My Favorite Two Theorems in Ramsey Theory

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Chapter 1

Van der Waerden's Theorem

1.1 Introduction

Traditionally, Baudet is credited with the following conjecture:

For any partition of the natural numbers into two sets, one of the sets will have arbitrarily long arithmetic progressions.

Van der Waerden's paper [6] is titled *Beweis einer Baudetschen Vermutung*, which translates as *Proof of a Conjecture of Baudet*; hence, van der Waerden thought he was solving a conjecture of Baudet. However, Soifer [5] gives compelling evidence that Baudet and Schur deserve joint credit for this conjecture.

As for who proved the conjecture there is no controversy: van der Waerden proved it [6]. The proof we give is essentially his. He has written an account of how he came up with the proof [7] which is reprinted in Soifer's book.

VDW is more general than Baudet's conjecture. VDW guarantees long APs within finite rather than infinite sets of natural numbers, and allows for the natural numbers to be divided up into any finite number of sets (by color) instead of just two.

In this chapter, we will prove VDW the same way van der Waerden did. We will express the proof in the color-focusing language of Walters [8].

Van der Waerden's Theorem: For all $k, c \in \mathbb{N}$ there exists W such that, for all c-colorings $\chi{:}[W] \to [c]$, there exists $a, d \in \mathbb{N}$ such that

$$\chi(a) = \chi(a + d) = \chi(a + 2d) = \cdots = \chi(a + (k - 1)d).$$

Def 1.1.1 Let $k, c \in \mathbb{N}$. The *van der Waerden number* $W(k,c)$ is the least number W that satisfies van der Waerden's Theorem with parameters k, c.

Before proving the theorem, let's look at a few simple base cases.

- $c = 1$: $W(k,1) = k$, because the sequence $1, 2, \ldots, k$ forms a k-AP.
- $k = 1$: $W(1,c) = 1$, because a 1-AP is any single term.
- $k = 2$: $W(2,c) = c + 1$, because any two numbers form a 2-AP.

Not bad— we have proven the theorem for an infinite number of cases. How many more could there be?

Notation 1.1.2 We use VDW(k,c) to mean the statement of VDW with the parameters k and c. Note that the two statements VDW(k,c) *holds* and $W(k,c)$ *exists* are equivalent.

The proof has three key ideas. We prove subcases that illustrate these ideas before proving the full theorem itself.

1.2 Proof of van der Waerden's Theorem

1.2.1 VDW$(3,2)$ and the first key idea

We show that there exists a W such that any 2-coloring of $[W]$ has a monochromatic 3-AP. By enumeration one can show $W(3,2) = 9$; however, we prefer to use a technique that generalizes to other values of k and c. The proof will show $W(3,2) \leq 325$.

For this section let $W \in \mathbb{N}$ and let $\chi : [W] \to \{R, B\}$ (where W will be determined later). Imagine breaking up the numbers $\{1, 2, 3, \ldots, W\}$ into blocks of five. We can assume W is divisible by 5.

$$\{1,2,3,4,5\}, \{6,7,8,9,10\}, \ldots, \{W-4, W-3, W-2, W-1, W\}$$

Let B_i be the ith block. Consider what happens *within* a block. Clearly for any block of five there must be three equally spaced elements for which the first two are the same color. We state this formally so we can refer to it.

Fact 1.2.1 *Let B be a block of five elements. If χ is restricted to B. then there exists a, d, $d \neq 0$, such that*

$$a, a+d, a+2d \in B$$

and

$$\chi(a) = \chi(a+d).$$

We need to view $\chi:[W] \rightarrow \{R, B\}$ differently. The mapping χ can be viewed as assigning to each block one of the 2^5 possible colorings of five numbers: $RRRRR$, $RRRRB$, …, $BBBBB$. This is…

The First Key Idea: We view χ as a 32-coloring of the blocks. The following viewpoint will be used over and over again in this book: View a c-coloring of $[W]$ as a c^B coloring of (W/B) blocks of size B.

The following is clear from the pigeonhole principle.

Lemma 1.2.2 *Assume $W \geq 5 \cdot 33 = 165$. There exists two blocks B_i and B_j ($1 \leq i < j \leq 33$) with the same coloring.*

Theorem 1.2.3 *Let $W \geq 325$. Let $\chi:[W] \rightarrow [2]$ be a 2-coloring of $[W]$. Then there exists $a, d \in \mathbb{N}$ such that*

$$\chi(a) = \chi(a+d) = \chi(a+2d).$$

Proof: Let the colors be RED and BLUE. Assume, by way of contradiction, that there is no monochromatic 3-AP. View $[W]$ as being 65 blocks of five. By Lemma 1.2.2 there exists two blocks B_i, B_j ($1 \leq i < j \leq 33$) with the same coloring. By Fact 1.2.1, within B_i, there exists a, d such that $\chi(a) = \chi(a+d)$. Since B_i and B_j are the same color and are D apart we have that there exists a, d, D such that, up to recoloring, the following holds.

- $\chi(a) = \chi(a+d) = \chi(a+D) = \chi(a+D+d) =$ RED.
- $\chi(a+2d) = \chi(a+D+2d) =$ BLUE.
- $a + 2D + 2d \in [W]$.

Figure 1.1 represents the situation.

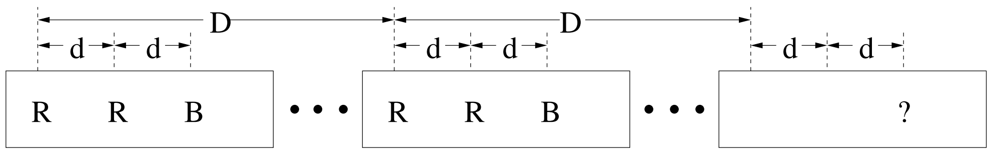

Figure 1.1: Three 5-Blocks

If $\chi(a+2D+2d) = \text{BLUE}$ then

$$\chi(a+2d) = \chi(a+D+2d) = \chi(a+2D+2d) = \text{BLUE}.$$

If $\chi(a+2D+2d) = \text{RED}$ then

$$\chi(a) = \chi(a+(D+d)) = \chi(a+2(D+d)) = \text{RED}.$$

In either case we get a monochromatic 3-AP, a contradiction. ▮

Exercise 1

1. How many 2-colorings of a 5-block are there that do not have a monochromatic 3-AP? Use the answer to obtain a smaller upper bound on $W(3,2)$ in the proof of Theorem 1.2.3.

2. Use 3-blocks instead of 5-blocks in a proof similar to that of Theorem 1.2.3 to obtain a smaller upper bound on $W(3,2)$ in the proof of Theorem 1.2.3.

3. Show that $W(3,2) = 9$. (Hint: Do not use anything like Theorem 1.2.3.)

4. Find all 3-colorings of $[8]$ that do not have a monochromatic 3-AP.

5. For $n = 10, 11, \ldots$ try to find a 3-coloring of $[n]$ that has no monochromatic 3-AP's by doing the following which is called the Greedy method. (By VDW there will be an n such that this is impossible.)

   - Color the numbers in order.

- If you can color a number RED without forming a monochromatic 3-AP, do so.
- If not, then if you can color that number BLUE without forming a monochromatic 3-AP, do so.
- If not, then if you can color that number GREEN without forming a monochromatic 3-AP, do so.
- If every color would form a monochromatic 3-AP, then stop.

6. (Open-ended) For $n = 10, 11, \ldots$, try to find a 3-coloring of $[n]$ that has no monochromatic 3-AP's. (By VDW there will be an n such that this is impossible.) Do this by whatever means necessary. Get as large an n as you can. Be all you can be!

1.2.2 $\mathrm{VDW}(3, 3)$ and the second key idea

We show that there exists a W such that any 3-coloring of $[W]$ has a monochromatic 3-AP. It is known, using a computer program, that $W(3, 3) = 27$ [2] We use a technique that generalizes to other values of k and c, but does not attain the exact bound.

For this section let $W \in \mathbb{N}$ and let $\chi\colon[W] \to \{R, B, G\}$ (where W is to be determined later). Imagine breaking up the numbers $\{1, 2, 3, \ldots, W\}$ into blocks of seven. We can assume W is divisible by 7.

$$\{1,2,3,4,5,6,7\},\{8,9,10,11,12,13,14\},\cdots,\{W-6,W-5,W-4,W-3,W-2,W-1,W\}$$

By techniques similar to those used in Section 1.2.1 we obtain that there is some number U such that, for all 3-colorings of $[U]$, up to recoloring, there exists a, d, D such that

- $\chi(a) = \chi(a + d) = \chi(a + D) = \chi(a + D + d) = \text{RED}$.
- $\chi(a + 2d) = \chi(a + D + 2d) = \text{BLUE}$.
- $a + 2D + 2d \in [W]$.

Figure 1.2 represents the situation.

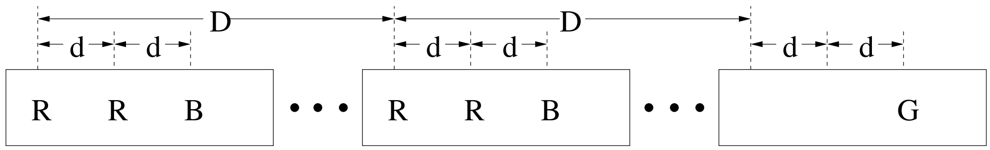

Figure 1.2: Three 7-Blocks

If $\chi(a+2D+2d) = $ BLUE then

$$\chi(a+2d) = \chi(a+D+2d) = \chi(a+2D+2d) = \text{BLUE}.$$

If $\chi(a+2D+2d) = $ RED then

$$\chi(a) = \chi(a+(D+d)) = \chi(a+2(D+d)) = \text{RED}.$$

Unfortunately all we can conclude is that $\chi(a+2D+2d) = $ GREEN.
We have sketched a proof of the following:

Lemma 1.2.4 *There exists U such that, up to recoloring, for all 3-colorings of $[U]$ one of the following must occur.*

1. *There exists a monochromatic 3-AP.*
2. *There exists two 3-AP's such that*
   - *One is colored* RED − RED − GREEN.
   - *One is colored* BLUE − BLUE − GREEN.
   - *They have the same third point.*

Let U be as in Lemma 1.2.4. Imagine breaking up the numbers $\{1, 2, 3, \ldots, W\}$ into blocks of U (we can assume W is divisible by U).

The Second Key Idea: We now take $[U]$ to be our block. We view $[W]$ as a sequence of blocks, each of length U. This viewpoint will be used over and over again in this book. First divide $[W]$ into blocks, then later take a block of blocks, and then a block of blocks of blocks, etc.

We resume our discussion. View the 3-coloring of $[W]$ as a 3^U coloring of the blocks. Take W large enough so that there are two blocks B_i, B_j that are the same color and a third block B_k such that B_i, B_j, B_k form an arithmetic progression of blocks.

Figure 1.3 represents the situation.

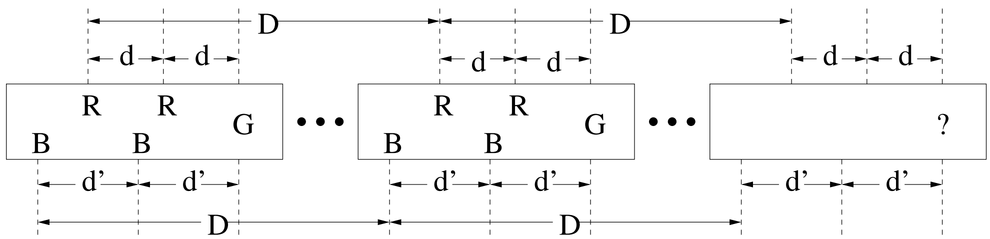

Figure 1.3: Three U-Blocks

We leave the formal proof of Lemma 1.2.4 and the proof of VDW$(3,3)$ to the reader.

Exercise 2

1. Use the ideas in this section to produce a rigorous proof of VDW$(3,3)$. Obtain an actual number that bounds $W(3,3)$.
2. Use the ideas in this section to produce a rigorous proof of VDW$(3,4)$.
3. Use the ideas in this section to produce a rigorous proof that, for all c, $vdw(3,c)$ holds.

1.2.3 VDW (4,2): and the third key idea

We show that there exists a W such that any 2-coloring of $[W]$ has a monochromatic 4-AP. It is known, using a computer program, that $W(4,2) = 35$ [2] We use a technique that generalizes to other values of k and c, but does not attain the exact bound.

For this section let $W \in \mathbb{N}$ and let $\chi{:}[W] \to \{R,B\}$ (W to be determined later). Imagine breaking up the numbers $\{1,2,3,\ldots,W\}$ into blocks of length $2W(3,2)$ (we can assume W is divisible by $2W(3,2)$).

$$\{1,2,3,4,5,\ldots,2W(3,2)\}, \{2W(3,2)+1,\ldots,4W(3,2)\}, \{4W(3,2)+1,\ldots,6W(3,2)\}, \cdots$$

We will use VDW$(3,c)$ for rather large values of c to prove VDW$(4,2)$. This is. . .

The Third Key Idea: To prove VDW$(k,2)$ we will use VDW$(k-1,c)$ for an enormous value of c. Formally, this is an ω^2 induction. We will discuss and use inductions on complicated orderings later in this book.

We leave the following easy lemma to the reader.

Lemma 1.2.5 *Let $c \in \mathbb{N}$. Let χ:$[2W(3,c)] \to [c]$. There exists $a, d \in \mathbb{N}$ such that*

- $\chi(a) = \chi(a+d) = \chi(a+2d)$, *and*
- $a + 3d \in [2W(3,c)]$ *so* $\chi(a+3d)$ *is defined, though we make no claims of its value.*

Theorem 1.2.6 *Let $W \geq 4W(3,2) \times W(3, 2^{2W(3,2)})$. Let χ:$[W] \to [2]$ be a 2-coloring of $[W]$. Then there exists $a, d \in \mathbb{N}$ such that*

$$\chi(a) = \chi(a+d) = \chi(a+2d) = \chi(a+3d).$$

Proof: Let the colors be RED and BLUE. Assume, by way of contradiction, that there is no monochromatic 4-AP. View $[W]$ as being $2W(3, 2^{2W(3,2)})$ blocks of size $2W(3,2)$. We view the 2-coloring of $[W]$ as a $2^{2W(3,2)}$-coloring of the blocks. We will use VDW$(3, 2^{2W(3,2)})$ on the block-coloring and VDW$(3,2)$ on the coloring of each block. By Lemma 1.2.5 applied to *both* the coloring of the blocks and the coloring within a block, and symmetry, we have the following: There exists $a, d, D \in \mathbb{N}$ such that

- $\chi(a) = \chi(a+d) = \chi(a+2d) =$ RED,
  $\chi(a+D) = \chi(a+D+d) = \chi(a+D+2d) =$ RED,
  $\chi(a+2D) = \chi(a+2D+d) = \chi(a+2D+2d) =$ RED.
- $\chi(a+3d) = \chi(a+D+3d) = \chi(a+2D+3d) =$ BLUE.
- $a + 3D + 3d \in [W]$.

Figure 1.4 represents the situation.

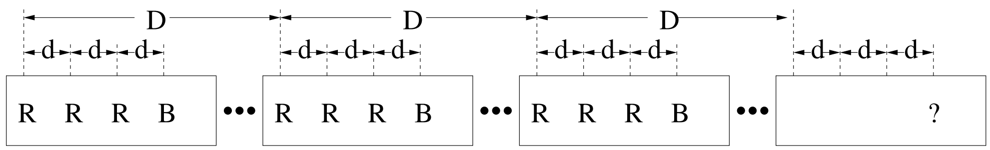

Figure 1.4: Four $2W(3,2)$-Blocks

If $\chi(a+3D+3d) = \text{BLUE}$ then

$$\chi(a+3d) = \chi(a+3d+D) = \chi(a+2D+3d) = \chi(a+3D+3d) = \text{BLUE}.$$

If $\chi(a+3D+3d) = \text{RED}$ then

$$\chi(a) = \chi(a+(D+d)) = \chi(a+2(D+d)) = \chi(a+3(D+d)) = \text{RED}.$$

In either case we get a monochromatic 4-AP, a contradiction. ▮

Exercise 3

1. Use the proof of Theorem 1.2.6 to obtain an actual bound on $W(4,2)$.
2. Fix k. Assume that, for all c, VDW$(k-1,c)$ is true. Prove VDW$(k,2)$.
3. Fix k. Assume that, for all c, VDW$(k-1,c)$ is true. Prove VDW$(k,3)$.
4. Prove the full VDW.

1.2.4 The full proof

Now that you know the Key Ideas you have all of the intuitions for the proof. We formalize them here. The method we use here, color focusing, will occur again and again in this book.

We will prove a lemma from which van der Waerden's Theorem will follow easily. Informally, the lemma states the following: if you c-color a large enough $[U]$, then *either* there will be a monochromatic k-AP *or* there will be an arbitrarily large number of monochromatic $(k-1)$-AP's, all of different colors. Once there are $c+1$ such $(k-1)$-AP's the latter cannot happen, so the former must.

Lemma 1.2.7 *Fix $k, c \in N$ with $k \geq 3$. Assume $(\forall c')[\mathrm{VDW}(k-1, c')] holds]$. Then, for all r, there exists $U = U(r)$*[1] *such that for all c-colorings $\chi:[U] \rightarrow [c]$, one of the following statements holds.*

Statement I: *There are $a, d \in \mathbb{N}$ such that*

$$\chi(a) = \chi(a+d) = \chi(a+2d) = \cdots = \chi(a+(k-1)d).$$

Statement II: *There exists an anchor $a \in \mathbb{N}$ and numbers $d_1, d_2, \ldots, d_r \in \mathbb{N}$, such that*

$$\chi(a+d_1) = \chi(a+2d_1) = \cdots = \chi(a+(k-1)d_1)$$

$$\chi(a+d_2) = \chi(a+2d_2) = \cdots = \chi(a+(k-1)d_2)$$

$$\vdots$$

$$\chi(a+d_r) = \chi(a+2d_r) = \cdots = \chi(a+(k-1)d_r)$$

and, for all $i \neq j$, $\chi(a+d_i) \neq \chi(a+d_j)$.

Proof:

We define $U(r)$ to be the least number such that this lemma holds. We will prove $U(r)$ exists by giving an upper bound on it.

Base Case: $r = 1$. We show that $U(1) \leq 2W(k-1, c)$. Let $\chi:[2W(k-1, c)] \rightarrow [c]$. Apply $\mathrm{VDW}(k-1, c)$ to the *last half of* $[U(1)]$ to obtain $a', d \in \mathbb{N}$ such that

$$\chi(a') = \chi(a'+d) = \cdots = \chi(a'+(k-2)d)$$

and

$$a' - d \in [U(1)].$$

Figure 1.5 represents the situation.

[1]Formally U depends on k, c, r; however, we suppress the dependence on k and c for ease of notation.

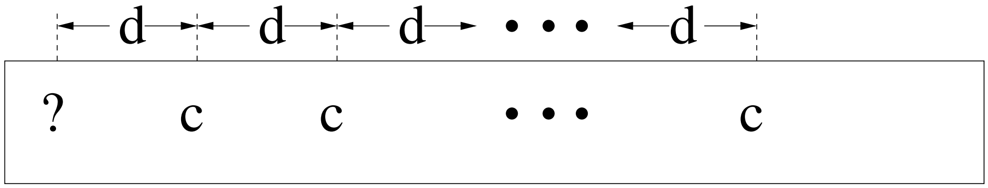

Figure 1.5: Base Case

Let $a = a' - d$. If $\chi(a) = \chi(a')$ then $a' - d, a', a' + d, \ldots, a' + (k-2)d$ is a monochromatic k-AP that satisfies Statement I. If $\chi(a) \neq \chi(a')$ then a, d satisfy Statement II.

Induction Step: By induction, assume $U(r)$ exists. We will show that $U(r+1) \leq 2U(r)W(k-1, c^{U(r)})$. Let

$$U = 2U(r)W(k-1, c^{U(r)}).$$

Let χ:$[U] \to [c]$ be an arbitrary c-coloring of $[U]$.

We view $[U]$ as being $U(r)W(k-1, c^{U(r)})$ numbers followed by $W(k-1, c^{U(r)})$ blocks of size $U(r)$. We denote these blocks by

$$B_1, B_2, \ldots, B_{W(k-1, c^{U(r)})}.$$

Just one of these block looks like Figure 1.6. Figure 1.7 represents the situation we have with $W(k-1, c^{U(r)})$ blocks.

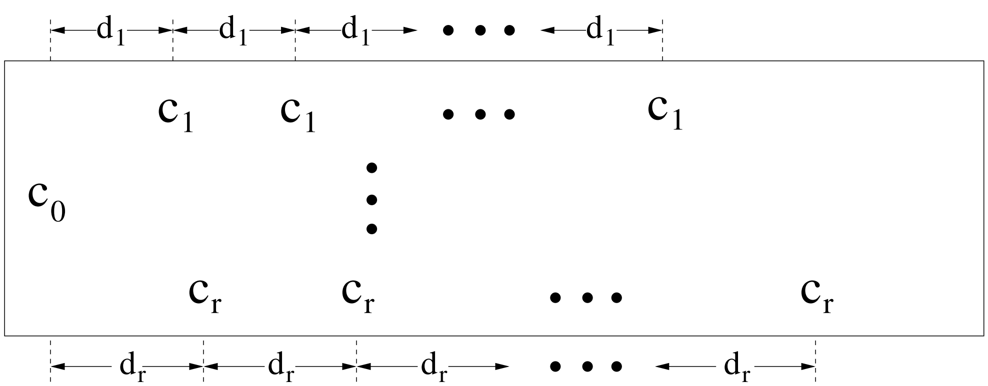

Figure 1.6: One Block of Size $U(r)$

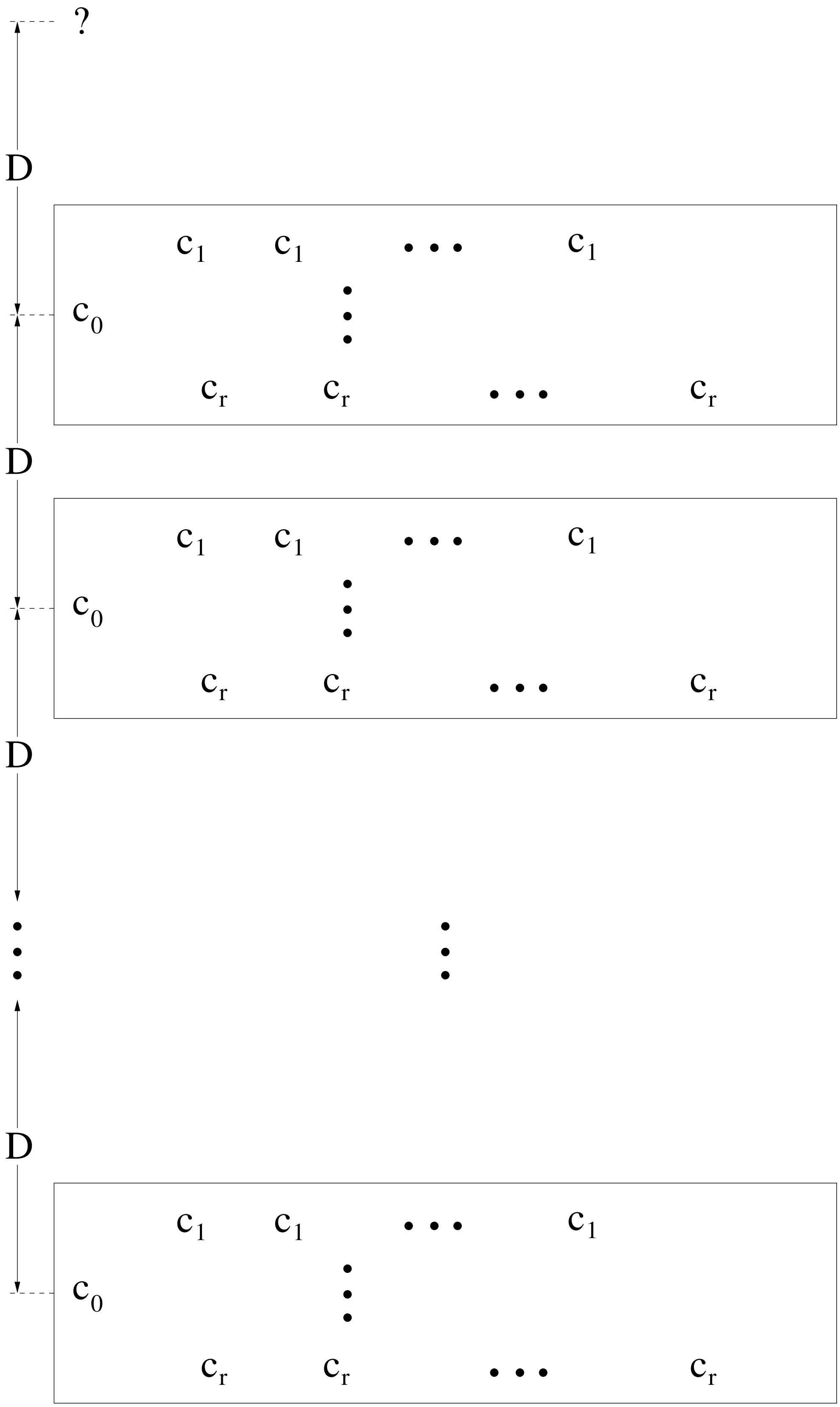

Figure 1.7: Many Blocks of $U(r)$

We view a c-coloring of the second half of $[U]$ as a $c^{U(r)}$-coloring of these blocks.

Let χ^* be this coloring. By the definition of $W(k-1, c^{U(r)})$, we get a monochromatic $(k-1)$-AP of blocks. Hence we have A, D' such that

$$\chi^*(B_A) = \chi^*(B_{A+D'}) = \cdots = \chi^*(B_{A+(k-2)D'})$$

Figure 1.7 represents the situation.

Now, consider block B_A. It is colored by χ. It has length $U(r)$, which tells us that either Statement I or II from the lemma holds. If Statement I holds — we have a monochromatic k-AP — then we are done. If not, then we have the following: $a', d_1, d_2, \ldots, d_r$ with $a' \in B_A$, and

$$\{a'+d_1, a'+2d_1, \ldots, a'+(k-1)d_1\} \subseteq B_A$$
$$\{a'+d_2, a'+2d_2, \ldots, a'+(k-1)d_2\} \subseteq B_A$$
$$\vdots$$
$$\{a'+d_r, a'+2d_r, \ldots, a'+(k-1)d_r\} \subseteq B_A$$

$$\chi(a'+d_1) = \chi(a'+2d_1) = \cdots = \chi(a'+(k-1)d_1)$$
$$\chi(a'+d_2) = \chi(a'+2d_2) = \cdots = \chi(a'+(k-1)d_2)$$
$$\vdots$$
$$\chi(a'+d_r) = \chi(a'+2d_r) = \cdots = \chi(a'+(k-1)d_r)$$

where $\chi(a'+d_i)$ are all different colors, and different from a' (or else there would already be a monochromatic k-AP). How far apart are corresponding elements in adjacent blocks? Since the blocks viewed as points are D' apart, and each block has $U(r)$ elements in it, corresponding elements in adjacent blocks are $D = D' \times U(r)$ apart. Hence

$$\chi(a'+d_1) = \chi(a'+D+d_1) = \cdots = \chi(a'+(k-2)D+d_1)$$
$$\chi(a'+d_2) = \chi(a'+D+d_2) = \cdots = \chi(a'+(k-2)D+d_2)$$

$$\vdots$$

$$\chi(a' + d_r) = \chi(a' + D + d_r) = \cdots = \chi(a' + (k-2)D + d_r)$$

We now note that we have only worked with the second half of $[U]$. Since we know that

$$a > \frac{1}{2}U = U(r)W(k-1, c^{U(r)})$$

and

$$D \leq \frac{1}{k-1}U(r)W(k-1, c^{U(r)}) \leq U(r)W(k-1, c^{U(r)})$$

so we find that $a = a' - D > 0$ and thus $a \in [U]$. The number a is going to be our **new anchor**.

So now we have

$$\begin{array}{c}
\chi(a + (D + d_1)) = \chi(a + 2(D + d_1)) = \cdots = \chi(a + (k-1)(D + d_1)) \\
\chi(a + (D + d_2)) = \chi(a + 2(D + d_2)) = \cdots = \chi(a + (k-1)(D + d_2)) \\
\vdots \\
\chi(a + (D + d_r)) = \chi(a + 2(D + d_r)) = \cdots = \chi(a + (k-1)(D + d_r))
\end{array}$$

Where each progression uses different color.

We need an $(r+1)^{\text{st}}$ monochromatic set of points. Consider

$$\{a + D, a + 2D, \ldots, a + (k-1)D\}.$$

These are corresponding points in blocks which have the same color under χ^*, hence

$$\chi(a + D) = \chi(a + 2D) = \cdots = \chi(a + (k-1)D)).$$

In addition, since

$$(\forall i)[\chi(a') \neq \chi(a' + d_i)]$$

the color of this new progression is different from the r progression above.

Hence we have $r+1$ monochromatic $(k-1)$-AP's, all of different colors, and all with projected first term a. Formally the new parameters are $a, D + d_1, \ldots, D + d_r$, and D. ∎

Theorem 1.2.8 ***(Van der Waerden's Theorem)*** $\forall k, c \in \mathbb{N}, \exists W = W(k,c)$ *such that, for all c-colorings* $\chi{:}[W] \rightarrow [c]$, $\exists a, d \in \mathbb{N}, d \neq 0$ *such that*

$$\chi(a) = \chi(a+d) = \chi(a+2d) = \cdots = \chi(a+(k-1)d)$$

Proof:

We prove this by induction on k. That is, we show that

- $(\forall c)[W(1,c)$ exists$]$
- $(\forall c)[W(k,c)$ exists$] \implies (\forall c)[W(k+1,c)$ exists$]$

Base Case: $k = 1$ As noted above $W(1,c) = 1$ suffices. In fact, we also know that $W(2,c) = c+1$ suffices.

Recall that VDW(k,c) means that Van der Waerden's Theorem holds with parameters k, c.

Induction Step: Assume $(\forall c)[\text{VDW}(k-1,c)$ holds$]$. Fix c. Consider what Lemma 1.2.7 says when $r = c$. In any c-coloring of $U = U(c)$, either there is a monochromatic k-AP or there are c monochromatic $(k-1)$-AP's which are all colored differently, and a number a whose color differs from all of them. Since there are only c colors, this cannot happen, so we must have a monochromatic k-AP. Hence $W(k,c) \leq U(c)$ and hence exists. ▮

Note that the proof of VDW(k,c) depends on VDW$(k-1,c')$ where c' is quite large. Formally the proof is an induction on the following order on $\mathbb{N} \times \mathbb{N}$.

$$(1,1) \prec (1,2) \prec \cdots \prec (2,1) \prec (2,2) \prec \cdots \prec (3,1) \prec (3,2) \cdots$$

This is an ω^2 ordering. It is well founded, so induction works.

Chapter 2

The Polynomial van der Waerden's Theorem

2.1 Introduction

In this Chapter we state and proof a generalization of van der Waerden's Theorem known as the *Polynomial van der Waerden's Theorem*. We rewrite van der Waerden's Theorem with an eye toward generalizing it.

Van der Waerden's Theorem: *For all $k, c \in \mathbb{N}$ there exists $W = W(k, c)$ such that, for all c-colorings $\chi:[W] \to [c]$, there exists $a, d \in [W]$, such that the following set is monochromatic:*

$$\{a\} \cup \{a + id \mid 1 \le i \le k - 1\}.$$

Note that van der Waerden's Theorem was really about the set of functions $\{id \mid 1 \le i \le k - 1\}$. Why this set of functions? Would other sets of functions work? What about sets of polynomials? The following statement is a natural generalization of van der Waerden's Theorem; however, it is not true.

False POLYVDW: *Fix $c \in \mathbb{N}$ and $P \subseteq \mathbb{Z}[x]$ finite. Then there exists $W = W(P, c)$ such that, for all c-colorings $\chi:[W] \to [c]$, there are $a, d \in \mathbb{N}$, $d \ne 0$, such that the following set is monochromatic:*

$$\{a\} \cup \{a + p_i(d) \mid p \in P\}.$$

The above statement is false since the polynomial $p(x) = 2$ and the coloring

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\cdots \\ R & R & B & B & R & R & B & B & R & R\cdots \end{array}$$

provides a counterexample. Hence we need a condition to rule out constant functions. The condition $(\forall p \in P)[p(0) = 0]$ suffices.

The Polynomial van der Waerden Theorem (POLYVDW) *Fix $c \in \mathbb{N}$ and $P \subseteq \mathbb{Z}[x]$ finite, with $(\forall p \in P)[p(0) = 0]$. Then there exists $W = W(P, c)$ such that, for all c-colorings χ:$[W] \rightarrow [c]$, there are $a, d \in [W]$, such that the following set is monochromatic:*

$$\{a\} \cup \{a + p_i(d) \mid p \in P\}.$$

This was proved for $k = 1$ by Fürstenberg [3] and (independently) Sarkozy [4]. The original proof of the full theorem by Bergelson and Leibman [1] used ergodic methods. A later proof by Walters [8] uses purely combinatorial techniques. We will present an expanded version of Walters' proof.

Note 2.1.1 Do we need the condition $d \in [W]$? For the classical van der Waerden Theorem $d \in [W]$ was obvious since

$$\{a\} \cup \{a + d, \ldots, a + (k-1)d\} \subseteq [W] \implies d \in [W].$$

For the Polynomial van der Waerden's Theorem one could have a polynomial with negative coefficients, hence it would be possible to have

$$\{a\} \cup \{a + p(d) \mid p \in P\} \subseteq [W] \text{ and } d \notin [W].$$

For the final result we do not care where d is; however, in order to prove POLYVDW inductively we will need the condition $d \in [W]$.

Recall that VDW was proven by induction on k and c. The main step was showing that if $(\forall c)[W(k, c)$ exists] then $(\forall c)[W(k+1, c)$ exists]. To prove POLYVDW we will do something similar. We will assign to every set of polynomials (that do not have a constant term) a type. The types will be ordered. We will then do an induction on the types of polynomials.

Def 2.1.2 Let $n_e, \ldots, n_1 \in \mathbb{N}$. Let $P \subseteq \mathbb{Z}[x]$. *$P$ is of type $(n_e, \ldots, n_1)$* if the following hold:

1. P is finite.

2. $(\forall p \in P)[p(0) = 0]$

3. The largest degree polynomial in P is of degree $\leq e$.

4. For all i, $1 \leq i \leq e$, There are $\leq n_i$ different lead coefficients of the polynomials of degree i. Note that there may be many more than n_i polynomials of degree i.

Note 2.1.3

1. Type $(0, n_e, \ldots, n_1)$ is the same as type $(n_e, \ldots, n_1)$.

2. We have no n_0. This is intentional. All the polynomials $p \in P$ have $p(0) = 0$.

3. By convention P will never have 0 in it. For example, if

$$Q = \{x^2, 4x\}$$

then

$$\{q - 4x : q \in Q\}$$

will be $\{x^2 - 4x\}$. We will just omit the 0.

Example 2.1.4

1. The set $\{x, 2x, 3x, 4x, \ldots, 100x\}$ is of type (100).

2. The set
$\{x^4+17x^3-65x, x^4+x^3+2x^2-x, x^4+14x^3, -x^4-3x^2+12x, -x^4+78x,$
$x^3 - x^2, x^3 + x^2, 3x, 5x, 6x, 7x\}$
is of type $(2, 1, 0, 4)$

3. The set

$$\{x^4 + b_3x^3 + b_2x^2 + b_1x \mid -10^{10} \leq b_1, b_2, b_3 \leq 10^{10}\ \}$$

is of type $(1, 0, 0, 0)$.

4. If P is of type $(1,0)$ then there exists $b \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that

$$P \subseteq \{bx^2 + ix \mid -k \le i \le k\}.$$

5. If P is of type $(1,1)$ then there exists $b_2, b_1 \in \mathbb{Z}$, and $k \in \mathbb{N}$ such that

$$P \subseteq \{b_2x^2 - kx, b_2x^2 - (k-1)x, \ldots, b_2x^2 + kx\} \cup \{b_1x\} \cup \{0\}.$$

6. If P is of type (n_3, n_2, n_1) then there exists $b_3^{(1)}, \ldots, b_3^{(n_3)} \in \mathbb{Z}$, $b_2^{(1)}, \ldots, b_2^{(n_2)} \in \mathbb{Z}$, $b_1^{(1)}, \ldots, b_1^{(n_1)} \in \mathbb{Z}$, $k_1, k_2 \in \mathbb{N}$, T_1 of type (k_1), and T_2 of type (k_2, k_1) such that

$$\begin{array}{ll} P \subseteq & \{b_3^i x^3 + p(x) \mid 1 \le i \le f, p \in T_2\} \cup \\ & \{b_2^i x^2 + p(x) \mid 1 \le i \le g, p \in T_1\} \cup \\ & \{b_1^i x \mid 1 \le i \le h\} \end{array}$$

7. Let
$$P = \{2x^2 + 3x, x^2 + 20x, 5x, 8x\}.$$
Let
$$Q = \{p(x) - 8x \mid p \in P\}.$$
Then
$$Q = \{2x^2 - 5x, x^2 + 12x, -3x, \}.$$
P is of type $(2,2)$ and Q is of type $(2,1)$. If we did not have out convention of omitting 0 then the type of Q would have been $(2,2)$. The type would not have gone "down" (in an ordering to be defined later). This is why we have the convention.

8. Let P be of type $(n_e, \ldots, n_i + 1, 0, \ldots, 0)$. Let bx^i be the leading term of some polynomial of degree i in P (note that we are not saying that $bx^i \in P$). Let
$$Q = \{p(x) - bx^i \mid p \in P\}.$$
There are numbers $n_{i-1}, \ldots, n_1$ such that Q is of type $(n_e, \ldots, n_i, n_{i-1}, \ldots, n_1)$. The type is decreasing in an ordering to be defined later.

Def 2.1.5

1. Let $P \subseteq \mathbb{Z}[x]$ such that $(\forall p \in P)[p(0) = 0]$. POLYVDW$(P)$ means that the following holds:

   For all $c \in \mathbb{N}$, there exists $W = W(P, c)$ such that for all c-colorings $\chi{:}[W] \rightarrow [c]$, there exists $a, d \in [W]$ such that

   $$\{a\} \cup \{a + p(d) \mid p \in P\} \textit{ is monochromatic.}$$

   (If we use this definition on a coloring of $\{s+1, \ldots, s+W\}$ then the conclusion would have $a \in \{s+1, \ldots, s+W\}$ and $d \in [W]$.)

2. Let $n_e, \ldots, n_1 \in \mathbb{N}$. POLYVDW$(n_e, \ldots, n_1)$ means that, for all $P \subseteq \mathbb{Z}[x]$ of type $(n_e, \ldots, n_1)$ POLYVDW(P) holds.

3. Let $(n_e, \ldots, n_i, \omega, \ldots, \omega)$ be the e-tuple that begins with $(n_e, \ldots, n_i)$ and then has $i - 1$ ω's.

   $$\text{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega)$$

   is the statement

   $$\bigwedge_{n_{i-1}, \ldots, n_1 \in \mathbb{N}} \text{POLYVDW}(n_e, \ldots, n_i, n_{i-1}, \ldots, n_1).$$

4. POLYVDW is the statement

   $$\bigwedge_{i=1}^{\infty} \text{POLYVDW}(\omega, \ldots, \omega)(\ \omega \text{ occurs } i \text{ times}).$$

   Note that POLYVDW is the complete Polynomial van der Waerden Theorem.

Example 2.1.6

1. The statement POLYVDW(ω) is equivalent to the ordinary van der Waerden's Theorem.

2. To prove POLYVDW$(1, 0)$ it will suffice to prove POLYVDW(P) for all P of the form

   $$\{bx^2 - kx, bx^2 - (k-1)x, \ldots, bx^2 + kx\}.$$

3. Assume that you know

$$\mathrm{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega)$$

and that you want to prove

$$\mathrm{POLYVDW}(n_e, \ldots, n_i + 1, 0, \ldots, 0).$$

Let P be of type $(n_e, \ldots, n_i + 1, 0 \ldots, 0)$. Let bx^i be the first term of some polynomial of degree i in P.

   (a) Let
   $$Q = \{p(x) - bx^i \mid p \in P\}.$$
   Then there exists $n_{i-1}, \ldots, n_1$, such that Q is of type
   $$(n_e, \ldots, n_i, n_{i-1}, \ldots, n_1).$$
   Since
   $$\mathrm{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega)$$
   holds by assumption, we can assert that POLYVDW(Q) holds.

   (b) Let $U \in \mathbb{N}$. Let
   $$Q = \{p(x + u) - p(u) - bx^i \mid p \in P, 0 \le u \le U\}.$$
   Note $q(0) = 0$ for all $q \in Q$. Then there exists $n_{i-1}, \ldots, n_1$, such that Q is of type
   $$(n_e, \ldots, n_i, n_{i-1}, \ldots, n_1).$$
   Since
   $$\mathrm{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega)$$
   holds by assumption, we can assert that POLYVDW(Q) holds.

We will prove the Polynomial van der Waerden's Theorem by an induction on a complicated structure. We will prove the following:

1. POLYVDW(1) (this will easily follow from the pigeon hole principle).

2. For all $n_e, \ldots, n_i \in \mathbb{N}$,

$$\text{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega) \implies \text{POLYVDW}(n_e, \ldots, n_i{+}1, 0, 0, \ldots, 0).$$

   Note that this includes the case

$$\text{POLYVDW}(n_e, \ldots, n_2, n_1) \implies \text{POLYVDW}(n_e, \ldots, n_2, n_1 + 1).$$

The ordering we use is formally defined as follows:

Def 2.1.7 $(n_e, \ldots, n_1) \preceq (m_{e'}, \ldots, m_1)$ if either

- $e < e'$, or
- $e = e'$ and, for some i, $1 \le i \le e$, $n_e = m_e$, $n_{e-1} = m_{e-1}$, …, $n_{i+1} = m_{i+1}$, but $n_i < m_i$.

This is an ω^ω ordering.

Example 2.1.8 We will use the following ordering on types.

$$(1) \prec (2) \prec (3) \prec \cdots$$

$$(1,0) \prec (1,1) \prec \cdots \prec (2,0) \prec (2,1) \prec \cdots \prec (3,0) \cdots \prec$$

$$(1,0,0) \prec (1,0,1) \prec \cdots \prec (1,1,0) \prec (1,1,1) \prec (1,2,0) \prec (1,2,1) \prec$$

$$(2,0,0) \prec \cdots \prec (3,0,0) \prec \cdots (4,0,0) \cdots .$$

2.2 The Proof of the Polynomial van der Waerden Theorem

2.2.1 POLYVDW($\{x^2, x^2 + x, \ldots, x^2 + kx\}$)

Def 2.2.1 Let $k \in \mathbb{N}$.

$$P_k = \{x^2, x^2 + x, \ldots, x^2 + kx\}.$$

We show POLYVDW(P_k). This proof contains many of the ideas used in the proof of POLYVDW.

We prove a lemma from which POLYVDW(P_k) will be obvious.

Lemma 2.2.2 *Fix k, c throughout. For all r there exists $U = U(r)$ such that for all c-colorings $\chi:[U] \to [c]$ one of the following statements holds.*
Statement I: *There exists $a, d \in [U]$, such that*

- $\{a\} \cup \{a + d^2, a + d^2 + d, \ldots, a + d^2 + kd\} \subseteq [U]$,
- $\{a\} \cup \{a + d^2, a + d^2 + d, \ldots, a + d^2 + kd\}$ *is monochromatic.*

Statement II: *There exists $a, d_1, \ldots, d_r \in [U]$ such that the following hold.*

- $\{a + d_1^2, a + d_1^2 + d_1, \ldots, a + d_1^2 + kd_1\} \subseteq [U]$.
  $\{a + d_2^2, a + d_2^2 + d_2, \ldots, a + d_2^2 + kd_2\} \subseteq [U]$.
  $$\vdots$$
  $\{a + d_r^2, a + d_r^2 + d_r, \ldots, a + d_r^2 + kd_r\} \subseteq [U]$.
  (The element a is called the anchor*)*
- $\{a + d_1^2, a + d_1^2 + d_1, \ldots, a + d_1^2 + kd_1\}$ *is monochromatic.*
  $\{a + d_2^2, a + d_2^2 + d_2, \ldots, a + d_2^2 + kd_2\}$ *is monochromatic.*
  $$\vdots$$
  $\{a + d_r^2, a + d_r^2 + d_r, \ldots, a + d_r^2 + kd_r\}$ *is monochromatic.*
  With each monochromatic set being colored differently and differently from a. We refer to a as the **anchor**.

Informal notes:

1. *We are saying that if you c-color $[U]$ either you will have a monochromatic set of the form*

$$\{a\} \cup \{a+d^2, a+d^2+d, \ldots, a+d^2+kd\}$$

or you will have many *monochromatic sets of the form*

$$\{a+d^2, a+d^2+d, \ldots, a+d^2+kd\},$$

all of different colors, and different from a. Once "many" is more than c, then the latter cannot happen, so the former must, and we have POLYVDW(P).

2. *If we apply this theorem to a coloring of $\{s+1, \ldots, s+U\}$ then we either have*

$$d \in [U] \text{ and } \{a\} \cup \{a+d^2+d, \ldots, a+d^2+kd\} \subseteq \{s+1, \ldots, s+U\}.$$

or

$$d_1, \ldots, d_r \in [U] \text{ and, for all } i \text{ with } 1 \le i \le r \text{ such that}$$

$$\{a\} \cup \{a+d_i^2+d_i, \ldots, a+d_i^2+kd_i\} \subseteq \{s+1, \ldots, s+U\}, \text{ and}$$

$$\{a+d_i^2+d_i, \ldots, a+d_i^2+kd_i\} \subseteq \{s+1, \ldots, s+U\} \text{ monochromatic for each } i.$$

Proof:

We define $U(r)$ to be the least number such that this Lemma holds. We will prove $U(r)$ exists by giving an upper bound on it.

Base Case: $r = 1$. $U(1) \le W(k+1,c)^2 + W(k+1,c)$.

Let χ be any c-coloring of $[W(k+1,c) + W(k+1,c)^2]$. Look at the coloring restricted to the last $W(k+1,c)$ elements. By van der Waerden's Theorem applied to the restricted coloring there exists

$$a' \in [(W(k+1,c))^2 + 1, \ldots, (W(k+1,c))^2 + W(k+1,c)]$$

and

$$d' \in [W(k+1,c)]$$

such that

$$\{a', a'+d', a'+2d', \ldots, a'+kd'\} \text{ is monochromatic .}$$

Let the anchor be $a = a' - (d')^2$ and let $d_1 = d'$.

$\{a', a'+d', a'+2d', \ldots, a'+kd'\} = \{a+d_1^2, a+d_1^2+d_1, \ldots, a+d_1^2+kd_1\}$ is monochromatic.

If a is the same color then Statement I holds. If a is a different color then Statement II holds. There is one more issue– do we have

$$a, d_1 \in [(W(k+1,c))^2 + W(k+1,c)]?$$

Since

$$a' \geq (W(k+1,c))^2 + 1$$

and

$$d' \leq W(k+1,c)$$

we have that

$$a \geq (W(k+1,c))^2 + 1 - (W(k+1,c))^2 = 1.$$

Clearly

$$a < a' \leq W(k+1,c) + (W(k+1,c))^2.$$

Hence

$$a \in [W(k+1,c) + (W(k+1,c))^2].$$

Since $d_1 = d' \in [W(k+1,c)]$ we clearly have

$$d_1 \in [W(k+1,c) + (W(k+1,c))^2].$$

Induction Step: Assume $U(r)$ exists, and let

$$X = W(k + 2U(r), c^{U(r)}).$$

(X stands for eXtremely large.)
We show that

$$U(r+1) \leq (X \times U(r))^2 + X \times U(r).$$

Let χ be a c-coloring of

$$[(X \times U(r))^2 + X \times U(r)].$$

View this set as $(X \times U(r))^2$ consecutive elements followed by X blocks of length $U(r)$. Let the blocks be

$$B_1, B_2, \ldots, B_X.$$

Restrict χ to the blocks. Let $\chi^*:[X] \rightarrow [c^{U(r)}]$ be the coloring viewed as a $c^{U(r)}$-coloring of the blocks. By VDW applied to χ^* and the choice of X there exists $A, D' \in [X]$ such that

- $\{A, A+D', \ldots, A+(k+2U(r))D'\} \subseteq [X]$,
- $\{B_A, B_{A+D'}, \ldots, B_{A+(k+2U(r))D'}\}$ is monochromatic. How far apart are corresponding elements in adjacent blocks? Since the blocks viewed as points are D' apart, and each block has $U(r)$ elements in it, corresponding elements in adjacent blocks are $D = D' \times U(r)$ numbers apart.

Consider the coloring of B_A. Since B_A is of size $U(r)$ either there exists $a, d \in U(r)$ such that

- $\{a\} \cup \{a+d^2, a+d^2+d, \ldots, a+d^2+kd\} \subseteq B_A$,
- $\{a\} \cup \{a+d^2, a+d^2+d, \ldots, a+d^2+kd\}$ is monochromatic

in which case Statement I holds so we are done, or there exists
$a' \in B_A$, $d'_1, \ldots, d'_r \in [U(r)]$
such that

- $\{a'+{d'_1}^2, a'+{d'_1}^2+d'_1, \ldots, a'+{d'_1}^2+kd'_1\} \subseteq B_A$
  $\{a'+{d'_2}^2, a'+{d'_2}^2+d'_2, \ldots, a'+{d'_2}^2+kd'_2\} \subseteq B_A$
  $$\vdots$$
  $\{a'+{d'_r}^2, a'+{d'_r}^2+d'_r, \ldots, a'+{d'_r}^2+kd'_r\} \subseteq B_A$
- $\{a'+{d'_1}^2, a'+{d'_1}^2+d'_1, \ldots, a'+{d'_1}^2+kd'_1\}$ is monochromatic.
  $\{a'+{d'_2}^2, a'+{d'_2}^2+d'_2, \ldots, a'+{d'_2}^2+kd'_2\}$ is monochromatic.
  $$\vdots$$
  $\{a'+{d'_r}^2, a'+{d'_r}^2+d'_r, \ldots, a'+{d'_r}^2+kd'_r\}$ is monochromatic.

with each monochromatic set colored differently from the others and from a'.

Since $\{B_A, B_{A+D}, \ldots, B_{A+(k+2U(r))D}\}$ is monochromatic we also have that, for all j with $0 \le j \le k + 2U(r)$,

NEED FIGURE

$$\{a' + {d'_1}^2 + jD, a' + {d'_1}^2 + d'_1 + jD, \ldots, a' + {d'_1}^2 + kd'_1 + jD \mid 0 \le j \le k + 2U(r)\}$$

is monochromatic

$$\{a' + {d'_2}^2 + jD, a' + {d'_2}^2 + d'_2 + jD, \ldots, a' + {d'_2}^2 + kd'_2 + jD\} \mid 0 \le j \le k + 2U(r)\}$$

is monochromatic

$$\vdots$$

$$\{a' + {d'_r}^2 + jD, a' + {d'_r}^2 + d'_r + jD, \ldots, a' + {d'_2}^2 + kd'_r + jD\} \mid 0 \le j \le k + 2U(r)\}$$

is monochromatic.

with each monochromatic set colored differently from the others and from a', but the same as their counterpart in B_A.

Let the new anchor be $a = a' - D^2$. Let $d_i = D + d'_i$ for all $1 \le i \le r$, and $d_{r+1} = D$. We first show that these parameters work and then show that $a, d_1, \ldots, d_r \in [U(r+1)]$.

For $1 \le i \le r$ we need to show that

$$\{a + (D + d'_i)^2, a + (D + d'_i)^2 + (D + d'_i), \ldots, a + (D + d'_i)^2 + k(D + d'_i)\}$$

is monochromatic. Let $0 \le j \le k$. Note that

$$a + (D + d'_i)^2 + j(D + d'_i) = (a' - D^2) + (D^2 + 2Dd'_i + {d'_i}^2) + (jD + jd'_i) = a' + {d'_i}^2 + jd'_i + (j + 2d'_i)D.$$

Notice that $0 \le j + 2d'_i \le k + 2U(r)$. Hence $a + d_i^2 + jd_i \in B_{A+(j+2d'_i)D'}$, the $(j + 2d'_i)$th block. Since B_A is the same color as $B_{A+(j+2d'_i)D'}$,

$$\chi(a + d_i^2) = \chi(a + d_i^2 + jd_i).$$

So we have that, for all $0 \le i \le r$, for all j, $0 \le j \le k$, the set

$$\{a + d_i^2, a + d_i^2 + d_i, \ldots, a + d_i^2 + kd_i\}$$

is monochromatic for each i. And, since the original sequences were different colors, so are our new sequences. Finally, if $\chi(a) = \chi(a + d_i^2)$ for some i, then we have $\{a, a+d_i^2, a+d_i^2+d_i, \ldots, a+d_i^2+kd_i\}$ monochromatic, satisfying Statement I. Otherwise, we satisfy Statement II.

We still need to show that $a, d_1, \ldots, d_r \in [X \times U(r))^2 + X \times U(r)]$. This is an easy exercise based on the lower bound on a' (since it came from the later $X \times U(r)$ coordinates) the inductive upper bound on the d_i's, and the upper bound $D \le U(r)$.

■

Theorem 2.2.3 *For all k,* POLYVDW(P_k)*.*

Proof: We show $W(P_k, c)$ exists by bounding it. Let $U(r)$ be the function from Lemma 2.2.2. We show $W(P_k, c) \le U(c)$. If χ is any c-coloring of $[U(c)]$ then second case of Lemma 2.2.2 cannot happen. Hence the first case must happen, so there exists $a, d \in [U(c)]$ such that

- $\{a\} \cup \{a + d^2, a + d^2 + d, \ldots, a + d^2 + kd\} \subseteq [U(c)]$
- $\{a\} \cup \{a + d^2, a + d^2 + d, \ldots, a + d^2 + kd\}$ is monochromatic.

Therefore $W(P_k, c) \le U(c)$. ■

Note 2.2.4 The proof of Theorem 2.2.3 used VDW. Hence it used POLYVDW(ω). The proof can be modified to proof POLYVDW$(1, 0)$. So the proof can be viewed as showing that POLYVDW$(\omega) \implies$ POLYVDW$(1, 0)$.

2.2.2 The Full Proof

We prove a lemma from which the implication

$$\text{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega) \implies \text{POLYVDW}(n_e, \ldots, n_i+1, 0, 0, \ldots, 0)$$

will be obvious.

Lemma 2.2.5 *Let $n_e, \ldots, n_i \in \mathbb{N}$. Assume that* POLYVDW$(n_e, \ldots, n_i, \omega, \ldots, \omega)$ *holds. Let $P \subseteq \mathbb{Z}[x]$ of type $(n_e, \ldots, n_i + 1, 0, \ldots, 0)$. Let $c \in \mathbb{N}$. We regard these as fixed. For all r, there exists $U = U(r)$[1] such that for all c-colorings $\chi:[U] \rightarrow [c]$ one of the following Statements holds.*
Statement I: *there exists $a, d \in [U]$, such that*

- $\{a\} \cup \{a + p(d) \mid p \in P\} \subseteq [U]$.
- $\{a\} \cup \{a + p(d) \mid p \in P\}$ *is monochromatic.*

Statement II: *there exists $a, d_1, \ldots, d_r \in [U]$ such that the following hold.*

- $\{a + p(d_1) \mid p \in P\} \subseteq [U]$
  $\{a + p(d_2) \mid p \in P\} \subseteq [U]$
  $\vdots$
  $\{a + p(d_r) \mid p \in P\} \subseteq [U]$
  (The number a is called **the anchor***)*
- $\{a + p(d_1) \mid p \in P\}$ *is monochromatic*
  $\{a + p(d_2) \mid p \in P\}$ *is monochromatic*
  $\vdots$
  $\{a + p(d_r) \mid p \in P\}$ *is monochromatic*

With each monochromatic set being colored differently and differently from a.

Informal notes:

1. *We are saying that if you c-color $[U]$ either you will have a monochromatic set of the form*

$$\{a\} \cup \{a + p(d) \mid p \in P\}$$

or you will have many monochromatic sets of the form

$$\{a + p(d) \mid p \in P\},$$

[1] Formally U depends on P, c, r; however, we suppress the dependence on P and c for notational ease.

all of different colors, and different from a. Once "many" is more than c, then the latter cannot happen, so the former must, and we have

$$\text{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega) \implies \text{POLYVDW}(n_e, \ldots, n_i{+}1, 0, \ldots, 0).$$

2. *If we apply this theorem to a coloring of $\{s+1, \ldots, s+U\}$ then we either have*

$$d \in [U] \text{ and } \{a\} \cup \{a + p(d) \mid p \in P\} \subseteq \{s+1, \ldots, s+U\}$$

or

$$d_1, \ldots, d_r \in [U] \text{ and, for all } i \text{ with } 1 \leq i \leq r$$

$$\{a\} \cup \{a + p(d_i) \mid p \in P\} \subseteq \{s+1, \ldots, s+U\}.$$

Proof: We define $U(r)$ to be the least number such that this Lemma holds. We will prove $U(r)$ exists by giving an upper bound on it. In particular, for each r, we will bound $U(r)$. We will prove this theorem by induction on r.

One of the fine points of this proof will be that we are careful to make sure that $a \in [U]$. The fact that we have inductively bounded the d_i's will help that.

Fix $P \subseteq \mathbb{Z}[x]$ of type $(n_e, \ldots, n_i+1, 0, \ldots, 0)$. Fix $c \in \mathbb{N}$. We can assume P actually has $n_i + 1$ lead coefficients for degree i polynomials (else P is of smaller type and hence POLYVDW(P, c) already holds and the lemma is true). In particular there exists some polynomial of degree i in P. We assume that x^i be the first term of some polynomial of degree i in P (the proof for bx^i with $b \in \mathbb{Z}$ is similar).

Base Case: $r = 1$. Let

$$Q = \{p(x) - x^i \mid p \in P\}.$$

It is easy to show that there exists $n_{i-1}, \ldots, n_1$ such that Q is of type $(n_e, \ldots, n_i, n_{i-1}, \ldots, n_1)$, and that $(\forall q \in Q)[q(0) = 0]$. Since POLYVDW$(n_e, \ldots, n_i, \omega, \ldots, \omega)$ is true, POLYVDW(Q) is true. Hence $W(Q, c)$ exists.

We show that

$$U(1) \leq W(Q, c)^i + W(Q, c).$$

Let χ be any c-coloring of $[W(Q, c)^i + W(Q, c)]$. Look at the coloring restricted to the last $W(Q, c)$ elements. By POLYVDW(Q) applied to the restricted coloring there exists $a' \in \{W(Q,c)^i + 1, \ldots, W(Q,c)^i + W(Q,c)\}$ and $d' \in [W(Q,c)]$ such that

$$\{a'\} \cup \{a' + q(d') \mid q \in Q\} \subseteq \{W(Q,c)^i + 1, \ldots, W(Q,c)^i + W(Q,c)\}$$

$$\{a'\} \cup \{a' + q(d') \mid q \in Q\} \text{ is monochromatic .}$$

(Note- we will only need that $\{a' + q(d') \mid q \in Q\}$ is monochromatic.)

Let the new anchor be $a = a' - b(d')^i$. Let $d_1 = d'$. (We will use $b > 0$ later to show that $a \in [U(1) \leq W(Q,c)^i + W(Q,c)]$.)

Then

$$\begin{aligned} \{a' + q(d') \mid q \in Q\} = & \ \{a' + p(d') - b(d')^i \mid p \in P\} \\ = & \ \{(a' - b(d_1)^i) + p(d_1) \mid p \in P\} \\ = & \ \{a + p(d_1) \mid p \in P\} \text{ is monochromatic.} \end{aligned}$$

If a is the same color then Statement I holds. If a is a different color then Statement II holds. There is one more issue– do we have $a, d \in [U(1)]$?

Since

$$a' \geq W(Q,c)^i + 1$$

and

$$d' \leq W(Q,c) \text{ (Recall that POLYVDW has the restriction } d \in [W].)$$

we have that

$$a = a' - b(d')^i \geq W(Q,c)^i + 1 - d(d')^i \geq W(Q,c)^i + 1 - W(Q,c)^i = 1$$

Clearly

$$a < a' \leq W(Q,c)^i + W(Q,c)$$

Hence

$$a \in [W(Q,c)^i + W(Q,c)].$$

Since $d_1 = d' \in [W(Q,c)]$ we clearly have

$$d_1 \in [W(Q,c)^i + W(Q,c)].$$

Induction Step: Assume $U(r)$ exists. Let

$$Q = \{p(x+u) - p(u) - x^i \mid p \in P, 0 \le u \le U(r)\}.$$

Note that

$$\{p(x) - x^i \mid p \in P\} \subseteq Q.$$

Clearly $(\forall q \in Q)[q(0) = 0]$. It is an easy exercise to show that, there exists $n_i, \ldots, n_1$ such that Q is of type $(n_e, \ldots, n_{i+1}, n_i, \ldots, n_1)$.

Now, let

$$Q' = \left\{ \frac{q(x \times U(r))}{U(r)} \mid q \in Q \right\}$$

Since every $q \in Q$ is an integer polynomial with $q(0) = 0$, it follows that $U(r)$ divides $q(xU(r))$, so we have $Q' \subseteq \mathbb{Z}[x]$. Moreover, it's clear that Q' has the same type as Q.

Since POLYVDW$(n_e, \ldots, n_i, \omega, \ldots, \omega)$ holds, we have POLYVDW(Q').

Hence $(\forall c')[W(Q', c')$ exists]. We show that

$$U(r+1) \le b\left(U(r)W(Q', c^{U(r)})\right)^i + U(r)W(Q', c^{U(r)}).$$

Let χ be a c-coloring of

$$\left[b\left(U(r)W(Q', c^{U(r)})\right)^i + U(r)W(Q', c^{U(r)}) \right].$$

View this set as $b\left(U(r)W(Q', c^{U(r)})\right)^i$ elements followed by $W(Q', c^{U(r)})$ blocks of size $U(r)$ each. Restrict χ to the blocks. Now **view the restricted c-coloring of numbers as a $c^{U(r)}$-coloring of blocks.** Call this coloring χ^*. Let the blocks be

$$B_1, B_2, \ldots, B_{W(Q', c^{U(r)})}.$$

By the definition of $W(Q', c^{U(r)})$ applied to χ^*, and the assumption that POLYVDW$(n_e, \ldots, n_i, \omega, \ldots, \omega)$ holds, there exists $A, D' \in [W(Q', c^{U(r)})]$ such that

$$\{B_{A+q'(D')} \mid q' \in Q'\} \text{ is monochromatic.}$$

Note that we are saying that the blocks are the same color. Let $D = D' \times U(r)$ be the distance between corresponding elements of the blocks.

Because each block is length $U(r)$, if we have an element $x \in B_A$, then in block $B_{A+q'(D')}$ we have a point x', where

CHECK NORMAL VDW WITH THIS POINT ABOUT BLOCKS

NEED FIGURE

$$\begin{aligned} x' &= x + q'(D')U(r) \\ &= x + q'\left(\frac{D}{U(r)}\right)U(r) \\ &= x + q(D) \text{ for some } q \in Q, \text{ by definition of } Q' \end{aligned}$$

This will be very convenient.

Consider the coloring of B_A. Since B_A is of size $U(r)$ one of the following holds.

I) There exists $a \in B_A$ and $d \in [U(r)]$ such that

- $\{a\} \cup \{a + p(d) \mid p \in P\} \subseteq B_A$
- $\{a\} \cup \{a + p(d) \mid p \in P\}$ is monochromatic (so we are done).

II) There exists $a' \in B_A$ (so $a' \geq W(Q', c^{U(r)})^i + 1$) and $d'_1, \ldots, d'_r \in [U(r)]$ such that

- $$\begin{gathered} \{a' + p(d'_1) \mid p \in P\} \subseteq B_A \\ \{a' + p(d'_2) \mid p \in P\} \subseteq B_A \\ \vdots \\ \{a' + p(d'_r) \mid p \in P\} \subseteq B_A \end{gathered}$$
- $$\begin{gathered} \{a' + p(d'_1) \mid p \in P\} \text{ is monochromatic} \\ \{a' + p(d'_2) \mid p \in P\} \text{ is monochromatic} \\ \vdots \\ \{a' + p(d'_r) \mid p \in P\} \text{ is monochromatic} \end{gathered}$$

  with each monochromatic set being colored differently from each other and from a'.

Since $\{B_{A+q'(D')} \mid q' \in Q'\}$ is monochromatic, and since we know that $x \in B_A$ corresponds to $x + q(D) \in B_{A+q'(D')}$, we discover that, for all $q \in Q$,

$$\begin{array}{c}\{a' + p(d'_1) + q(D) \mid p \in P\} \text{ is monochromatic} \\ \{a' + p(d'_2) + q(D) \mid p \in P\} \text{ is monochromatic} \\ \vdots \\ \{a' + p(d'_r) + q(D) \mid p \in P\} \text{ is monochromatic.}\end{array}$$

with each monochromatic set being colored differently from each other, and from a', but the same as their counterpart in B_A.

Our new **anchor** is $a = a' - D^i$. Note that since

$$a' \geq W(Q', c^{U(r)})^i + 1$$

and

$$D \leq W(Q', c^{U(r)})$$

we have

$$a = a' - D^i \geq W(Q', c^{U(r)})^i + 1 - W(Q', c^{U(r)})^i = 1$$

Clearly $a \leq a' \leq W(Q', c^{U(r)} + U(r)W(Q', c^{U(r)})$. Hence

$$a \in [W(Q', c^{U(r)})^i + U(r)W(Q', c^{U(r)})].$$

Since

$$\{B_{A+q'(D')} \mid q' \in Q'\}$$

is monochromatic (viewing the coloring on blocks) we know that

$$\{a' + q(D) \mid q \in Q\}$$

is monochromatic (viewing the coloring on numbers). Remember that the following is a subset of Q:

$$\{p(x) - x^i \mid p \in P\}.$$

Hence the following set is monochromatic:

$$\begin{aligned}\{a' + p(D) - D^i \mid p \in P\} &= \{a + D^i + p(D) - D^i \mid p \in P\} \\ &= \{a + p(D) \mid p \in P\}.\end{aligned}$$

If a is the same color then Statement I holds and we are done. If a is a different color then we have one value of d, namely $d_{r+1} = D$. We seek r additional ones to show that Statement II holds.

For each i we want to find a new d_i that works with the new anchor a. Consider the monochromatic set $\{a' + p(d_i') \mid p \in P\}$. We will take each element of it and shift it $q(D)$ elements for some $q \in Q$. The resulting set is still monochromatic. We will pick $q \in Q$ carefully so that the resulting set, together with the new anchor a and the new values $d_i = d_i' + D$ work.

CHECK VDW AND QVDW FOR THIS POINT

For each $p \in P$ we want to find a $q \in Q$ such that $a + p(d_i' + D)$ is of the form $a' + p(d_i') + q(D)$, and hence the color is the same as $a' + p(d_i')$.

$$\begin{aligned} a' + p(d_i') + q(D) &= a + p(d_i' + D) \\ a' + p(d_i') + q(D) - a &= p(d_i' + D) \\ D^i + p(d_i') + q(D) &= p(d_i' + D) \\ q(D) &= p(d_i' + D) - p(d_i') - D^i \end{aligned}$$

Take $q(x) = p(x + d_i') - p(d_i') - D^i$. Note that $d_i' \leq U(Q, c, r)$ so that $q \in Q$.

— Put bounds on d_i in here.

BILL - CHECK THIS

Let $d_i = d_i' + D$ for $1 \leq i \leq r$, and $d_{r+1} = D$.

We have seen that

$$\{a + p(d_1) \mid p \in P\} \text{ is monochromatic}$$

$$\vdots$$

$$\{a + p(d_r) \mid p \in P\} \text{ is monochromatic}$$

$$\text{AND}$$

$$\{a + p(d_{r+1}) \mid p \in P\} \text{ is monochromatic}$$

The first r are guaranteed to be different colors by the inductive assumption. The $(r+1)^{st}$ is yet another color, because it shares a color with the anchor of our original sequences, which we assumed had its own color. So here we see that the Lemma is satisfied with parameters $a, d_1, \ldots, d_r, d_{r+1}$. ∎

Lemma 2.2.6 *For all* $n_e, \ldots, n_i$

$$\text{POLYVDW}(n_e, \ldots, n_i, \omega, \ldots, \omega) \implies \text{POLYVDW}(n_e, \ldots, n_i{+}1, 0, 0, \ldots, 0).$$

Proof: Assume POLYVDW($n_e, \ldots, n_i, \omega, \ldots, \omega$). Let P be of type POLYVDW($n_e, \ldots, n_i + 1, 0, 0, \ldots, 0$). Apply Lemma 2.2.5 to P with $r = c$. Statement II cannot hold, so statement I must, and we are done. ∎

We can now prove the Polynomial van der Waerden Theorem.

Theorem 2.2.7 *For all $P \subseteq \mathbb{Z}[x]$ finite, such that $(\forall p \in P)[p(0) = 0]$, for all $c \in \mathbb{N}$, there exists $W = W(P, c)$ such that for all c-colorings χ:$[W] \to [c]$, there exists $a, d \in [W]$ such that*

- $\{a\} \cup \{a + p(d) \mid p \in P\} \subseteq [W]$,
- $\{a\} \cup \{a + p(d) \mid p \in P\}$ *is monochromatic.*

Proof:

We use the ordering from Definition 2.1.7. The least element of this set is (0). POLYVDW(0) is the base case. The only sets of polynomials of type (0) are $\emptyset$. For each of these sets, the Polynomial van der Waerden Theorem requires only one point to be monochromatic (the anchor), so of course POLYVDW (0) holds.

Lemma 2.2.5 is the induction step.

This proves the theorem.

∎

Note 2.2.8

1. Our proof of POLYVDW did not use van der Waerden's Theorem. The base case for POLYVDW was POLYVDW(0) which is trivial.

2. Let $p(x) = x^2 - x$ and $P = \{p(x)\}$. Note that $p(1) = 0$. The statement POLYVDW(P, 2012) is true but stupid: if χ is an 2012-coloring of [1] then let $a = 0$ and $d = 1$. Then $a, a + p(d)$ are the same color since they are the same point. Hence POLYVDW(P, 2012) holds. The proof of POLYVDW we gave can be modified to obtain a d so that not only is $d \neq 0$ but
$$\{a\} \cup \{a + p(d) \mid p \in P\}$$
has all distinct elements. Once this is done POLYVDW(P, 2012) is true in a way that is not stupid.

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