SECTION 4.1

The derivative of a function

Objectives

At the end of this section you should be able to:

- Find the slope of a straight line given any two points on the line.
- Detect whether a line is uphill, downhill or horizontal using the sign of the slope.
- Recognise the notation $f'(x)$ and dy/dx for the derivative of a function.
- Estimate the derivative of a function by measuring the slope of a tangent.
- Differentiate power functions.

This introductory section is designed to get you started with differential calculus in a fairly painless way. There are really only three things that we are going to do. We discuss the basic idea of something called a derived function, give you two equivalent pieces of notation to describe it, and finally show you how to write down a formula for the derived function in simple cases.

In Chapter 1 the slope of a straight line was defined to be the change in the value of y brought about by a 1 unit increase in x. In fact, it is not necessary to restrict the change in x to a 1 unit increase. More generally, the **slope**, or **gradient**, of a line is taken to be the change in y divided by the corresponding change in x as you move between any two points on the line. It is customary to denote the change in y by Δy, where Δ is the Greek letter 'delta'. Likewise, the change in x is written Δx. In this notation we have

$$\boxed{\text{slope} = \frac{\Delta y}{\Delta x}}$$

Example

Find the slope of the straight line passing through

(a) A (1, 2) and B (3, 4) **(b)** A (1, 2) and C (4, 1) **(c)** A (1, 2) and D (5, 2)

Solution

(a) Points A and B are sketched in Figure 4.1. As we move from A to B, the y coordinate changes from 2 to 4, which is an increase of 2 units, and the x coordinate changes from 1 to 3, which is also an increase of 2 units. Hence

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{4-2}{3-1} = \frac{2}{1} = 1$$

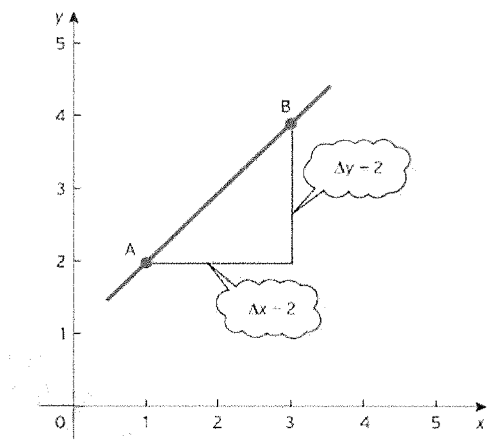

Figure 4.1

(b) Points A and C are sketched in Figure 4.2. As we move from A to C, the y coordinate changes from 2 to 1, which is a decrease of 1 unit, and the x coordinate changes from 1 to 4, which is an increase of 3 units. Hence

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{1-2}{4-1} = \frac{-1}{3}$$

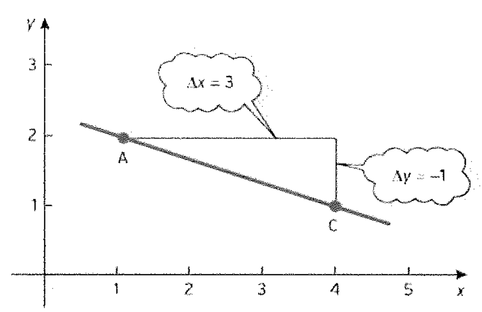

Figure 4.2

(c) Points A and D are sketched in Figure 4.3. As we move from A to D, the y coordinate remains fixed at 2, and the x coordinate changes from 1 to 5, which is an increase of 4 units. Hence

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{2-2}{5-1} = \frac{0}{4} = 0$$

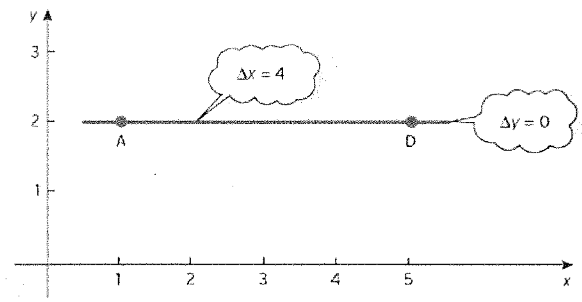

Figure 4.3

Practice Problem

1. Find the slope of the straight line passing through

(a) E (−1, 3) and F (3, 11) **(b)** E (−1, 3) and G (4, −2) **(c)** E (−1, 3) and H (49, 3)

From these examples we see that the gradient is positive if the line is uphill, negative if the line is downhill and zero if the line is horizontal.

Unfortunately, not all functions in economics are linear, so it is necessary to extend the definition of slope to include more general curves. To do this we need the idea of a tangent, which is illustrated in Figure 4.4.

A straight line which passes through a point on a curve and which just touches the curve at this point is called a **tangent**. The slope, or gradient, of a curve at $x = a$ is then defined to be that of the tangent at $x = a$. Since we have already seen how to find the slope of a straight line, this gives us a precise way of measuring the slope of a curve. A simple curve together with a selection of tangents at various points is shown in Figure 4.5. Notice how each tangent passes through exactly one point on the curve and strikes a glancing blow. In this case, the slopes of the tangents increase as we move from left to right along the curve. This reflects the fact that the curve is flat at $x = 0$ but becomes progressively steeper further away.

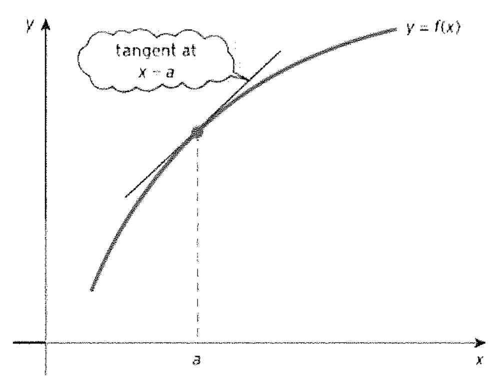

Figure 4.4

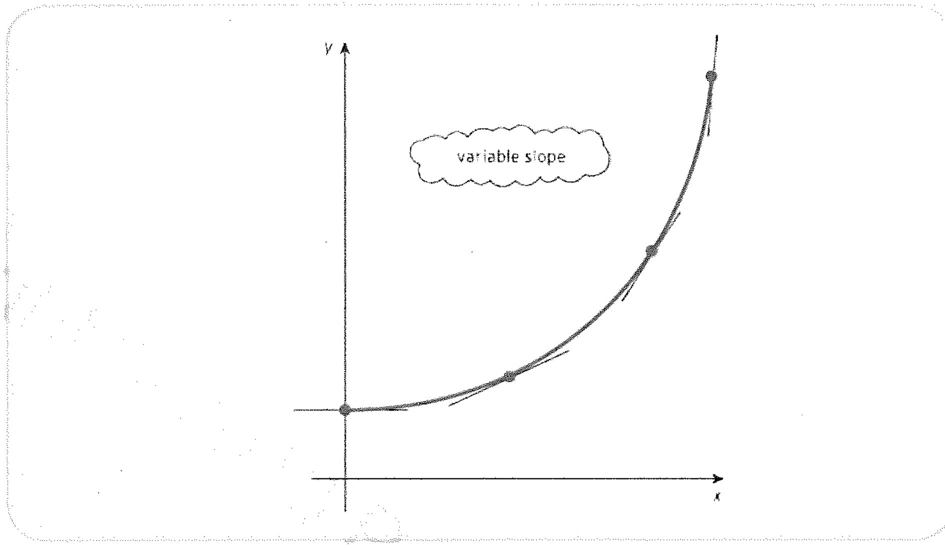

Figure 4.5

This highlights an important difference between the slope of a straight line and the slope of a curve. In the case of a straight line, the gradient is fixed throughout its length and it is immaterial which two points on a line are used to find it. For example, in Figure 4.6 all of the ratios $\Delta y/\Delta x$ have the value $^1/_2$. However, as we have just seen, the slope of a curve varies as we move along it. In mathematics we use the symbol

$f'(a)$ (read 'f dashed of a')

to represent the slope of the graph of a function f at $x = a$. This notation conveys the maximum amount of information with the minimum of fuss. As usual, we need the label f to denote which function we are considering. We certainly need the a to tell us at which point on the curve the gradient is being measured. Finally, the 'prime' symbol $'$ is used to distinguish the gradient from the function value. The notation $f(a)$ gives the height of the curve above the x axis at $x = a$, whereas $f'(a)$ gives the gradient of the curve at this point.

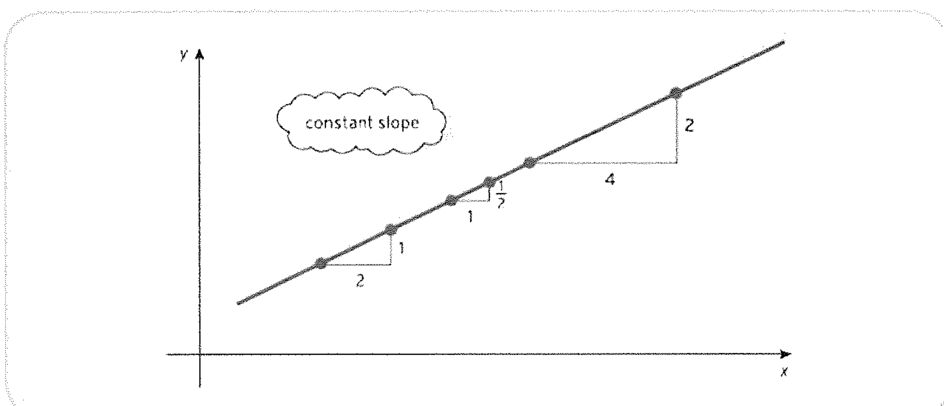

Figure 4.6

The slope of the graph of a function is called the **derivative** of the function. It is interesting to notice that corresponding to each value of x there is a uniquely defined derivative $f'(x)$. In other words, the rule 'find the slope of the graph of f at x' defines a function. This slope function is usually referred to as the **derived function**. An alternative notation for the derived function is

$$\frac{dy}{dx}$$ (read 'dee y by dee x')

Historically, this symbol arose from the corresponding notation $\Delta y/\Delta x$ for the gradient of a straight line; the letter 'd' is the English equivalent of the Greek letter Δ. However, it is important to realise that

$$\frac{dy}{dx}$$

does not mean 'dy divided by dx'. It should be thought of as a single symbol representing the derivative of y with respect to x. It is immaterial which notation is used, although the context may well suggest which is more appropriate. For example, if we use

$$y = x^2$$

to identify the square function then it is natural to use

$$\frac{dy}{dx}$$

for the derived function. On the other hand, if we use

$$f(x) = x^2$$

then $f'(x)$ seems more appropriate.

Example

Complete the following table of function values and hence sketch an accurate graph of $f(x) = x^2$.

x	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
$f(x)$									

Draw the tangents to the graph at $x = -1.5, -0.5, 0, 0.5$ and 1.5. Hence estimate the values of $f'(-1.5), f'(-0.5), f'(0), f'(0.5)$ and $f'(1.5)$.

Solution

Using a calculator we obtain

x	−2.0	−1.5	−1.0	−0.5	0.0	0.5	1.0	1.5	2.0
$f(x)$	4	2.25	1	0.25	0	0.25	1	2.25	4

The corresponding graph of the square function is sketched in Figure 4.7. From the graph we see that the slopes of the tangents are

$$f'(-1.5) = \frac{-15}{0.5} = -3$$

$$f'(-0.5) = \frac{-0.5}{0.5} = -1$$

$$f'(0) = 0$$

$$f'(0.5) = \frac{0.5}{0.5} = 1$$

$$f'(1.5) = \frac{1.5}{0.5} = 3$$

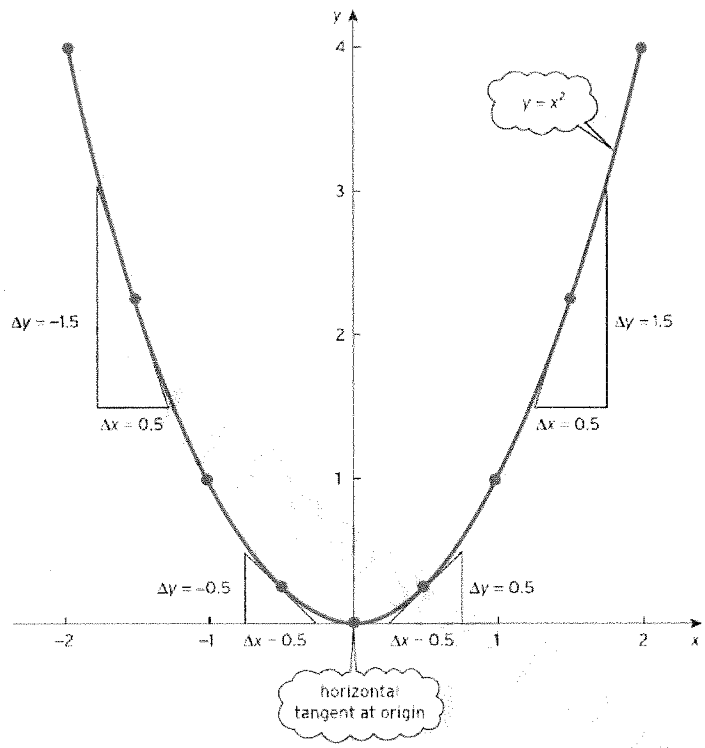

Figure 4.7

The value of $f'(0)$ is zero because the tangent is horizontal at $x = 0$. Notice that

$$f'(-1.5) = -f'(1.5) \quad \text{and} \quad f'(-0.5) = -f'(0.5)$$

This is to be expected because the graph is symmetric about the y axis. The slopes of the tangents to the left of the y axis have the same size as those of the corresponding tangents to the right. However, they have opposite signs since the curve slopes downhill on one side and uphill on the other.

Practice Problem

2. Complete the following table of function values and hence sketch an accurate graph of $f(x) = x^3$.

x	−1.50	−1.25	−1.00	−0.75	−0.50	−0.25	0.00
$f(x)$		−1.95			−0.13		

x	0.25	0.50	0.75	1.00	1.25	1.50
$f(x)$		0.13			1.95	

Draw the tangents to the graph at $x = -1, 0$ and 1. Hence estimate the values of $f'(-1)$, $f'(0)$ and $f'(1)$.

Practice Problem 2 should convince you how hard it is in practice to calculate $f'(a)$ exactly using graphs. It is impossible to sketch a perfectly smooth curve using graph paper and pencil, and it is equally difficult to judge, by eye, precisely where the tangent should be. There is also the problem of measuring the vertical and horizontal distances required for the slope of the tangent. These inherent errors may compound to produce quite inaccurate values for $f'(a)$. Fortunately, there is a really simple formula that can be used to find $f'(a)$ when f is a power function. It can be proved that

$$\boxed{\text{if } f(x) = x^n \text{ then } f'(x) = nx^{n-1}}$$

or, equivalently,

$$\boxed{\text{if } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}}$$

The process of finding the derived function symbolically (rather than using graphs) is known as **differentiation**. In order to differentiate x^n all that needs to be done is to bring the power down to the front and then to subtract 1 from the power:

subtract 1 from the power

x^n differentiates to nx^{n-1}

bring down the power

To differentiate the square function we set $n = 2$ in this formula to deduce that

subtract 1

$f(x) = x^2$ differentiates to $f'(x) = 2x^{2-1}$

the 2 comes down

that is,

$$f'(x) = 2x^1 = 2x$$

Using this result we see that

$$f'(-1.5) = 2 \times (-1.5) = -3$$
$$f'(-0.5) = 2 \times (-0.5) = -1$$
$$f'(0) = 2 \times (0) = 0$$
$$f'(0.5) = 2 \times (0.5) = 1$$
$$f'(1.5) = 2 \times (1.5) = 3$$

which are in agreement with the results obtained graphically in the preceding example.

Practice Problem

3. If $f(x) = x^3$ write down a formula for $f'(x)$. Calculate $f'(-1)$, $f'(0)$ and $f'(1)$. Confirm that these are in agreement with your rough estimates obtained in Practice Problem 2.

Example

Differentiate

(a) $y = x^4$ **(b)** $y = x^{10}$ **(c)** $y = x$ **(d)** $y = 1$ **(e)** $y = 1/x^4$ **(f)** $y = x$

Solution

(a) To differentiate $y = x^4$ we bring down the power (that is, 4) to the front and then subtract 1 from the power (that is, $4 - 1 = 3$) to deduce that

$$\frac{dy}{dx} = 4x^3$$

(b) Similarly,

$$\text{if } y = x^{10} \quad \text{then} \quad \frac{dy}{dx} = 10x^9$$

(c) To use the general formula to differentiate x we first need to express $y = x$ in the form $y = x^n$ for some number n. In this case $n = 1$ because $x^1 = x$, so

$$\frac{dy}{dx} = 1x^0 = 1 \quad \text{since} \quad x^0 = 1$$

This result is also obvious from the graph of $y = x$ sketched in Figure 4.8.

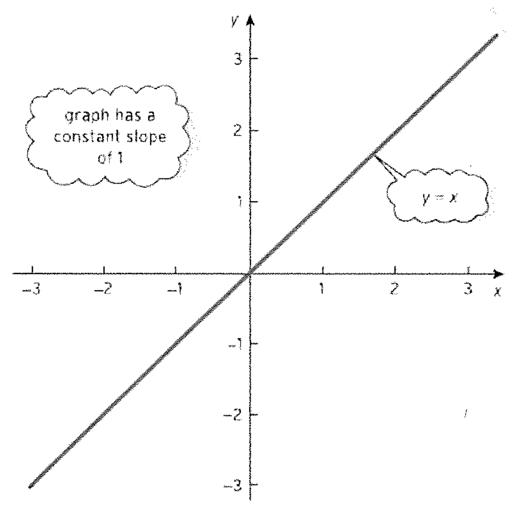

Figure 4.8

(d) Again, to differentiate 1 we need to express $y = 1$ in the form $y = x^n$. In this case $n = 0$ because $x^0 = 1$, so

$$\frac{dy}{dx} = 0x^{-1} = 0$$

This result is also obvious from the graph of $y = 1$ sketched in Figure 4.9.

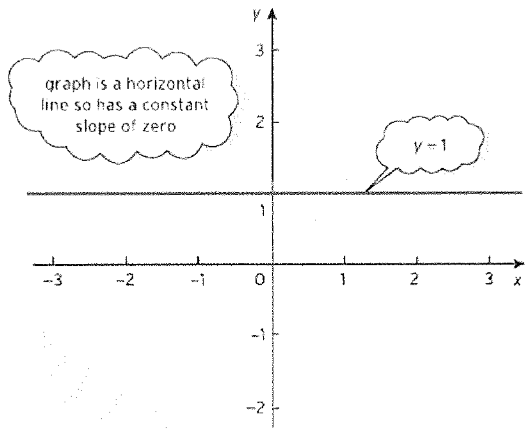

Figure 4.9

(e) Noting that $1/x^4 = x^{-4}$ it follows that

$$\text{if } y = \frac{1}{x^4} \quad \text{then} \quad \frac{dy}{dx} = -4x^{-5} = -\frac{4}{x^5}$$

The power has decreased to –5 because –4 – 1 = –5.

(f) Noting that $\sqrt{x} = x^{1/2}$ it follows that if

$$y = \sqrt{x} \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2x^{1/2}}$$ negative powers denote reciprocals

$$= \frac{1}{2\sqrt{x}}$$ fractional powers denote roots

The power has decreased to $-\frac{1}{2}$ because $\frac{1}{2} - 1 = -\frac{1}{2}$.

Practice Problem

4. Differentiate

(a) $y = x^5$ **(b)** $y = x^6$ **(c)** $y = x^{100}$ **(d)** $y = 1/x$ **(e)** $y = 1/x^2$

[Hint: in parts (d) and (e) note that $1/x = x^{-1}$ and $1/x^2 = x^{-2}$]

Key Terms

Derivative The gradient of the tangent to a curve at a point. The derivative at $x = a$ is written $f'(a)$.

Derived function The rule, f', which gives the gradient of a function, f, at a general point.

Differentiation The process or operation of determining the first derivative of a function.

Gradient The gradient of a line measures steepness and is the vertical change divided by the horizontal change between any two points on the line. The gradient of a curve at a point is that of the tangent at that point.

Slope An alternative word for gradient.

Tangent A line that just touches a curve at a point.

Exercise 4.1

1. Find the slope of the straight line passing through

(a) (2, 5) and (4, 9) **(b)** (3, −1) and (7, −5) **(c)** (7, 19) and (4, 19)

2. Verify that the points (0, 2) and (3, 0) lie on the line

$$2x + 3y = 6$$

Hence find the slope of this line. Is the line uphill, downhill or horizontal?

3. Sketch the graph of the function

$$f(x) = 5$$

Explain why it follows from this that

$$f'(x) = 0$$

4. Differentiate the function

$$f(x) = x^7$$

Hence calculate the slope of the graph of

$$y = x^7$$

at the point $x = 2$.

5. Differentiate

(a) $y = x^8$ **(b)** $y = x^{50}$ **(c)** $y = x^{19}$ **(d)** $y = x^{999}$

6. Differentiate the following functions, giving your answer in a similar form, without negative or fractional indices:

(a) $f(x) = \frac{1}{x^3}$ **(b)** $f(x) = \sqrt{x}$ **(c)** $f(x) = \frac{1}{\sqrt{x}}$ **(d)** $y = x\sqrt{x}$

7. Complete the following table of function values for the function, $f(x) = x^2 - 2x$:

x	−1	−0.5	0	0.5	1	1.5	2	2.5
$x^2 - 2x$								

Sketch the graph of this function and, by measuring the slope of the tangents, estimate

(a) $f'(-0.5)$ **(b)** $f'(1)$ **(c)** $f'(1.5)$

SECTION 4.2

Rules of differentiation

Objectives

At the end of this section you should be able to:

- Use the constant rule to differentiate a function of the form $cf(x)$.
- Use the sum rule to differentiate a function of the form $f(x) + g(x)$.
- Use the difference rule to differentiate a function of the form $f(x) - g(x)$.
- Evaluate and interpret second-order derivatives.

Advice

In this section we consider three elementary rules of differentiation. Subsequent sections of this chapter describe various applications to economics. However, before you can tackle these successfully, you must have a thorough grasp of the basic techniques involved. The problems in this section are repetitive in nature. This is deliberate. Although the rules themselves are straightforward, it is necessary for you to practise them over and over again before you can become proficient in using them. In fact, you will not be able to get much further with the rest of this book until you have mastered the rules of this section.

Rule 1 The constant rule

$$\text{If } h(x) = cf(x) \quad \text{then} \quad h'(x) = cf'(x)$$

for any constant c.

This rule tells you how to find the derivative of a constant multiple of a function:

differentiate the function and multiply by the constant

Example

Differentiate

(a) $y = 2x^4$ **(b)** $y = 10x$

Solution

(a) To differentiate $2x^4$ we first differentiate x^4 to get $4x^3$ and then multiply by 2. Hence

$$\text{if } y = 2x^4 \quad \text{then} \quad \frac{dy}{dx} = 2(4x^3) = 8x^3$$

(b) To differentiate $10x$ we first differentiate x to get 1 and then multiply by 10. Hence

$$\text{if } y = 10x \quad \text{then} \quad \frac{dy}{dx} = 10(1) = 10$$

Practice Problem

1. Differentiate

(a) $y = 4x^3$ **(b)** $y = 2/x$

The constant rule can be used to show that

constants differentiate to zero

To see this, note that the equation

$$y = c$$

is the same as

$$y = cx^0$$

because $x^0 = 1$. By the constant rule we first differentiate x^0 to get $0x^{-1}$ and then multiply by c. Hence

$$\text{if } y = c \quad \text{then} \quad \frac{dy}{dx} = c(0x^{-1}) = 0$$

This result is also apparent from the graph of $y = c$, sketched in Figure 4.10, which is a horizontal line c units away from the x axis. It is an important result and explains why lone constants lurking in mathematical expressions disappear when differentiated.

Rule 2 The sum rule

$$\text{If } h(x) = f(x) + g(x) \quad \text{then} \quad h'(x) = f'(x) + g'(x)$$

This rule tells you how to find the derivative of the sum of two functions:

differentiate each function separately and add

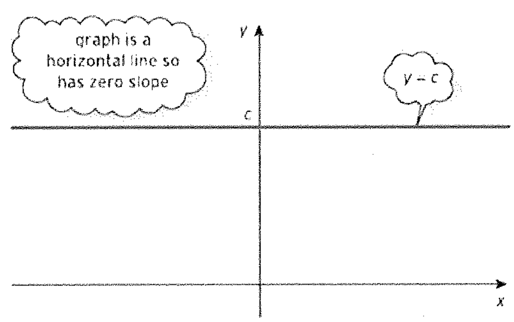

Figure 4.10

Example

Differentiate

(a) $y = x^2 + x^{50}$ **(b)** $y = x^3 + 3$

Solution

(a) To differentiate $x^2 + x^{50}$ we need to differentiate x^2 and x^{50} separately and add. Now

x^2 differentiates to $2x$

and

x^{50} differentiates to $50x^{49}$

so

$$\text{if } y = x^2 + x^{50} \quad \text{then} \quad \frac{dy}{dx} = 2x + 50x^{49}$$

(b) To differentiate $x^3 + 3$ we need to differentiate x^3 and 3 separately and add. Now

x^3 differentiates to $3x^2$

and

3 differentiates to 0

constants differentiate to zero

so

$$\text{if } y = x^3 + 3 \quad \text{then} \quad \frac{dy}{dx} = 3x^2 + 0 = 3x^2$$

Practice Problem

2. Differentiate

(a) $y = x^5 + x$ **(b)** $y = x^2 + 5$

Rule 3 The difference rule

$$\text{If } h(x) = f(x) - g(x) \quad \text{then} \quad h'(x) = f'(x) - g'(x)$$

This rule tells you how to find the derivative of the difference of two functions:

differentiate each function separately and subtract

Example

Differentiate

(a) $y = x^5 - x^2$ **(b)** $y = x - \frac{1}{x^2}$

Solution

(a) To differentiate $x^5 - x^2$ we need to differentiate x^5 and x^2 separately and subtract. Now

x^5 differentiates to $5x^4$

and

x^2 differentiates to $2x$

so

$$\text{if } y = x^5 - x^2 \quad \text{then} \quad \frac{dy}{dx} = 5x^4 - 2x$$

(b) To differentiate $x - \frac{1}{x^2}$ we need to differentiate x and $\frac{1}{x^2}$ separately and subtract. Now

x differentiates to 1

and

$\frac{1}{x^2}$ differentiates to $-\frac{2}{x^3}$

x^{-2} differentiates to $-2x^{-3}$

so

$$\text{if } y = x - \frac{1}{x^2} \quad \text{then} \quad \frac{dy}{dx} = 1 - \left(-\frac{2}{x^3}\right) = 1 + \frac{2}{x^3}$$

Practice Problem

3. Differentiate

(a) $y = x^2 - x^3$ **(b)** $y = 50 - \frac{1}{x^3}$

It is possible to combine these three rules and so to find the derivative of more involved functions, as the following example demonstrates.

Example

Differentiate

(a) $y = 3x^5 + 2x^3$ **(b)** $y = x^3 + 7x^2 - 2x + 10$ **(c)** $y = 2\sqrt{x} + \frac{3}{x}$

Solution

(a) The sum rule shows that to differentiate $3x^5 + 2x^3$ we need to differentiate $3x^5$ and $2x^3$ separately and add. By the constant rule

$3x^5$ differentiates to $3(5x^4) = 15x^4$

and

$2x^3$ differentiates to $2(3x^2) = 6x^2$

so

$$\text{if } y = 3x^5 + 2x^3 \text{ then } \frac{dy}{dx} = 15x^4 + 6x^2$$

With practice you will soon find that you can just write the derivative down in a single line of working by differentiating term by term. For the function

$y = 3x^5 + 2x^3$

we could just write

$$\frac{dy}{dx} = 3(5x^4) + 2(3x^2) = 15x^4 + 6x^2$$

(b) So far we have only considered expressions comprising at most two terms. However, the sum and difference rules still apply to lengthier expressions, so we can differentiate term by term as before. For the function

$y = x^3 + 7x^2 - 2x + 10$

we get

$$\frac{dy}{dx} = 3x^2 + 7(2x) - 2(1) + 0 = 3x^2 + 14x - 2$$

(c) To differentiate

$$y = 2\sqrt{x} + \frac{3}{x}$$

we first rewrite it using the notation of indices as

$y = 2x^{1/2} + 3x^{-1}$

Differentiating term by term then gives

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-1/2} + 3(-1)x^{-2} = x^{-1/2} - 3x^{-2}$$

which can be written in the more familiar form

$$\frac{1}{\sqrt{x}} - \frac{3}{x^2}$$

Practice Problem

4. Differentiate

(a) $y = 9x^5 + 2x^2$ **(b)** $y = 5x^8 - \dfrac{3}{x}$

(c) $y = x^2 + 6x + 3$ **(d)** $y = 2x^4 + 12x^3 - 4x^2 + 7x - 400$

Whenever a function is differentiated, the thing that you end up with is itself a function. This suggests the possibility of differentiating a second time to get the 'slope of the slope function'. This is written as

$f''(x)$ (read 'f double dashed of x')

or

$\dfrac{d^2y}{dx^2}$ (read 'dee two y by dee x squared')

For example, if

$$f(x) = 5x^2 - 7x + 12$$

then differentiating once gives

$$f'(x) = 10x - 7$$

and if we now differentiate $f'(x)$ we get

$$f''(x) = 10$$

The function $f'(x)$ is called the **first-order derivative** and $f''(x)$ is called the **second-order derivative**.

Example

Evaluate $f''(1)$ where

$$f(x) = x^7 + \frac{1}{x}$$

Solution

To find $f''(1)$ we need to differentiate

$$f(x) = x^7 + x^{-1}$$

twice and put $x = 1$ into the end result. Differentiating once gives

$$f'(x) = 7x^6 + (-1)x^{-2} = 7x^6 - x^{-2}$$

and differentiating a second time gives

$$f''(x) = 7(6x^5) - (-2)x^{-3} = 42x^5 + 2x^{-3}$$

Finally, substituting $x = 1$ into

$$f''(x) = 42x^5 + \frac{2}{x^3}$$

gives

$$f''(1) = 42 + 2 = 44$$

Practice Problem

5. Evaluate $f''(6)$ where

$$f(x) = 4x^3 - 5x^2$$

It is possible to give a graphical interpretation of the sign of the second-order derivative. Remember that the first-order derivative, $f'(x)$, measures the gradient of a curve. If the derivative of $f'(x)$ is positive (that is, if $f''(x) > 0$) then $f'(x)$ is increasing so the graph gets steeper as you move from left to right. The curve bends upwards and the function is said to be **convex**. On the other hand, if $f''(x) < 0$, the gradient, $f'(x)$ must be decreasing, so the curve bends downwards. The function is said to be **concave**. It is perfectly possible for a curve to be convex for a certain range of values of x and concave for others. This is illustrated in Figure 4.11. For this function, $f''(x) < 0$ to the left of $x = a$, and $f''(x) > 0$ to the right of $x = a$. At $x = a$ itself, the curve changes from bending downwards to bending upwards and at this point, $f''(a) = 0$.

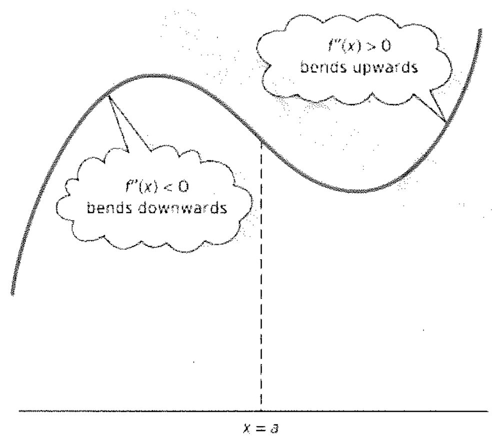

Figure 4.11

Example

Use the second-order derivative to show that the quadratic

$$y = ax^2 + bx + c$$

is always convex when $a > 0$ and concave when $a < 0$.

Solution

If $y = ax^2 + bx + c$ then $\frac{dy}{dx} = 2ax + b$ and $\frac{d^2y}{dx^2} = 2a$

If $a > 0$ then $\frac{d^2y}{dx^2} = 2a > 0$ so the parabola bends upwards.

If $a < 0$ then $\frac{d^2y}{dx^2} = 2a < 0$ so the parabola bends downwards.

Of course, if $a = 0$, the equation reduces to $y = bx + c$, which is the equation of a straight line, so the graph bends neither upwards nor downwards.

Throughout this section the functions have all been of the form $y = f(x)$, where the letters x and y denote the variables involved. In economic functions, different symbols are used. It should be obvious, however, that we can still differentiate such functions by applying the rules of this section. For example, if a supply function is given by

$$Q = P^2 + 3P + 1$$

and we need to find the derivative of Q with respect to P then we can apply the sum and difference rules to obtain

$$\frac{dQ}{dP} = 2P + 3$$

Key Terms

Concave Graph bends downwards when $f''(x) < 0$

Convex Graph bends upwards when $f''(x) > 0$

First-order derivative The rate of change of a function with respect to its independent variable. It is the same as the 'derivative' of a function, $y = f(x)$, and is written as $f'(x)$ or dy/dx.

Second-order derivative The derivative of the first-order derivative. The expression obtained when the original function, $y = f(x)$, is differentiated twice in succession and is written as $f''(x)$ or d^2y/dx^2.

Exercise 4.2

1. Differentiate

(a) $y = 5x^2$ **(b)** $y = \frac{3}{x}$ **(c)** $y = 2x + 3$

(d) $y = x^2 + x + 1$ **(e)** $y = x^2 - 3x + 2$ **(f)** $y = 3x - \frac{7}{x}$

(g) $y = 2x^3 - 6x^2 + 49x - 54$ **(h)** $y = ax + b$ **(i)** $y = ax^2 + bx + c$

(j) $y = 4x - \frac{3}{x} + \frac{7}{x^2}$

2. Evaluate $f'(x)$ for each of the following functions at the given point:

(a) $f(x) = 3x^9$ at $x = 1$

(b) $f(x) = x^2 - 2x$ at $x = 3$

(c) $f(x) = x^3 - 4x^2 + 2x - 8$ at $x = 0$

(d) $f(x) = 5x^4 - \frac{4}{x^4}$ at $x = -1$

(e) $f(x) = \sqrt{x} - \frac{2}{x}$ at $x = 4$

3. By writing $x^2\left(x^2 + 2x - \frac{5}{x^2}\right) = x^4 + 2x^3 - 5$ differentiate $x^2\left(x^2 + 2x - \frac{5}{x^2}\right)$.
Use a similar approach to differentiate

(a) $x^2(3x - 4)$

(b) $x(3x^3 - 2x^2 + 6x - 7)$

(c) $(x + 1)(x - 6)$

(d) $\frac{x^2 - 3}{x}$

(e) $\frac{x - 4x^2}{x^3}$

(f) $\frac{x^2 - 3x + 5}{x^2}$

4. Find expressions for d^2y/dx^2 in the case when

(a) $y = 7x^2 - x$

(b) $y = \frac{1}{x^2}$

(c) $y = ax + b$

5. Evaluate $f''(2)$ for the function

$$f(x) = x^3 - 4x^2 + 10x - 7$$

6. If $f(x) = x^2 - 6x + 8$, evaluate $f'(3)$. What information does this provide about the graph of $y = f(x)$ at $x = 3$?

7. By writing $\sqrt{4x} = \sqrt{4} \times \sqrt{x} = 2\sqrt{x}$, differentiate $\sqrt{4x}$.
Use a similar approach to differentiate

(a) $\sqrt{25x}$ (b) $\sqrt[3]{27x}$ (c) $\sqrt[4]{16x^3}$ (d) $\sqrt{\frac{25}{x}}$

8. Find expressions for

(a) $\frac{dQ}{dP}$ for the supply function $Q = P^2 + P + 1$

(b) $\frac{d(TR)}{dQ}$ for the total revenue function $TR = 50Q - 3Q^2$

(c) $\frac{d(AC)}{dQ}$ for the average cost function $AC = \frac{30}{Q} + 10$

(d) $\frac{dC}{dY}$ for the consumption function $C = 3Y + 7$

(e) $\frac{dQ}{dL}$ for the production function $Q = 10\sqrt{L}$

(f) $\frac{d\pi}{dQ}$ for the profit function $\pi = -2Q^3 + 15Q^2 - 24Q - 3$

Exercise 4.2*

1. Find the value of the first-order derivative of the function

$$y = 3\sqrt{x} - \frac{81}{x} + 13$$

when $x = 9$.

2. Find expressions for

(a) $\frac{dQ}{dP}$ for the supply function $Q = 2P^2 + P + 1$

(b) $\frac{d(TR)}{dQ}$ for the total revenue function $TR = 40Q - 3Q\sqrt{Q}$

(c) $\frac{d(AC)}{dQ}$ for the average cost function $AC = \frac{20}{Q} + 7Q + 25$

(d) $\frac{dC}{dY}$ for the consumption function $C = Y(2Y + 3) + 10$

(e) $\frac{dC}{dL}$ for the production function $Q = 200L - 4\sqrt[3]{L}$

(f) $\frac{d\pi}{dQ}$ for the profit function $\pi = -Q^3 + 20Q^2 - 7Q - 1$

3. Find the value of the second-order derivative of the following function at the point $x = 4$:

$$f(x) = -2x^3 + 4x^2 + x - 3$$

What information does this provide about the shape of the graph of $f(x)$ at this point?

4. Consider the graph of the function

$$f(x) = 2x^5 - 3x^4 + 2x^2 - 17x + 31$$

at $x = -1$.

Giving reasons for your answers,

(a) state whether the tangent slopes uphill, downhill or is horizontal

(b) state whether the graph is concave or convex at this point.

5. Use the second-order derivative to show that the graph of the cubic,

$$f(x) = ax^3 + bx^2 + cx + d \ (a > 0)$$

is convex when $x > -b/3a$ and concave when $x = -b/3a$.

6. Find the equation of the tangent to the curve

$$y = 4x^3 - 5x^2 + x - 3$$

at the point where it crosses the y axis.

SECTION 4.3
Marginal functions

Objectives

At the end of this section you should be able to:

- Calculate marginal revenue and marginal cost.
- Derive the relationship between marginal and average revenue for both a monopoly and perfect competition.
- Calculate marginal product of labour.
- State the law of diminishing marginal productivity using the notation of calculus.
- Calculate marginal propensity to consume and marginal propensity to save.

At this stage you may be wondering what on earth differentiation has got to do with economics. In fact, we cannot get very far with economic theory without making use of calculus. In this section we concentrate on three main areas that illustrate its applicability:

- revenue and cost
- production
- consumption and savings.

We consider each of these in turn.

4.3.1 Revenue and cost

In Chapter 2 we investigated the basic properties of the revenue function, TR. It is defined to be PQ, where P denotes the price of a good and Q denotes the quantity demanded. In practice, we usually know the demand equation, which provides a relationship between P and Q. This enables a formula for TR to be written down solely in terms of Q. For example, if

$$P = 100 - 2Q$$

then

$$\text{TR} = PQ = (100 - 2Q)Q = 100Q - 2Q^2$$

The formula can be used to calculate the value of TR corresponding to any value of Q. Not content with this, we are also interested in the effect on TR of a change in the value of Q from some existing level. To do this we introduce the concept of marginal revenue. The **marginal revenue**, MR, of a good is defined by

$$\text{MR} = \frac{\text{d(TR)}}{\text{d}Q}$$

marginal revenue is the derivative of total revenue with respect to demand

For example, the marginal revenue function corresponding to

$$\text{TR} = 100Q - 2Q^2$$

is given by

$$\frac{\text{d(TR)}}{\text{d}Q} = 100 - 4Q$$

If the current demand is 15, say, then

$$\text{MR} = 100 - 4(15) = 40$$

You may be familiar with an alternative definition often quoted in elementary economics textbooks. Marginal revenue is sometimes taken to be the change in TR brought about by a 1 unit increase in Q. It is easy to check that this gives an acceptable approximation to MR, although it is not quite the same as the exact value obtained by differentiation. For example, substituting $Q = 15$ into the total revenue function considered previously gives

$$\text{TR} = 100(15) - 2(15)^2 = 1050$$

An increase of 1 unit in the value of Q produces a total revenue

$$\text{TR} = 100(16) - 2(16)^2 = 1088$$

This is an increase of 38, which, according to the non-calculus definition, is the value of MR when Q is 15. This compares with the exact value of 40 obtained by differentiation.

It is instructive to give a graphical interpretation of these two approaches. In Figure 4.12 the point A lies on the TR curve corresponding to a quantity Q_0. The exact value of MR at this point is equal to the derivative

$$\frac{\text{d(TR)}}{\text{d}Q}$$

and so is given by the slope of the tangent at A. The point B also lies on the curve but corresponds to a 1 unit increase in Q. The vertical distance from A to B therefore equals the change in TR when Q increases by 1 unit. The slope of the line joining A and B (known as a **chord**) is

$$\frac{\Delta(\text{TR})}{\Delta Q} = \frac{\Delta(\text{TR})}{1} = \Delta(\text{TR})$$

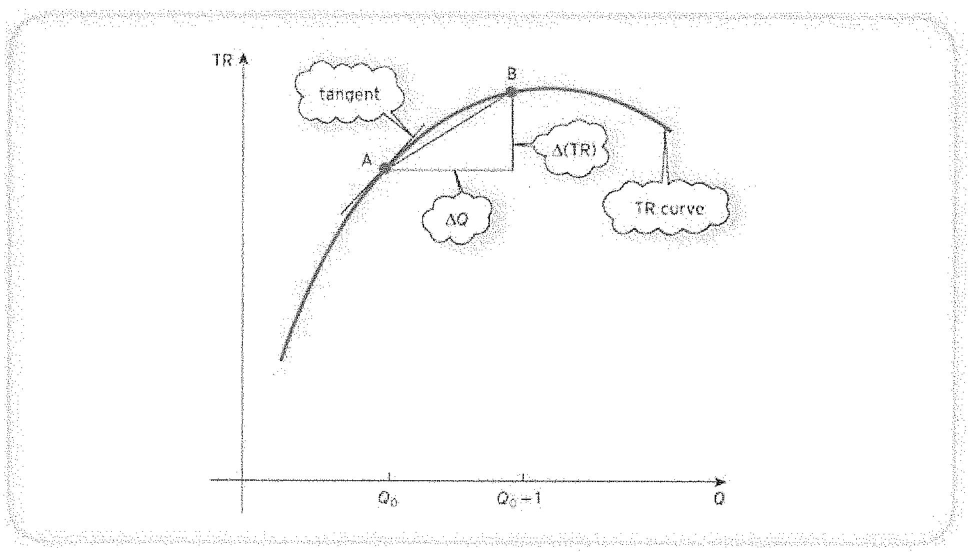

Figure 4.12

In other words, the slope of the chord is equal to the value of MR obtained from the non-calculus definition. Inspection of the diagram reveals that the slope of the tangent is approximately the same as that of the chord joining A and B. In this case the slope of the tangent is slightly the larger of the two, but there is not much in it. We therefore see that the 1 unit increase approach produces a reasonable approximation to the exact value of MR given by

$$\frac{d(TR)}{dQ}$$

Example

If the demand function is

$$P = 120 - 3Q$$

find an expression for TR in terms of Q.

Find the value of MR at $Q = 10$ using

(a) differentiation

(b) the 1 unit increase approach

Solution

$$TR = PQ = (120 - 3Q)Q = 120Q - 3Q^2$$

(a) The general expression for MR is given by

$$\frac{d(TR)}{dQ} = 120 - 6Q$$

so at $Q = 10$,

$$MR = 120 - 6 \times 10 = 60$$

(b) From the non-calculus definition we need to find the change in TR as Q increases from 10 to 11.

Putting $Q = 10$ gives $TR = 120 \times 10 - 3 \times 10^2 = 900$

Putting $Q = 11$ gives $TR = 120 \times 11 - 3 \times 11^2 = 957$

and so $MR = 57$

Practice Problem

1. If the demand function is

$$P = 60 - Q$$

find an expression for TR in terms of Q.

(1) Differentiate TR with respect to Q to find a general expression for MR in terms of Q. Hence write down the exact value of MR at $Q = 50$.

(2) Calculate the value of TR when

(a) $Q = 50$ **(b)** $Q = 51$

and hence confirm that the 1 unit increase approach gives a reasonable approximation to the exact value of MR obtained in part (1).

The approximation indicated by Figure 4.12 holds for any value of ΔQ. The slope of the tangent at A is the marginal revenue, MR. The slope of the chord joining A and B is $\Delta(\mathrm{TR})/\Delta Q$. It follows that

$$\mathrm{MR} \cong \frac{\Delta(\mathrm{TR})}{\Delta Q}$$

This equation can be transposed to give

$$\Delta(\mathrm{TR}) \cong MR \times \Delta Q$$

multiply both sides by ΔQ

that is,

$$\boxed{\text{change in total revenue}} \cong \boxed{\text{marginal revenue}} \times \boxed{\text{change in demand}}$$

Moreover, Figure 4.12 shows that the smaller the value of ΔQ, the better the approximation becomes.

Example

If the total revenue function of a good is given by

$$100Q - Q^2$$

write down an expression for the marginal revenue function. If the current demand is 60, estimate the change in the value of TR due to a 2 unit increase in Q.

Solution

If

$$\mathrm{TR} = 100Q - Q^2$$

then

$$\mathrm{MR} = \frac{\mathrm{d(TR)}}{\mathrm{d}Q} = 100 - 2Q$$

When $Q = 60$

$$\mathrm{MR} = 100 - 2(60) = -20$$

If Q increases by 2 units, $\Delta Q = 2$ and the formula

$$\Delta(\mathrm{TR}) \cong \mathrm{MR} \times \Delta Q$$

shows that the change in total revenue is approximately

$$(-20) \times 2 = -40$$

A 2 unit increase in Q therefore leads to a decrease in TR of about 40.

Practice Problem

2. If the total revenue function of a good is given by

$$1000Q - 4Q^2$$

write down an expression for the marginal revenue function. If the current demand is 30, find the approximate change in the value of TR due to a

(a) 3 unit increase in Q

(b) 2 unit decrease in Q.

The simple model of demand, originally introduced in Section 1.5, assumed that price, P, and quantity, Q, are linearly related according to an equation

$$P = aQ + b$$

where the slope, a, is negative and the intercept, b, is positive. A downward-sloping demand curve such as this corresponds to the case of a **monopolist**. A single firm, or possibly a group of firms forming a cartel, is assumed to be the only supplier of a particular product and so has control over the market price. As the firm raises the price, so demand falls. The associated total revenue function is given by

$$\begin{aligned}\text{TR} &= PQ \\ &= (aQ + b)Q \\ &= aQ^2 + bQ\end{aligned}$$

An expression for marginal revenue is obtained by differentiating TR with respect to Q to get

$$\text{MR} = 2aQ + b$$

It is interesting to notice that, on the assumption of a linear demand equation, the marginal revenue is also linear with the same intercept, b, but with slope $2a$. The marginal revenue curve slopes downhill exactly twice as fast as the demand curve. This is illustrated in Figure 4.13(a).

The **average revenue**, AR, is defined by

$$\text{AR} = \frac{\text{TR}}{Q}$$

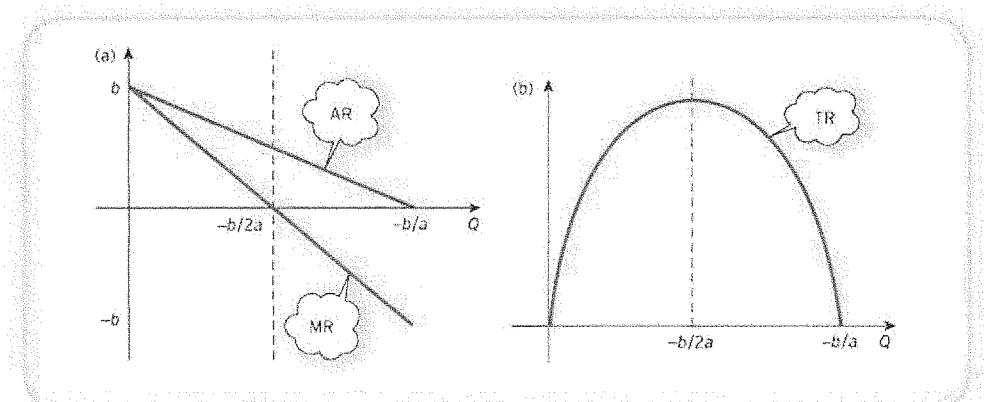

Figure 4.13

and, since TR = PQ, we have

$$AR = \frac{PQ}{Q} = P$$

For this reason the demand curve is labelled average revenue in Figure 4.13(a). The above derivation of the result AR = P is independent of the particular demand function. Consequently, the terms 'average revenue curve' and 'demand curve' are synonymous.

Figure 4.13(a) shows that the marginal revenue takes both positive and negative values. This is to be expected. The total revenue function is a quadratic and its graph has the familiar parabolic shape indicated in Figure 4.13(b). To the left of $-b/2a$ the graph is uphill, corresponding to a positive value of marginal revenue, whereas to the right of this point it is downhill, giving a negative value of marginal revenue. More significantly, at the maximum point of the TR curve, the tangent is horizontal with zero slope and so MR is zero.

At the other extreme from a monopolist is the case of **perfect competition**. For this model we assume that there are a large number of firms all selling an identical product and that there are no barriers to entry into the industry. Since any individual firm produces a tiny proportion of the total output, it has no control over price. The firm can sell only at the prevailing market price and, because the firm is relatively small, it can sell any number of goods at this price. If the fixed price is denoted by b then the demand function is

$$P = b$$

and the associated total revenue function is

$$TR = PQ = bQ$$

An expression for marginal revenue is obtained by differentiating TR with respect to Q and, since b is just a constant, we see that

$$MR = b$$

In the case of perfect competition, the average and marginal revenue curves are the same. They are horizontal straight lines, b units above the Q axis as shown in Figure 4.14.

So far we have concentrated on the total revenue function. Exactly the same principle can be used for other economic functions. For instance, we define the **marginal cost**, MC, by

$$MC = \frac{d(TC)}{dQ}$$

marginal cost is the derivative of total cost with respect to output

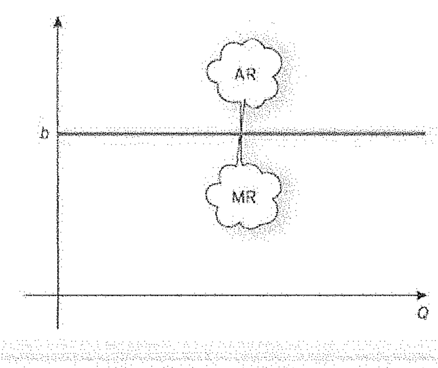

Figure 4.14

Again, using a simple geometrical argument, it is easy to see that if Q changes by a small amount ΔQ then the corresponding change in TC is given by

$$\Delta(\text{TC}) \cong \text{MC} \times \Delta Q$$

$$\boxed{\text{change in total cost}} \cong \boxed{\text{marginal cost}} \times \boxed{\text{change in output}}$$

In particular, putting $\Delta Q = 1$ gives

$$\Delta(\text{TC}) \cong \text{MC}$$

so that MC gives the approximate change in TC when Q increases by 1 unit.

Example

If the average cost function of a good is

$$\text{AC} = 2Q + 6 + \frac{13}{Q}$$

find an expression for MC. If the current output is 15, estimate the effect on TC of a 3 unit decrease in Q.

Solution

We first need to find an expression for TC using the given formula for AC. Now we know that the average cost is just the total cost divided by Q: that is,

$$\text{AC} = \frac{\text{TC}}{Q}$$

Hence

$$\begin{aligned}\text{TC} &= (\text{AC})Q \\ &= \left(2Q + 6 + \frac{13}{Q}\right)Q\end{aligned}$$

and, after multiplying out the brackets, we get

$$\text{TC} = 2Q^2 + 6Q + 13$$

In this formula the last term, 13, is independent of Q so must denote the fixed costs. The remaining part, $2Q^2 + 6Q$, depends on Q so represents the total variable costs. Differentiating gives

$$\begin{aligned}\text{MC} &= \frac{\text{d(TC)}}{\text{d}Q} \\ &= 4Q + 6\end{aligned}$$

Notice that because the fixed costs are constant they differentiate to zero and so have no effect on the marginal cost. When $Q = 15$,

$$\text{MC} = 4(15) + 6 = 66$$

Also, if Q decreases by 3 units then $\Delta Q = -3$. Hence the change in TC is given by

$$\Delta(\text{TC}) \cong \text{MC} \times \Delta Q = 66 \times (-3) = -198$$

so TC decreases by 198 units approximately.

Practice Problem

3. Find the marginal cost given the average cost function

$$AC = \frac{100}{Q} + 2$$

Deduce that a 1 unit increase in Q will always result in a 2 unit increase in TC, irrespective of the current level of output.

4.3.2 Production

Production functions were introduced in Section 2.3. In the simplest case output, Q, is assumed to be a function of labour, L, and capital, K. Moreover, in the short run the input K can be assumed to be fixed, so Q is then only a function of one input L. (This is not a valid assumption in the long run and in general Q must be regarded as a function of at least two inputs. Methods for handling this situation are considered in the next chapter.) The variable L is usually measured in terms of the number of workers or possibly in terms of the number of worker hours. Motivated by our previous work, we define the **marginal product of labour**, MP_L, by

$$MP_L = \frac{dQ}{dL}$$

marginal product of labour is the derivative of output with respect to labour

As before, this gives the approximate change in Q that results from using 1 more unit of L.

Example

If the production function is

$$Q = 300\sqrt{L} - 4L$$

where Q denotes output and L denotes the size of the workforce, calculate the value of MP_L when

(a) $L = 1$

(b) $L = 9$

(c) $L = 100$

(d) $L = 2500$

and discuss the implications of these results.

Solution

If

$$Q = 300\sqrt{L} - 4L = 300L^{1/2} - 4L$$

then

$$\begin{aligned} MP_L &= \frac{dQ}{dL} \\ &= 300(\tfrac{1}{2}L^{-1/2}) - 4 \\ &= 150L^{-1/2} - 4 \\ &= \frac{150}{\sqrt{L}} - 4 \end{aligned}$$

(a) When $L = 1$

$$MP_L = \frac{150}{\sqrt{9}} - 4 = 146$$

(b) When $L = 9$

$$MP_L = \frac{150}{\sqrt{1}} - 4 = 46$$

(c) When $L = 100$

$$MP_L = \frac{150}{\sqrt{100}} - 4 = 11$$

(d) When $L = 2500$

$$MP_L = \frac{150}{\sqrt{2500}} - 4 = -1$$

Notice that the values of MP_L decline with increasing L. Part (a) shows that if the workforce consists of only one person then to employ two people would increase output by approximately 146. In part (b) we see that to increase the number of workers from 9 to 10 would result in about 46 additional units of output. In part (c) we see that a 1 unit increase in labour from a level of 100 increases output by only 11. In part (d) the situation is even worse. This indicates that to increase staff actually reduces output! The latter is a rather surprising result, but it is borne out by what occurs in real production processes. This may be due to problems of overcrowding on the shopfloor or to the need to create an elaborate administration to organise the larger workforce.

This example illustrates the **law of diminishing marginal productivity** (sometimes called the **law of diminishing returns**). It states that the increase in output due to a 1 unit increase in labour will eventually decline. In other words, once the size of the workforce has reached a certain threshold level, the marginal product of labour will get smaller. In the previous example, the value of MP_L continually goes down with rising L. This is not always so. It is possible for the marginal product of labour to remain constant or to go up to begin with for small values of L. However, if it is to satisfy the law of diminishing marginal productivity then there must be some value of L above which MP_L decreases.

A typical product curve is sketched in Figure 4.15, which has slope

$$\frac{dQ}{dL} = MP_L$$

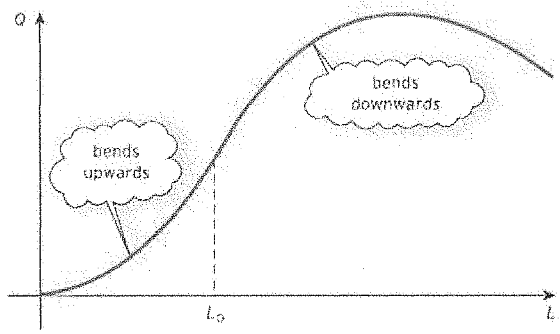

Figure 4.15

Between 0 and L_0 the curve bends upwards, becoming progressively steeper, and so the slope function, MP_L, increases. Mathematically, this means that the slope of MP_L is positive: that is,

$$\frac{d(MP_L)}{dQ} > 0$$

Now MP_L is itself the derivative of Q with respect to L, so we can use the notation for the second order derivative and write this as

$$\frac{d^2Q}{dL^2} > 0$$

Similarly, if L exceeds the threshold value of L_0, then Figure 4.15 shows that the product curve bends downwards and the slope decreases. In this region, the slope of the slope function is negative, so that

$$\frac{d^2Q}{dL^2} < 0$$

The law of diminishing returns states that this must happen eventually: that is,

$$\frac{d^2Q}{dL^2} < 0$$

for sufficiently large L.

Practice Problem

4. A Cobb–Douglas production function is given by

$$Q = 5L^{1/2}K^{1/2}$$

Assuming that capital, K, is fixed at 100, write down a formula for Q in terms of L only. Calculate the marginal product of labour when

(a) $L = 1$ **(b)** $L = 9$ **(c)** $L = 10\ 000$

Verify that the law of diminishing marginal productivity holds in this case.