SECTION 4.4
Further rules of differentiation

Objectives

At the end of this section you should be able to:

- Use the chain rule to differentiate a function of a function.
- Use the product rule to differentiate the product of two functions.
- Use the quotient rule to differentiate the quotient of two functions.
- Differentiate complicated functions using a combination of rules.

Section 4.2 introduced you to the basic rules of differentiation. Unfortunately, not all functions can be differentiated using these rules alone. For example, we are unable to differentiate the functions

$$x\sqrt{(2x-3)} \quad \text{and} \quad \frac{x}{x^2+1}$$

using just the constant, sum or difference rules. The aim of the present section is to describe three further rules which allow you to find the derivative of more complicated expressions. Indeed, the totality of all six rules will enable you to differentiate any mathematical function. Although you may find that the rules described in this section take you slightly longer to grasp than before, they are vital to any understanding of economic theory.

The first rule that we investigate is called the chain rule and it can be used to differentiate functions such as

$$y = (2x+3)^{10} \quad \text{and} \quad y = \sqrt{(1+x^2)}$$

The distinguishing feature of these expressions is that they represent a 'function of a function'. To understand what we mean by this, consider how you might evaluate

$$y = (2x+3)^{10}$$

on a calculator. You would first work out an intermediate number u, say, given by

$$u = 2x + 3$$

and then raise it to the power of 10 to get

$$y = u^{10}$$

This process is illustrated using the flow chart in Figure 4.16 (overleaf). Note how the incoming number x is first processed by the inner function, 'double and add 3'. The output u from this is then passed on to the outer function, 'raise to the power of 10', to produce the final outgoing number y.

The function

$$y = \sqrt{(1+x^2)}$$

Figure 4.16

can be viewed in the same way. To calculate y you perform the inner function, 'square and add 1', followed by the outer function, 'take square roots'.

The chain rule for differentiating a function of a function may now be stated.

Rule 4 The chain rule

If y is a function of u, which is itself a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

differentiate the outer function and multiply by the derivative of the inner function

To illustrate this rule, let us return to the function

$$y = (2x + 3)^{10}$$

in which

$$y = u^{10} \quad \text{and} \quad u = 2x + 3$$

Now

$$\frac{dy}{du} = 10u^9 = 10(2x + 3)^9$$

$$\frac{du}{dx} = 2$$

The chain rule then gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10(2x + 3)^9(2) = 20(2x + 3)^9$$

With practice it is possible to perform the differentiation without explicitly introducing the variable u. To differentiate

$$y = (2x + 3)^{10}$$

we first differentiate the outer power function to get

$$10(2x + 3)^9$$

and then multiply by the derivative of the inner function, $2x + 3$, which is 2, so

$$\frac{dy}{dx} = 20(2x + 3)^9$$

Example

Differentiate

(a) $y = (3x^2 - 5x + 2)^4$

(b) $y = \dfrac{1}{3x + 7}$

(c) $y = \sqrt{(1 + x^2)}$

Solution

(a) The chain rule shows that to differentiate $(3x^2 - 5x + 2)^4$ we first differentiate the outer power function to get

$$4(3x^2 - 5x + 2)^3$$

and then multiply by the derivative of the inner function, $3x^2 - 5x + 2$, which is $6x - 5$. Hence if

$$y = (3x^2 - 5x + 2)^4 \quad \text{then} \quad \frac{dy}{dx} = 4(3x^2 - 5x + 2)^3(6x - 5)$$

(b) To use the chain rule to differentiate

$$y = \frac{1}{3x + 7}$$

recall that reciprocals are denoted by negative powers, so that

$$y = (3x + 7)^{-1}$$

The outer power function differentiates to get

$$-(3x + 7)^{-2}$$

and the inner function, $3x + 7$, differentiates to get 3. By the chain rule we just multiply these together to deduce that

$$\text{if } y = \frac{1}{3x + 7} \quad \text{then} \quad \frac{dy}{dx} = -(3x + 7)^{2}(3) = \frac{-3}{(3x + 7)^2}$$

(c) To use the chain rule to differentiate

$$y = \sqrt{(1 + x^2)}$$

recall that roots are denoted by fractional powers, so that

$$y = (1 + x^2)^{1/2}$$

The outer power function differentiates to get

$$\frac{1}{2}(1 + x^2)^{-1/2}$$

and the inner function, $1 + x^2$, differentiates to get $2x$. By the chain rule we just multiply these together to deduce that

$$\text{if } y = \sqrt{(1 + x^2)} \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{2}(1 + x^2)^{-1/2}(2x) = \frac{x}{\sqrt{(1 + x^2)}}$$

Practice Problem

1. Differentiate

(a) $y = (3x - 4)^5$ (b) $y = (x^2 + 3x + 5)^3$ (c) $y = \frac{1}{2x - 3}$ (d) $y = \sqrt{(4x - 3)}$

The next rule is used to differentiate the product of two functions, $f(x)g(x)$. In order to give a clear statement of this rule, we write

$$u = f(x) \quad \text{and} \quad v = g(x)$$

Rule 5 The product rule

$$\text{If } y = uv \quad \text{then} \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This rule tells you how to differentiate the product of two functions:

multiply each function by the derivative of the other and add

Example

Differentiate

(a) $y = x^2(2x + 1)^3$ (b) $x\sqrt{(6x + 1)}$ (c) $y = \frac{x}{1 + x}$

Solution

(a) The function $x^2(2x + 1)^3$ involves the product of two simpler functions, namely x^2 and $(2x + 1)^3$, which we denote by u and v respectively. (It does not matter which function we label u and which we label v. The same answer is obtained if u is $(2x + 1)^3$ and v is x^2. You might like to check this for yourself later.) Now if

$$u = x^2 \quad \text{and} \quad v = (2x + 1)^3$$

then

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = 6(2x + 1)^2$$

where we have used the chain rule to find dv/dx. By the product rule,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x^2[6(2x + 1)^2] + (2x + 1)^3(2x)$$

The first term is obtained by leaving u alone and multiplying it by the derivative of v. Similarly, the second term is obtained by leaving v alone and multiplying it by the derivative of u.

If desired, the final answer may be simplified by taking out a common factor of $2x(2x + 1)^2$. This factor goes into the first term $3x$ times and into the second $2x + 1$ times. Hence

$$\frac{dy}{dx} = 2x(2x+1)^2[3x + (2x+1)] = 2x(2x+1)^2(5x+1)$$

(b) The function $x\sqrt{(6x+1)}$ involves the product of the simpler functions

$$u = x \quad \text{and} \quad v = \sqrt{6x+1} = (6x+1)^{1/2}$$

for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{2}(6x+1)^{-1/2} \times 6 = 3(6x+1)^{-1/2}$$

where we have used the chain rule to find dv/dx. By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= x[3(6x+1)^{-1/2}] + (6x+1)^{1/2}(1)\\ &= \frac{3x}{\sqrt{(6x+1)}} + \sqrt{(6x+1)}\end{aligned}$$

If desired, this can be simplified by putting the second term over a common denominator

$$\sqrt{(6x+1)}$$

To do this we multiply the top and bottom of the second term by $\sqrt{6x+1}$ to get

$$\frac{(6x+1)}{\sqrt{(6x+1)}}$$

$\sqrt{(6x+1)} \times \sqrt{(6x+1)}$
$= 6x + 1$

Hence

$$\frac{dy}{dx} = \frac{3x + (6x+1)}{\sqrt{(6x+1)}} = \frac{9x+1}{\sqrt{(6x+1)}}$$

(c) At first sight it is hard to see how we can use the product rule to differentiate

$$\frac{x}{1+x}$$

since it appears to be the quotient and not the product of two functions. However, if we recall that reciprocals are equivalent to negative powers, we may rewrite it as

$$x(1+x)^{-1}$$

It follows that we can put

$$u = x \quad \text{and} \quad v = (1+x)^{-1}$$

which gives

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{du}{dx} = -(1+x)^{-2}$$

where we have used the chain rule to find dv/dx. By the product rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x[-(1+x)^{-2}] + (1+x)^{-1}(1)$$

$$= \frac{-x}{(1+x)^2} + \frac{1}{1+x}$$

If desired, this can be simplified by putting the second term over a common denominator

$(1 + x)^2$

To do this we multiply the top and bottom of the second term by $1 + x$ to get

$$\frac{1+x}{(1+x)^2}$$

Hence

$$\frac{dy}{dx} = \frac{-x}{(1+x)^2} + \frac{1+x}{(1+x)^2} = \frac{-x+(1+x)}{(1+x)^2} = \frac{1}{(1+x)^2}$$

Practice Problem

2. Differentiate

(a) $y = x(3x-1)^6$ **(b)** $y = x^3\sqrt{(2x+3)}$ **(c)** $y = \frac{x}{x-2}$

Advice

You may have found the product rule the hardest of the rules so far. This may have been due to the algebraic manipulation that is required to simplify the final expression. If this is the case, do not worry about it at this stage. The important thing is that you can use the product rule to obtain some sort of an answer even if you cannot tidy it up at the end. This is not to say that the simplification of an expression is pointless. If the result of differentiation is to be used in a subsequent piece of theory, it may well save time in the long run if it is simplified first.

One of the most difficult parts of Practice Problem 2 is part (c), since this involves algebraic fractions. For this function, it is necessary to manipulate negative indices and to put two individual fractions over a common denominator. You may feel that you are unable to do either of these processes with confidence. For this reason we conclude this section with a rule that is specifically designed to differentiate this type of function. The rule itself is quite complicated. However, as will become apparent, it does the algebra for you, so you may prefer to use it rather than the product rule when differentiating algebraic fractions.

Rule 6 The quotient rule

$$\text{If } y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2}$$

This rule tells you how to differentiate the quotient of two functions:

> bottom times derivative of top, minus top times derivative of bottom, all over bottom squared

Example

Differentiate

(a) $y = \frac{x}{1+x}$ **(b)** $y = \frac{1+x^2}{2-x^3}$

Solution

(a) In the quotient rule, u is used as the label for the numerator and v is used for the denominator, so to differentiate

$$\frac{x}{1+x}$$

we must take

$$u = x \quad \text{and} \quad v = 1 + x$$

for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = 1$$

By the quotient rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\,du/dx - u\,du/dx}{v^2}\\ &= \frac{(1+x)(1) - x(1)}{(1+x^2)}\\ &= \frac{1+x-x}{(1+x^2)}\\ &= \frac{1}{(1+x^2)}\end{aligned}$$

Notice how the quotient rule automatically puts the final expression over a common denominator. Compare this with the algebra required to obtain the same answer using the product rule in part (c) of the previous example.

(b) The numerator of the algebraic fraction

$$\frac{1+x^2}{2-x^3}$$

is $1 + x^2$ and the denominator is $2 - x^3$, so we take

$$u = 1 + x^2 \quad \text{and} \quad v = 2 - x^3$$

for which

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = -3x^2$$

By the quotient rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\,du/dx - u\,dv/dx}{v^2} \\ &= \frac{(2 - x^3)(2x) - (1 + x^2)(-3x^2)}{(2 - x^3)^2} \\ &= \frac{4x - 2x^4 + 3x^2 + 3x^4}{(2 - x^3)^2} \\ &= \frac{x^4 + 3x^2 + 4x}{(2 - x^3)^2}\end{aligned}$$

Practice Problem

3. Differentiate

(a) $y = \dfrac{x}{x - 2}$ **(b)** $y = \dfrac{x - 1}{x + 1}$

[You might like to check that your answer to part (a) is the same as that obtained in Practice Problem 2(c).]

Advice

The product and quotient rules give alternative methods for the differentiation of algebraic fractions. It does not matter which rule you go for; use whichever rule is easiest for you.

Exercise 4.4

1. Use the chain rule to differentiate

(a) $y = (5x + 1)^3$ (b) $y = (2x - 7)^8$ (c) $y = (x + 9)^5$

(d) $y = (4x^2 - 7)^3$ (e) $y = (x^2 + 4x - 3)^4$ (f) $y = \sqrt{(2x + 1)}$

(g) $y = \frac{1}{3x + 1}$ (h) $y = \frac{1}{(4x - 3)^2}$ (i) $y = \frac{1}{\sqrt{(2x + 5)}}$

2. Use the product rule to differentiate

(a) $x(3x + 4)^2$ (b) $x^2(x - 2)^3$ (c) $x\sqrt{(x + 2)}$

(d) $(x - 1)(x + 6)^3$ (e) $(2x + 1)(x + 5)^3$ (f) $x^3(2x - 5)^4$

3. Use the quotient rule to differentiate

(a) $y \frac{x}{x - 5}$ (b) $y = \frac{x}{(x + 7)}$ (c) $y = \frac{x + 3}{x - 2}$

(d) $y = \frac{2x + 9}{3x + 1}$ (e) $\frac{x}{(5x + 6)}$ (f) $y = \frac{x + 4}{3x - 7}$

4. Differentiate

$$y = x^5(x + 2)^2$$

(a) by using the chain rule

(b) by first multiplying out the brackets and then differentiating term by term.

5. Differentiate

$$y = x^5(x + 2)^2$$

(a) by using the product rule

(b) by first multiplying out the brackets and then differentiating term by term.

6. Find expressions for marginal revenue in the case when the demand equation is given by

(a) $P = (100 - Q)^3$ (b) $P = \frac{1000}{Q + 4}$

7. If the consumption function is

$$C = \frac{300 + 2Y^2}{1 + Y}$$

calculate MPC and MPS when $Y = 36$ and give an interpretation of these results.

Exercise 4.4*

1. Use the chain rule to differentiate

(a) $y = (2x + 1)^{10}$ (b) $y = (x^2 + 3x - 5)^3$ (c) $y = \dfrac{1}{7x - 3}$

(d) $y = \dfrac{1}{x^2 + 1}$ (e) $y = \sqrt{(8x - 1)}$ (f) $y = \dfrac{1}{\sqrt[3]{(6x - 5)}}$

2. Use the product rule to differentiate

(a) $y = x^2(x + 5)^3$ (b) $y = x^5(4x + 5)^2$ (c) $y = x\sqrt[4]{(x + 1)}$

3. Use the quotient rule to differentiate

(a) $y = \dfrac{x^2}{x + 4}$ (b) $y = \dfrac{2x - 1}{x + 1}$ (c) $y = \dfrac{x^3}{\sqrt{(x - 1)}}$

4. Differentiate

(a) $y = x(x - 3)^4$ (b) $y = x\sqrt{(2x - 3)}$ (c) $y = \dfrac{x^3}{(3x + 5)^2}$ (d) $y = \dfrac{x}{x^2 + 1}$

(e) $y = \dfrac{ax + b}{cx + d}$ (f) $y = (ax + b)^m(cx + d)^n$ (g) $y = x(x + 2)^2(x + 3)^3$

5. Find an expression, simplified as far as possible, for the second-order derivative of the function, $y = \dfrac{x}{2x + 1}$.

6. Find expressions for marginal revenue in the case when the demand equation is given by

(a) $P = \sqrt{(100 - 2Q)}$ (b) $P = \dfrac{100}{\sqrt{2 + Q}}$

7. Determine the marginal propensity to consume for the consumption function

$$C = \frac{650 + 2Y^2}{9 + Y}$$

when $Y = 21$, correct to 3 decimal places.

Deduce the corresponding value of the marginal propensity to save and comment on the implications of these results.

SECTION 4.6

Optimisation of economic functions

Objectives

At the end of this section you should be able to:

- Use the first-order derivative to find the stationary points of a function.
- Use the second-order derivative to classify the stationary points of a function.
- Find the maximum and minimum points of an economic function.
- Use stationary points to sketch graphs of economic functions.

In Section 2.1 a simple three-step strategy was described for sketching graphs of quadratic functions of the form

$$f(x) = ax^2 + bx + c$$

The basic idea is to solve the corresponding equation

$$ax^2 + bx + c = 0$$

to find where the graph crosses the x axis. Provided that the quadratic equation has at least one solution, it is then possible to deduce the coordinates of the maximum or minimum point of the parabola. For example, if there are two solutions, then by symmetry the graph turns round at the point exactly halfway between these solutions. Unfortunately, if the quadratic equation has no solution then only a limited sketch can be obtained using this approach.

In this section we show how the techniques of calculus can be used to find the coordinates of the turning point of a parabola. The beauty of this approach is that it can be used to locate the maximum and minimum points of any economic function, not just those represented by quadratics. Look at the graph in Figure 4.21. Points B, C, D, E, F and G are referred to as the **stationary points** (sometimes called **critical points, turning points** or **extrema**) of the function. At a stationary point the tangent to the graph is horizontal and so has zero slope.

Consequently, at a stationary point of a function $f(x)$,

$$f'(x) = 0$$

The reason for using the word 'stationary' is historical. Calculus was originally used by astronomers to predict planetary motion. If a graph of the distance travelled by an object is sketched against time then the speed of the object is given by the slope, since this represents the rate of change of distance with respect to time. It follows that if the graph is horizontal at some point then the speed is zero and the object is instantaneously at rest: that is, stationary.

Stationary points are classified into one of three types: local maxima, local minima and stationary points of inflection.

At a **local maximum** (sometimes called a relative maximum) the graph falls away on both sides. Points B and E are the local maxima for the function sketched in Figure 4.21. The word

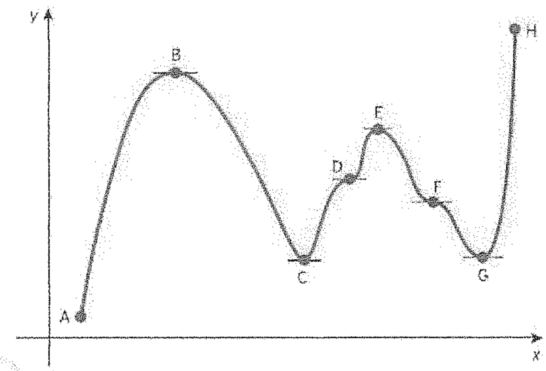

Figure 4.21

'local' is used to highlight the fact that, although these are the maximum points relative to their locality or neighbourhood, they may not be the overall or global maximum. In Figure 4.21 the highest point on the graph actually occurs at the right-hand end, H, which is not a stationary point, since the slope is not zero at H.

At a **local minimum** (sometimes called a relative minimum) the graph rises on both sides. Points C and G are the local minima in Figure 4.21. Again, it is not necessary for the global minimum to be one of the local minima. In Figure 4.21 the lowest point on the graph occurs at the left-hand end, A, which is not a stationary point.

At a **stationary point of inflection** the graph rises on one side and falls on the other. The stationary points of inflection in Figure 4.21 are labelled D and F. These points are of little value in economics, although they do sometimes assist in sketching graphs of economic functions. Maxima and minima, on the other hand, are important. The calculation of the maximum points of the revenue and profit functions is clearly worthwhile. Likewise, it is useful to be able to find the minimum points of average cost functions.

For most examples in economics, the local maximum and minimum points coincide with the global maximum and minimum. For this reason we shall drop the word 'local' when describing stationary points. However, it should always be borne in mind that the global maximum and minimum could actually be attained at an end point and this possibility may need to be checked. This can be done by comparing the function values at the end points with those of the stationary points and then deciding which of them gives rise to the largest or smallest values.

Two obvious questions remain. How do we find the stationary points of any given function and how do we classify them? The first question is easily answered. As we mentioned earlier, stationary points satisfy the equation

$$f'(x) = 0$$

so all we need do is to differentiate the function, to equate to zero and to solve the resulting algebraic equation. The classification is equally straightforward. It can be shown that if a function has a stationary point at $x = a$ then

- if $f''(a) > 0$ then $f(x)$ has a minimum at $x = a$
- if $f''(a) < 0$ then $f(x)$ has a maximum at $x = a$.

Therefore, all we need do is to differentiate the function a second time and to evaluate this second-order derivative at each point. A point is a minimum if this value is positive and a maximum if this value is negative. These facts are consistent with our interpretation of the second-order derivative in Section 4.2. If $f''(a) > 0$ the graph bends upwards at $x = a$ (points C and G in Figure 4.21). If $f''(a) < 0$ the graph bends downwards at $x = a$ (points B and E in Figure 4.21). There is, of course, a third possibility, namely $f''(a) = 0$. Sadly, when this happens it provides no information whatsoever about the stationary point. The point $x = a$ could be a maximum, minimum or inflection. This situation is illustrated in Question 2 in Exercise 4.6* at the end of this section.

Advice

If you are unlucky enough to encounter this case, you can always classify the point by tabulating the function values in the vicinity and use these to produce a local sketch.

To summarise, the method for finding and classifying stationary points of a function, $f(x)$, is as follows:

Step 1

Solve the equation $f'(x) = 0$ to find the stationary points, $x = a$.

Step 2

If

- $f''(a) > 0$ then the function has a minimum at $x = a$
- $f''(a) < 0$ then the function has a maximum at $x = a$
- $f''(a) = 0$ then the point cannot be classified using the available information.

Example

Find and classify the stationary points of the following functions. Hence sketch their graphs.

(a) $f(x) = x^2 - 4x + 5$ **(b)** $f(x) = 2x^3 + 3x^2 - 12x + 4$

Solution

(a) In order to use steps 1 and 2 we need to find the first- and second-order derivatives of the function

$$f'(x) = x^2 - 4x + 5$$

Differentiating once gives

$$f'(x) = 2x - 4$$

and differentiating a second time gives

$$f'(x) = 2$$

Step 1

The stationary points are the solutions of the equation

$$f'(x) = 0$$

so we need to solve

$$2x - 4 = 0$$

This is a linear equation so has just one solution. Adding 4 to both sides gives

$$2x = 4$$

and dividing through by 2 shows that the stationary point occurs at

$$x = 2$$

Step 2

To classify this point we need to evaluate

$$f''(2)$$

In this case

$$f''(x) = 2$$

for all values of x, so in particular

$$f''(2) = 2$$

This number is positive, so the function has a minimum at $x = 2$.

We have shown that the minimum point occurs at $x = 2$. The corresponding value of y is easily found by substituting this number into the function to get

$$y = (2)^2 - 4(2) + 5 = 1$$

so the minimum point has coordinates (2, 1). A graph of $f(x)$ is shown in Figure 4.22.

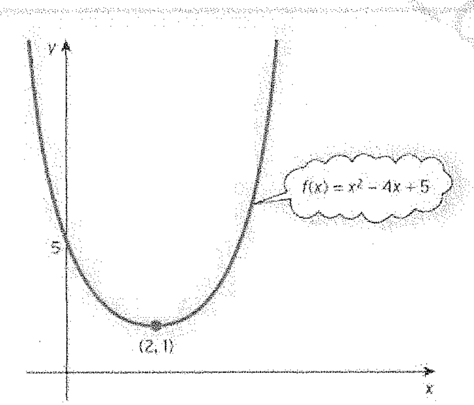

Figure 4.22

(b) In order to use steps 1 and 2 we need to find the first- and second-order derivatives of the function

$$f(x) = 2x^3 + 3x^2 - 12x + 4$$

Differentiating once gives

$$f'(x) = 6x^2 + 6x - 12$$

and differentiating a second time gives

$$f''(x) = 12x + 6$$

Step 1

The stationary points are the solutions of the equation

$$f'(x) = 0$$

so we need to solve

$$6x^2 + 6x - 12 = 0$$

This is a quadratic equation and so can be solved using 'the formula'. However, before doing so, it is a good idea to divide both sides by 6 to avoid large numbers. The resulting equation

$$x^2 + x - 2 = 0$$

has solution

$$x = \frac{-1 \pm \sqrt{(1^2 - 4(1)(-2))}}{2(1)} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = -2, 1$$

In general, whenever $f(x)$ is a cubic function the stationary points are the solutions of a quadratic equation, $f'(x) = 0$. Moreover, we know from Section 2.1 that such an equation can have two, one or no solutions. It follows that a cubic equation can have two, one or no stationary points. In this particular example we have seen that there are two stationary points, at $x = -2$ and $x = 1$.

Step 2

To classify these points we need to evaluate $f''(-2)$ and $f''(1)$. Now

$$f''(-2) = 12(-2) + 6 = -18$$

This is negative, so there is a maximum at $x = -2$. When $x = -2$,

$$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 4 = 24$$

so the maximum point has coordinates (−2, 24). Now

$$f''(1) = 12(1) + 6 = 18$$

This is positive, so there is a minimum at $x = 1$. When $x = 1$,

$$y = 2(1)^3 + 3(1)^2 - 12(1) + 4 = -3$$

so the minimum point has coordinates (1, −3).

This information enables a partial sketch to be drawn as shown in Figure 4.23. Before we can be confident about the complete picture it is useful to plot a few more points such as those below:

x	−10	0	10
y	−1816	4	2184

This table indicates that when x is positive the graph falls steeply downwards from a great height. Similarly, when x is negative the graph quickly disappears off the bottom of the page. The curve cannot wiggle and turn round except at the two stationary points already plotted (otherwise it would have more stationary points, which we know is not the case). We now have enough information to join up the pieces and so sketch a complete picture as shown in Figure 4.24.

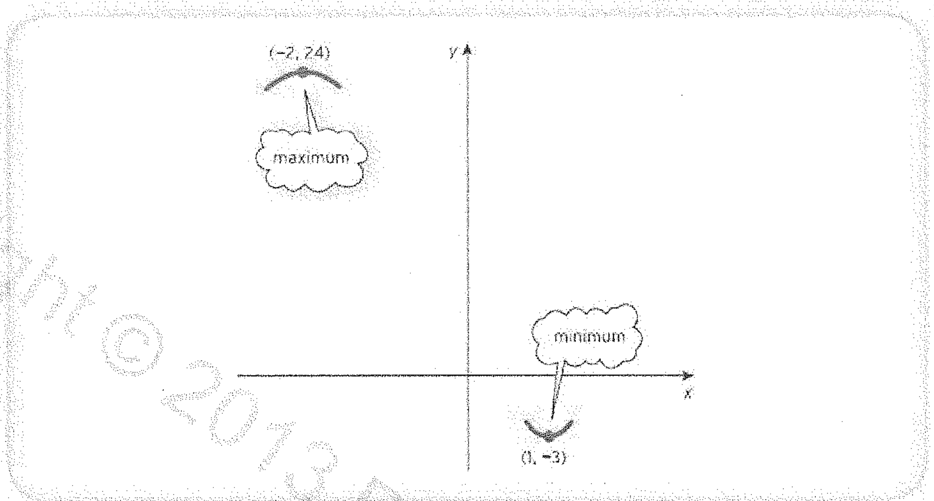

Figure 4.23

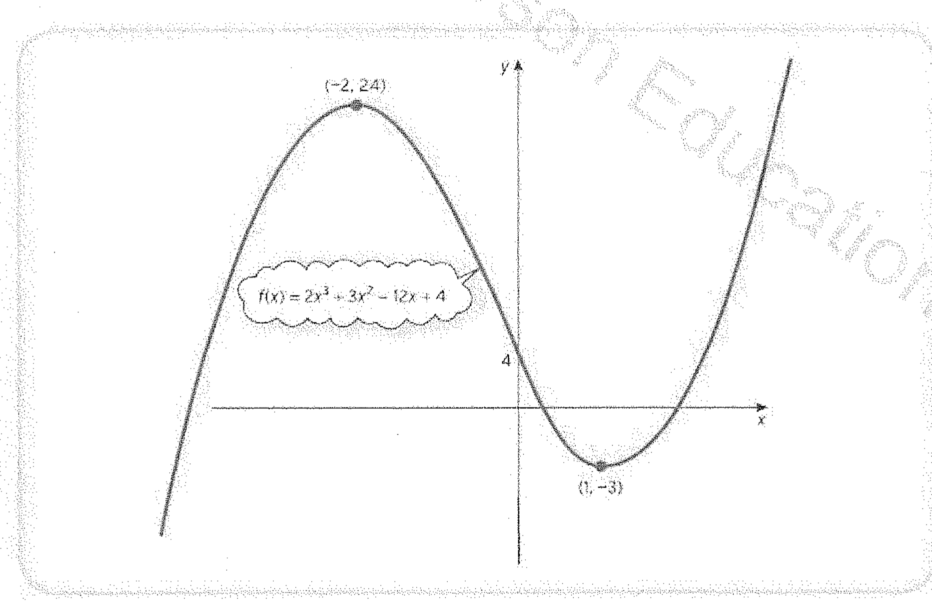

Figure 4.24

In an ideal world it would be nice to calculate the three points at which the graph crosses the x axis. These are the solutions of

$$2x^3 + 3x^2 - 12x + 4 = 0$$

There is a formula for solving cubic equations, just as there is for quadratic equations, but it is extremely complicated and is beyond the scope of this book.

Practice Problem

1. Find and classify the stationary points of the following functions. Hence sketch their graphs.

   (a) $y = 3x^2 + 12x - 35$ (b) $y = -2x^3 + 15x^2 - 36x + 27$

The task of finding the maximum and minimum values of a function is referred to as **optimisation**. This is an important topic in mathematical economics. It provides a rich source of examination questions and we devote the remaining part of this section and the whole of the next to applications of it. In this section we demonstrate the use of stationary points by working through four 'examination-type' problems in detail. These problems involve the optimisation of specific revenue, cost, profit and production functions. They are not intended to exhaust all possibilities, although they are fairly typical. The next section describes how the mathematics of optimisation can be used to derive general theoretical results.

Example

A firm's short-run production function is given by

$$Q = 6L^2 - 0.2L^3$$

where L denotes the number of workers.

(a) Find the size of the workforce that maximises output and hence sketch a graph of this production function.

(b) Find the size of the workforce that maximises the average product of labour. Calculate MP_L and AP_L at this value of L. What do you observe?

Solution

(a) In the first part of this example we want to find the value of L which maximises

$$Q = 6L^2 - 0.2L^3$$

Step 1

At a stationary point

$$\frac{dQ}{dL} = 12L - 0.6L^2 = 0$$

This is a quadratic equation and so we could use 'the formula' to find L. However, this is not really necessary in this case because both terms have a common factor of L and the equation may be written as

$$L(12 - 0.6L) = 0$$

It follows that either

$$L = 0 \text{ or } 12 - 0.6L = 0$$

that is, the equation has solutions

$$L = 0 \text{ and } L = 12/0.6 = 20$$

Step 2

It is obvious on economic grounds that $L = 0$ is a minimum and presumably $L = 20$ is the maximum. We can, of course, check this by differentiating a second time to get

$$\frac{d^2Q}{dL^2} = 12 - 1.2L$$

When $L = 0$,

$$\frac{d^2Q}{dL^2} = 12 > 0$$

which confirms that $L = 0$ is a minimum. The corresponding output is given by

$$Q = 6(0)^2 - 0.2(0)^3 = 0$$

as expected. When $L = 20$,

$$\frac{d^2Q}{dL^2} = -12 > 0$$

which confirms that $L = 20$ is a maximum.

The firm should therefore employ 20 workers to achieve a maximum output

$$Q = 6(20)^2 - 0.2(20)^3 = 800$$

We have shown that the minimum point on the graph has coordinates (0, 0) and the maximum point has coordinates (20, 800). There are no further turning points, so the graph of the production function has the shape sketched in Figure 4.25.

It is possible to find the precise values of L at which the graph crosses the horizontal axis. The production function is given by

$$Q = 6L^2 - 0.2L^3$$

so we need to solve

$$6L^2 - 0.2L^3 = 0$$

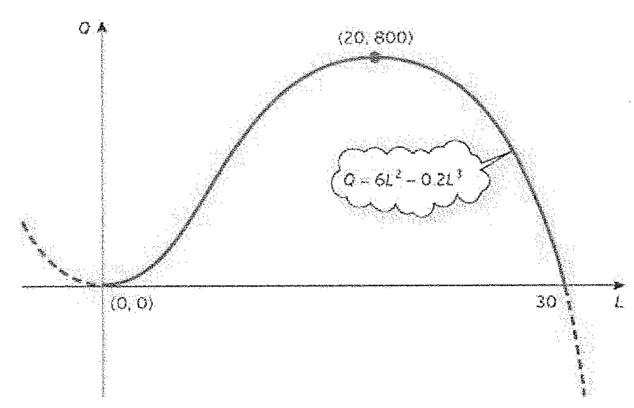

Figure 4.25

We can take out a factor of L^2 to get

$$L^2(6 - 0.2L) = 0$$

Hence, either

$$L^2 = 0 \text{ or } 6 - 0.2L = 0$$

The first of these merely confirms the fact that the curve passes through the origin, whereas the second shows that the curve intersects the L axis at $L = 6/0.2 = 30$.

(b) In the second part of this example we want to find the value of L which maximises the average product of labour. This is a concept that we have not met before in this book, although it is not difficult to guess how it might be defined.

The **average product of labour**, AP_L, is taken to be total output divided by labour, so that in symbols

$$AP_L = \frac{Q}{L}$$

This is sometimes called **labour productivity**, since it measures the average output per worker.

In this example,

$$AP_L = \frac{6L^2 - 0.2L^3}{L} = 6L - 0.2L^2$$

Step 1

At a stationary point

$$\frac{d(AP_L)}{dL} = 0$$

so

$$6 - 0.4L = 0$$

which has solution $L = 6/0.4 = 15$.

Step 2

To classify this stationary point we differentiate a second time to get

$$\frac{d(AP_L)}{dL} = -0.4 < 0$$

which shows that it is a maximum.

The labour productivity is therefore greatest when the firm employs 15 workers. In fact, the corresponding labour productivity, AP_L, is

$$6(15) - 0.2(15)^2 = 45$$

In other words, the largest number of goods produced per worker is 45.

Finally, we are invited to calculate the value of MP_L at this point. To find an expression for MP_L we need to differentiate Q with respect to L to get

$$MP_L = 12L - 0.6L^2$$

When $L = 15$,

$$MP_L = 12(15) - 0.6(15)^2 = 45$$

We observe that at $L = 15$ the values of MP_L and AP_L are equal.

In this particular example we discovered that at the point of maximum average product of labour

marginal product of labour = average product of labour

There is nothing special about this example and in the next section we show that this result holds for any production function.

Practice Problem

2. A firm's short-run production function is given by

$$Q = 300L^2 - L^4$$

where L denotes the number of workers. Find the size of the workforce that maximises the average product of labour and verify that at this value of L

$$MP_L = AP_L$$

Example

The demand equation of a good is

$$P + Q = 30$$

and the total cost function is

$$TC = \frac{1}{2}Q^2 + 6Q + 7$$

(a) Find the level of output that maximises total revenue.

(b) Find the level of output that maximises profit. Calculate MR and MC at this value of Q. What do you observe?

Solution

(a) In the first part of this example we want to find the value of Q which maximises total revenue. To do this we use the given demand equation to find an expression for TR and then apply the theory of stationary points in the usual way.

The total revenue is defined by

$$TR = PQ$$

We seek the value of Q which maximises TR, so we express TR in terms of the variable Q only. The demand equation

$$P + Q = 30$$

can be rearranged to get

$$P = 30 - Q$$

Hence

$$\begin{aligned} TR &= (30 - Q)Q \\ &= 30Q - Q^2 \end{aligned}$$

Step 1

At a stationary point

$$\frac{d(TR)}{dQ} = 0$$

so

$$30 - 2Q = 0$$

which has solution $Q = 30/2 = 15$.

Step 2

To classify this point we differentiate a second time to get

$$\frac{d^2(TR)}{dQ^2} = -2$$

This is negative, so TR has a maximum at $Q = 15$.

(b) In the second part of this example we want to find the value of Q which maximises profit. To do this we begin by determining an expression for profit in terms of Q. Once this has been done, it is then a simple matter to work out the first- and second-order derivatives and so to find and classify the stationary points of the profit function.

The profit function is defined by

$$\pi = TR - TC$$

From part (a)

$$TR = 30Q - Q^2$$

We are given the total cost function

$$TC = \tfrac{1}{2}Q^2 + 6Q + 7$$

Hence

$$\begin{aligned}\pi &= (30Q - Q^2) - (\tfrac{1}{2}Q^2 + 6Q + 7)\\ &= 30Q - Q^2 - \tfrac{1}{2}Q^2 - 6Q - 7\\ &= -\tfrac{3}{2}Q^2 + 24Q - 7\end{aligned}$$

Step 1

At a stationary point

$$\frac{d\pi}{dQ} = 0$$

so

$$-3Q + 24 = 0$$

which has solution $Q = 24/3 = 8$.

Step 2

To classify this point we differentiate a second time to get

$$\frac{d^2\pi}{dQ^2} = -3$$

This is negative, so π has a maximum at $Q = 8$. In fact, the corresponding maximum profit is

$$\pi = -\tfrac{3}{2}(8)^2 + 24(8) - 7 = 89$$

Finally, we are invited to calculate the marginal revenue and marginal cost at this particular value of Q. To find expressions for MR and MC we need only differentiate TR and TC, respectively. If

$$TR = 30Q - Q^2$$

then

$$MR = \frac{d(TR)}{dQ} = 30 - 2Q$$

so when $Q = 8$

$$MR = 30 - 2(8) = 14$$

If

$$TC = \tfrac{1}{2}Q^2 + 6Q + 7$$

then

$$MC = \frac{d(TC)}{dQ} = Q + 6$$

so when $Q = 8$

$$MC = 8 + 6 = 14$$

We observe that at $Q = 8$, the values of MR and MC are equal.

In this particular example we discovered that at the point of maximum profit,

$$\boxed{\text{marginal revenue}} = \boxed{\text{marginal cost}}$$

There is nothing special about this example and in the next section we show that this result holds for any profit function.

Practice Problem

3. The demand equation of a good is given by

$$P + 2Q = 20$$

and the total cost function is

$$Q^3 - 8Q^2 + 20Q + 2$$

(a) Find the level of output that maximises total revenue.

(b) Find the maximum profit and the value of Q at which it is achieved. Verify that, at this value of Q, MR = MC.

Example

The cost of building an office block, x floors high, is made up of three components:

(1) \$10 million for the land

(2) \$1/4 million per floor

(3) specialised costs of \$10 000x per floor.

How many floors should the block contain if the average cost per floor is to be minimised?

Solution

The \$10 million for the land is a fixed cost because it is independent of the number of floors. Each floor costs \$1/4 million, so if the building has x floors altogether then the cost will be 250 000x.

In addition there are specialised costs of 10 000x per floor, so if there are x floors this will be

$$(10\,000x)x = 10\,000x^2$$

Notice the square term here, which means that the specialised costs rise dramatically with increasing x. This is to be expected, since a tall building requires a more complicated design. It may also be necessary to use more expensive materials.

The total cost, TC, is the sum of the three components: that is,

$$\text{TC} = 10\,000\,000 + 250\,000x + 10\,000x^2$$

The average cost per floor, AC, is found by dividing the total cost by the number of floors: that is,

$$\begin{aligned}\text{AC} = \frac{\text{TC}}{x} &= \frac{10\,000\,000 + 250\,000x + 10\,000x^2}{x} \\ &= \frac{10\,000\,000}{x} + 250\,000 + 10\,000x \\ &= 10\,000\,000x^{-1} + 250\,000 + 10\,000x\end{aligned}$$

Step 1

At a stationary point

$$\frac{\text{d(AC)}}{\text{d}x} = 0$$

In this case

$$\frac{\text{d(AC)}}{\text{d}x} = -10\,000\,000x^{-2} + 10\,000 = \frac{-10\,000\,000}{x^2} + 10\,000$$

so we need to solve

$$10\,000 = \frac{10\,000\,000}{x^2} \text{ or equivalently } 10\,000x^2 = 10\,000\,000$$

Hence

$$x^2 = \frac{10\,000\,000}{10\,000} = 1000$$

This has solution

$$x = \pm\sqrt{1000} = \pm 31.6$$

We can obviously ignore the negative value because it does not make sense to build an office block with a negative number of floors, so we can deduce that $x = 31.6$.

Step 2

To confirm that this is a minimum we need to differentiate a second time. Now

$$\frac{d(AC)}{dx} = -10\,000\,000x^{-2} + 10\,000$$

so

$$\frac{d^2(AC)}{dx^2} = -2(-10\,000\,000)x^{-3} = \frac{20\,000\,000}{x^3}$$

When $x = 31.6$ we see that

$$\frac{d^2(AC)}{dx^2} = \frac{20\,000\,000}{(31.6)^3} = 633.8$$

It follows that $x = 31.6$ is indeed a minimum because the second-order derivative is a positive number.

At this stage it is tempting to state that the answer is 31.6. This is mathematically correct but is a physical impossibility since x must be a whole number. To decide whether to take x to be 31 or 32 we simply evaluate AC for these two values of x and choose the one that produces the lower average cost.

When $x = 31$,

$$AC = \frac{10\,000\,000}{31} + 250\,000 + 10\,000(31) = \$882\,581$$

When $x = 32$,

$$AC = \frac{10\,000\,000}{32} + 250\,000 + 10\,000(32) = \$882\,500$$

Therefore an office block 32 floors high produces the lowest average cost per floor.

Practice Problem

4. The total cost function of a good is given by

$$TC = Q^2 + 3Q + 36$$

Calculate the level of output that minimises average cost. Find AC and MC at this value of Q. What do you observe?

Example

The supply and demand equations of a good are given by

$$P = Q_S + 8$$

and

$$P = -3Q_D + 80$$

respectively.

The government decides to impose a tax, t, per unit. Find the value of t which maximises the government's total tax revenue on the assumption that equilibrium conditions prevail in the market.

Solution

The idea of taxation was first introduced in Chapter 1. In Section 1.5 the equilibrium price and quantity were calculated from a given value of t. In this example t is unknown but the analysis is exactly the same. All we need to do is to carry the letter t through the usual calculations and then to choose t at the end so as to maximise the total tax revenue.

To take account of the tax we replace P by $P - t$ in the supply equation. This is because the price that the supplier actually receives is the price, P, that the consumer pays less the tax, t, deducted by the government. The new supply equation is then

$$P - t = Q_S + 8$$

so that

$$P = Q_S + 8 + t$$

In equilibrium

$$Q_S = Q_D$$

If this common value is denoted by Q then the supply and demand equations become

$$P = Q + 8 + t$$
$$P = -3Q + 80$$

Hence

$$Q + 8 + t = -3Q + 80$$

since both sides are equal to P. This can be rearranged to give

$$Q = -3Q + 72 - t \quad \text{(subtract } 8 + t \text{ from both sides)}$$
$$4Q = 72 - t \quad \text{(add } 3Q \text{ to both sides)}$$
$$Q = 18 - \tfrac{1}{4}t \quad \text{(divide both sides by 4)}$$

Now, if the number of goods sold is Q and the government raises t per good then the total tax revenue, T, is given by

$$T = tQ$$
$$= t(18 - \tfrac{1}{4}t)$$
$$= 18t - \tfrac{1}{4}t^2$$

This then is the expression that we wish to maximise.

Step 1

At a stationary point

$$\frac{dT}{dt} = 0$$

so

$$18 - \frac{1}{2}t = 0$$

which has solution

$$t = 36$$

Step 2

To classify this point we differentiate a second time to get

$$\frac{d^2T}{dt^2} = -\frac{1}{2} < 0$$

which confirms that it is a maximum.

Hence the government should impose a tax of $36 on each good.

Practice Problem

5. The supply and demand equations of a good are given by

$$P = \tfrac{1}{2}Q_S + 25$$

and

$$P = -2Q_D + 50$$

respectively.

The government decides to impose a tax, *t*, per unit. Find the value of *t* which maximises the government's total tax revenue on the assumption that equilibrium conditions prevail in the market.

In theory a spreadsheet such as Excel could be used to solve optimisation problems, although it cannot handle the associated mathematics. The preferred method is to use a symbolic computation system such as Maple, Matlab, Mathcad or Derive which can not only sketch the graphs of functions but also differentiate and solve equations. Consequently it is possible to obtain the exact solution using one of these packages.

The Online Resources describe how to get started with Maple and an example is given which shows how to find the exact coordinates of the maximum and minimum points of a cubic function.

Key Terms

Average product of labour (labour productivity) Output per worker: $AP_L = Q/L$.

Maximum (local) point A point on a curve which has the highest function value in comparison with other values in its neighbourhood; at such a point the first-order derivative is zero and the second-order derivative is either zero or negative.

Minimum (local) point A point on a curve which has the lowest function value in comparison with other values in its neighbourhood; at such a point the first-order derivative is zero and the second-order derivative is either zero or positive.

Optimisation The determination of the optimal (usually stationary) points of a function.

Stationary point of inflection A stationary point that is neither a maximum nor a minimum; at such a point both the first- and second-order derivatives are zero.

Stationary points (critical points, turning points, extrema) Points on a graph at which the tangent is horizontal; at a stationary point the first-order derivative is zero.

Exercise 4.6

1. Find and classify the stationary points of the following functions. Hence give a rough sketch of their graphs.

(a) $y = -x^2 + x + 1$ (b) $y = x^2 - 4x + 4$ (c) $y = x^2 - 20x + 105$ (d) $y = -x^3 + 3x$

2. If the demand equation of a good is

$$P = 40 - 2Q$$

find the level of output that maximises total revenue.

3. A firm's short-run production function is given by

$$Q = 30L^2 - 0.5L^3$$

Find the value of L which maximises AP_L and verify that $MP_L = AP_L$ at this point.

4. If the fixed costs are 13 and the variable costs are $Q + 2$ per unit, show that the average cost function is

$$AC = \frac{13}{Q} + Q + 2$$

(a) Calculate the values of AC when $Q = 1, 2, 3, \ldots, 6$. Plot these points on graph paper and hence produce an accurate graph of AC against Q.

(b) Use your graph to estimate the minimum average cost.

(c) Use differentiation to confirm your estimate obtained in part (b).

5. The demand and total cost functions of a good are

$$4P + Q - 16 = 0$$

and

$$TC = 4 + 2Q - \frac{3Q^2}{10} + \frac{Q^3}{20}$$

respectively.

(a) Find expressions for TR, π, MR and MC in terms of Q.

(b) Solve the equation

$$\frac{d\pi}{dQ} = 0$$

and hence determine the value of Q which maximises profit.

(c) Verify that, at the point of maximum profit, MR = MC.

6. The supply and demand equations of a good are given by

$$3P - Q_S = 3$$

and

$$2P + Q_D = 14$$

respectively.

The government decides to impose a tax, t, per unit. Find the value of t (in dollars) which maximises the government's total tax revenue on the assumption that equilibrium conditions prevail in the market.

7. A manufacturer has fixed costs of \$200 each week, and the variable costs per unit can be expressed by the function, VC = $2Q - 36$

(a) Find an expression for the total cost function and deduce that the average cost function is given by

$$AC = \frac{200}{Q} + 2Q - 36$$

(b) Find the stationary point of this function and show that this is a minimum.

(c) Verify that, at this stationary point, average cost is the same as marginal cost.

Exercise 4.6*

1. A firm's demand function is

$$P = 60 - 0.5Q$$

If fixed costs are 10 and variable costs are $Q + 3$ per unit, find the maximum profit.

2. Show that all of the following functions have a stationary point at $x = 0$. Verify in each case that $f''(0) = 0$. Classify these points by producing a rough sketch of each function.

(a) $f(x) = x^3$ (b) $f(x) = x^4$ (c) $f(x) = -x^6$

3. If fixed costs are 15 and the variable costs are $2Q$ per unit, write down expressions for TC, AC and MC. Find the value of Q which minimises AC and verify that AC = MC at this point.

4. An electronic components firm launches a new product on 1 January. During the following year a rough estimate of the number of orders, S, received t days after the launch is given by

$$S = t^2 - 0.002t^3$$

(a) What is the maximum number of orders received on any one day of the year?

(b) After how many days does the firm experience the greatest increase in orders?

5. If the demand equation of a good is

$$P = \sqrt{(1000 - 4Q)}$$

find the value of Q which maximises total revenue.

6. A firm's total cost and demand functions are given by

$$TC = Q^2 + 50Q + 10 \text{ and } P = 200 - 4Q$$

respectively.

(a) Find the level of output needed to maximise the firm's profit.

(b) The government imposes a tax of $\$t$ per good. If the firm adds this tax to its costs and continues to maximise profit, show that the price of the good increases by two-fifths of the tax, irrespective of the value of t.

7. Given that the cubic function, $f(x) = x^3 + ax^2 + bx + c$ has a stationary point at (2, 5), and that it passes through (1, 3), find the values of a, b and c.