x+1)+4x}{x(x+1)} = \frac{5x+1}{x(x+1)}$$

(b) The quickest way to differentiate

$$y = \ln\left(\frac{x}{\sqrt{(x+5)}}\right)$$

is to expand first to get

$$y = \ln x - \ln(x+5)^{1/2} \quad \text{(rule 2)}$$

$$= \ln x - \frac{1}{2}\ln(x+5) \quad \text{(rule 3)}$$

Again this expression is easy to differentiate:

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2(x+5)}$$

If desired, this can be written as a single fraction:

$$\frac{1}{x} - \frac{1}{2(x+5)} = \frac{2(x+5)-x}{2x(x+5)} = \frac{x+10}{2x(x+5)}$$

Practice Problem

4. Differentiate the following functions by first expanding each expression using the rules of logs:

(a) $y = \ln(x^3(x+2)^4)$ **(b)** $y = \ln\left(\frac{x^2}{2x+3}\right)$

Exponential and natural logarithm functions provide good mathematical models in many areas of economics and we conclude this chapter with some illustrative examples.

Example

A firm's short-run production function is given by

$$Q = L^2 e^{-0.01L}$$

Find the value of L that maximises the average product of labour.

Solution

The average product of labour is given by

$$AP_L = \frac{Q}{L} = \frac{L^2 e^{-0.01L}}{L} = Le^{-0.01L}$$

To maximise this function we adopt the strategy described in Section 4.6.

Step 1

At a stationary point

$$\frac{d(AP_L)}{dL} = 0$$

To differentiate $Le^{-0.01L}$, we use the product rule. If

$$u = L \quad \text{and} \quad v = e^{-0.01L}$$

then

$$\frac{du}{dL} = 1 \quad \text{and} \quad \frac{dv}{dL} = -0.01e^{-0.01L}$$

e^{mx} differentiates to me^{mx}

By the product rule

$$\frac{d(AP_L)}{dL} = u\frac{dv}{dL} + v\frac{du}{dL} = L(-0.01e^{-0.01L}) + e^{-0.01L} = (1 - 0.01L)e^{-0.01L}$$

We know that a negative exponential is never equal to zero. (Although $e^{-0.01L}$ gets ever closer to zero as L increases, it never actually reaches it for finite values of L.) Hence the only way that

$$(1 - 0.01L)e^{-0.01L}$$

can equal zero is when

$$1 - 0.01L = 0$$

which has solution $L = 100$.

Step 2

To show that this is a maximum we need to differentiate a second time. To do this we apply the product rule to

$$(1 - 0.01L)e^{-0.01L}$$

taking

$$u = 1 - 0.01L \quad \text{and} \quad v = e^{-0.01L}$$

for which

$$\frac{du}{dL} = -0.01 \quad \text{and} \quad \frac{dv}{dL} = -0.01e^{-0.01L}$$

Hence

$$\frac{d^2(AP_L)}{dL^2} = u\frac{dv}{dL} + v\frac{du}{dL} = (1 - 0.01L)(-0.01e^{-0.01L}) + e^{-0.01L}(-0.01) = (-0.02 + 0.0001L)e^{-0.01L}$$

Finally, putting $L = 100$ into this gives

$$\frac{d^2(AP_L)}{dL^2} = -0.0037$$

The fact that this is negative shows that the stationary point, $L = 100$, is indeed a maximum.

Practice Problem

5. The demand function of a good is given by

$$Q = 1000e^{-0.2P}$$

If fixed costs are 100 and the variable costs are 2 per unit, show that the profit function is given by

$$\pi = 1000Pe^{-0.2P} - 2000e^{-0.2P} - 100$$

Find the price needed to maximise profit.

Example

A firm estimates that the total revenue received from the sale of Q goods is given by

$$TR = \ln(1 + 1000Q^2)$$

Calculate the marginal revenue when $Q = 10$.

Solution

The marginal revenue function is obtained by differentiating the total revenue function. To differentiate $\ln(1 + 1000Q^2)$ we use the chain rule. We first differentiate the outer log function to get

$$\frac{1}{1 + 1000Q^2}$$

natural logs differentiate to reciprocals

and then multiply by the derivative of the inner function, $1 + 1000Q^2$, to get $2000Q$. Hence

$$MR = \frac{d(TR)}{dQ} = \frac{2000Q}{1 + 1000Q^2}$$

At $Q = 10$,

$$MR = \frac{2000(10)}{1 + 1000(10)^2} = 0.2$$

Practice Problem

6. If the demand equation is

$$P = 200 - 40\ln(Q + 1)$$

calculate the price elasticity of demand when $Q = 20$.

Exercise 4.8

1. Write down the derivative of

(a) $y = e^{6x}$ (b) $y = e^{-342x}$ (c) $y = 2e^{-x} + 4e^{x}$ (d) $y = 10e^{4x} - 2x^2 + 7$

2. If \$4000 is saved in an account offering a return of 4% compounded continuously the future value, S, after t years is given by

$$S = 4000e^{0.04t}$$

(1) Calculate the value of S when

(a) $t = 5$ (b) $t = 5.01$

and hence estimate the rate of growth at $t = 5$. Round your answers to 2 decimal places.

(2) Write down an expression for $\frac{dS}{dt}$ and hence find the exact value of the rate of growth after 5 years.

3. Write down the derivative of

(a) $y = \ln(3x)$ $(x > 0)$ (b) $y = \ln(-13x)$ $(x < 0)$

4. Use the chain rule to differentiate

(a) $y = e^{x^3}$ (b) $y = \ln(x^4 + 3x^2)$

5. Use the product rule to differentiate

(a) $y = x^4e^{2x}$ (b) $y = x \ln x$

6. Use the quotient rule to differentiate

(a) $y = \dfrac{e^{4x}}{x^2 + 2}$ (b) $y = \dfrac{e^{x}}{\ln x}$

7. Find and classify the stationary points of

(a) $y = xe^{-x}$ (b) $y = \ln x - x$

Hence sketch their graphs.

8. Find the output needed to maximise profit given that the total cost and total revenue functions are

$$TC = 2Q \text{ and } TR = 100 \ln(Q + 1)$$

respectively.

9. If a firm's production function is given by

$$Q = 700Le^{-0.02L}$$

find the value of L that maximises output.

10. The demand function of a good is given by

$$P = 100e^{-0.1Q}$$

Show that demand is unit elastic when $Q = 10$.

Exercise 4.8*

1. Differentiate:

   (a) $y = e^{2x} - 3e^{-4x}$ (b) xe^{4x} (c) $\dfrac{e^{-x}}{x^2}$ (d) $x^m \ln x$ (e) $x(\ln x - 1)$

   (f) $\dfrac{x^n}{\ln x}$ (g) $\dfrac{e^{mx}}{(ax+b)^n}$ (h) $\dfrac{e^{ax}}{(\ln bx)^n}$ (i) $\dfrac{e^x - 1}{e^x + 1}$

2. Use the rules of logarithms to expand each of the following functions. Hence write their derivatives.

   (a) $y = \ln\left(\dfrac{x}{x+1}\right)$ (b) $y = \ln(x\sqrt{(3x-1)})$ (c) $y = \ln\sqrt{\dfrac{x+1}{x-1}}$

3. The growth rate of an economic variable, y, is defined to be $\dfrac{dy}{dt} \div y$.

   Use this definition to find the growth rate of the variable, $y = Ae^{kt}$.

4. Differentiate the following functions with respect to x, simplifying your answers as far as possible:

   (a) $y = x^4 e^{-2x^2}$ (b) $y = \ln\left(\dfrac{3}{(x+1)^2}\right)$

5. Find and classify the stationary points of

   (a) $y = xe^{ax}$ (b) $y = \ln(ax^2 + bx)$

   where $a < 0$.

6. (a) Use the quotient rule to show that the derivative of the function

   $$y = \frac{2x+1}{\sqrt{4x+3}}$$

   is given by

   $$\frac{4(x+1)}{(4x+3)\sqrt{4x+3}}$$

   (b) Use the chain rule to differentiate the function

   $$y = \ln\left(\frac{2x+1}{\sqrt{4x+3}}\right)$$

   (c) Confirm that your answer to part (b) is correct by first expanding

   $$\ln\left(\frac{2x+1}{\sqrt{4x+3}}\right)$$

   using the rules of logs and then differentiating.

7. A firm's short-run production function is given by

   $$Q = L^3 e^{-0.02L}$$

   Find the value of L that maximises the average product of labour.

8. Find an expression for the price elasticity of demand for each of the demand curves:

(a) $P = 100e^{-Q}$ (b) $P = 500 - 75\ln(2Q + 1)$

9. Find an expression for the marginal revenue for each of the following demand curves:

(a) $P = \dfrac{e^{Q^2}}{Q^2}$ (b) $P = \ln\left(\dfrac{2Q}{3Q + 1}\right)$