Lines Imply Spaces in Density Ramsey Theory

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Abstract

Some results of geometric Ramsey theory assert that if F is a finite field (respectively, set) and n is sufficiently large, then in any coloring of the points of F^n there is a monochromatic k-dimensional affine (respectively, combinatorial) subspace (see [9]). We prove that the density version of this result for lines (i.e., $k = 1$) implies the density version for arbitrary k. By using results in [2, 6] we obtain various consequences: a "group-theoretic" version of Roth's Theorem, a proof of the density assertion for arbitrary k in the finite field case when $|F| = 3$, and a proof of the density assertion for arbitrary k in the combinatorial case when $|F| = 2$.

1 Results

In this section we will state and discuss the main results and prove some corollaries. The proofs of the main results are in the following section. Throughout q denotes a prime power.

Let $\mathbb{F}_q$ be the field with q elements and let V be an n-dimensional vector space over $\mathbb{F}_q$. For each positive integer k and positive real number ε let $n(\varepsilon, k, q)$ denote the smallest integer (if one exists) such that

$$n = \dim_{\mathbb{F}_q} V \geq n(\varepsilon, k, q), \quad A \subset V, \quad |A| > \varepsilon |V|,$$

imply that A contains an affine k-space. (By an affine k-space we mean any translate of a k-dimensional vector subspace; the purist will note that we only use the structure of V as an affine space.)

The "Affine Line Conjecture" is the assertion that $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$ and all q. The existence of $n(\varepsilon, k, q)$ would be a density version of the results in [9] on Ramsey theorems in geometric contexts.

The main assertion of this paper is that if, for a fixed q, $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$, then $n(\varepsilon, k, q)$ exists for all k and all $\varepsilon > 0$. We will also reinterpret this result in the context of "combinatorial" k-spaces and "lattices" in abelian groups. We include a number of corollaries and remarks.

(It is not hard to see that if $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$ and *all* q, then $n(\varepsilon, k, q)$ exists for all k, ε, and q. Indeed, if ε, k, and q are given, let F be the extension of $\mathbb{F}_q$ of degree k. An affine line in an F-vector space is a k-space over $\mathbb{F}_q$ if we "restrict scalars" to $\mathbb{F}_q$; from this it is easy to see that the existence of an affine line in a large enough subset of F^n implies the existence of an affine k-space in a large enough subset of $\mathbb{F}_q^{kn}$.)

Theorem 1. *Suppose that $\mathbb{F}_q$ is a fixed finite field and that $n(\varepsilon,1,q)$ exists for all $\varepsilon > 0$. Then $n(\varepsilon,k,q)$ exists for all $\varepsilon > 0$ and all k.*

Corollary. *The integers $n(\varepsilon,k,2)$ and $n(\varepsilon,k,3)$ exist for all $\varepsilon > 0$ and all k.*

Proof of the corollary. Any two-element subset of an $\mathbb{F}_2$ vector space is an affine line so it is trivial that $n(\varepsilon,1,q)$ exists. The theorem then implies that $n(\varepsilon,k,2)$ exists for all k (see the corollary to Lemma 1 in [2] for a different proof of the existence of $n(\varepsilon,k,2)$). The existence of $n(\varepsilon,k,3)$ follows from Theorem 1 and the existence of $n(\varepsilon,1,3)$ which is the central result of [2]. This finishes the proof of the corollary. □

A set $\{x_1,\dots,x_k\}$ of the elements in an abelian group G is said to be independent if $c_1x_1+c_2x_2+\cdots = c_kx_k = 0$ implies that $c_ix_i = 0$ for each i. An (m,k)-lattice in an abelian group G is a set of the form

$$M = \{a + c_1x_1 + \cdots + c_kx_k : c_i = 0,1,\dots,m-1\},$$

where a is an element of G and the x_i are independent. If V is a vector space over a finite field, then by an (m,k)-lattice in V we mean an (m,k)-lattice in its underlying additive group.]

Let $n'(\varepsilon,k,q)$ denote the smallest integer (if one exists) such that if

$$n = \dim_{\mathbb{F}_q} V \geq n'(\varepsilon,k,q), \quad A \subset V, \quad |A| > \varepsilon|V|,$$

then A contains a $(3,k)$-lattice.

Theorem 2. *$n'(\varepsilon,k,q)$ exists for all $\varepsilon > 0$, k, and q.*

Corollary. *For each $\varepsilon > 0$ and positive integer k there is an integer $m(\varepsilon,k)$ such that if G is any finite abelian group with more than $m(\varepsilon,k)$ elements and A is any subset of G with more that $\varepsilon|G|$ elements, then there is a $(3,k)$-lattice inside A.*

Proof of the corollary. Let k and ε be given. Choose by Szemerédi's theorem [10] a large enough n so that any subset of $\{1,2,\dots,n\}$ with more than εn elements contains an arithmetic progression with 3^k terms. Choose $m(\varepsilon,k)$ large enough so that any finite abelian group G with more than $m(\varepsilon,k)$ elements must contain either a cyclic subgroup H of order at least n, or a subgroup H which is the direct product of at least $n'(\varepsilon,k,p)$ cyclic groups of order p for some prime $p < n$.

Now let G be a finite abelian group with more than $m(\varepsilon,k)$ elements and let A be a subset of G with $|A| > \varepsilon|G|$. Let H be the subgroup whose existence is guaranteed by the choice of $m(\varepsilon,k)$. Then $|A \cap a+H| > \varepsilon|H|$ for some coset $a+H$ of H. If H is cyclic, then $A-a$ contains the set

$$\{a_0 + c_1d + c_2(3d) + \cdots c_k(3^{k-1}d) : c_i = 0,1,2\},$$

where d is the difference of the arithmetic progression whose existence is guaranteed by the choice of n above. If H is the direct product of at least $n'(\varepsilon,k,q)$ cyclic groups of order p, then $A-a$ contains

$$\{a_0 + c_1x_1 + \cdots + c_kx_k : c_i = 0,1,2\}$$

for an independent set of x_i. Thus in either case A contains a $(3,k)$-lattice and we are finished. □

Remarks. (1) Roth's special case of Szemerédi's theorem asserts that if n is sufficiently large and A is a subset of $\{1,2,\dots,n\}$ with more than εn elements then A contains a set of the form $\{a,a+x,a+2x\}$. This is equivalent to the case $k=1$ of the corollary in the case in which G is cyclic. Indeed, it is not hard to check that one has

$$m(\varepsilon,1) \le n \le \frac{1}{2}m\left(\frac{\varepsilon}{2},1\right)+1$$

(to verify the second inequality consider subsets of the "first half" of a sufficiently large cyclic group). Thus the corollary could be thought of as a group-theoretic generalization of Roth's Theorem.

(2) Since sufficiently large groups contain large abelian subgroups [4], we could actually delete the requirement that G be abelian in the statement of the corollary.

(3) If the Affine Line Conjecture is valid, then the results here imply the obvious "group-theoretic generalization" of Szemerédi's Theorem: For every $\varepsilon>0$, k, and l there exists an integer $m(\varepsilon,k,l)$ such that if G is any finite abelian group with more than $m(\varepsilon,k,l)$ elements and A is any subset of G with more $\varepsilon|G|$ elements, then there exists an (l,k)-lattice in A.

Finally, we remove the algebraic structure on the underlying set, replacing $\mathbb{F}_q$ with an arbitrary finite set. Thus we consider combinatorial subspaces; we briefly recall the definition (see [6] for further details).

Let F be the finite set $\{0,1,\dots,t-1\}$ with t elements. A subset W of F^n is a *combinatorial k-space* if it satisfies the following. There is a partition

$$\{1,\dots,n\} = B_0 \cup B_1 \cup \dots \cup B_k$$

such that $B_1,\dots,B_k$ are nonempty. There is a function $f: B_0 \mapsto F$. A function $\bar{f}: F^k \mapsto F^n$ is defined by $\bar{f}(y_1,\dots,y_k) = (x_1,\dots,x_n)$ where

$$\begin{aligned} x_i &= f(i) && \text{for } i \text{ in } B_0,\\ x_i &= y_j && \text{for } i \text{ in } B_j, 1 \le j \le k. \end{aligned}$$

W is the range of $\bar{f}$.

The definition is complicated, but it captures a notion of subspace when the only structure on F is that of a finite set. We remark that the Hales-Jewett Theorem [6, 7] asserts that if n is large enough, then in any coloring of F^n there is a monochromatic combinatorial 1-space (usually called a combinatorial line).

Let $n''(\varepsilon,k,t)$ be the smallest integer (if one exists) such that if

$$n \ge n''(\varepsilon,k,t), \quad A \subset F^n, \quad |A| > \varepsilon|F^n|,$$

then A contains a combinatorial k-space.

Theorem 3. *Let t be fixed. If $n''(\varepsilon,1,t)$ exists for all $\varepsilon>0$, then $n''(\varepsilon,k,t)$ exists for all $\varepsilon>0$ and all k.*

Corollary. *$n''(\varepsilon,k,2)$ exists for all $\varepsilon>0$ and all k.*

Proof of the corollary. The existence of $n''(\varepsilon,1,2)$ is a simple consequence of Sperner's Lemma (see [1] or [6]). □

Remarks. (1) In [1] it is shown that if there is a fixed $\varepsilon_0 < 1$ such that $n''(\varepsilon_0, 1, t)$ exists for all t, then $n''(\varepsilon, 1, t)$ exists for all $\varepsilon > 0$ and all t. The corresponding result for $n(\varepsilon, 1, q)$ is proved in [3].

(2) The existence of $n''(\varepsilon, 1, t)$ is a "density version" of the Hales-Jewett Theorem. Graham has offered a reward for a proof of the existence (or non-existence!) of the numbers $n''(\varepsilon, 1, 3)$.

2 Proofs

The following lemma contains the crucial idea underlying Theorems 1, 2, and 3.

Lemma. *Let $\mathbb{F}_q$ be a fixed finite field and k a fixed positive integer. Assume that $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$. Then for each positive integer r, if $n(1/(r+1), k, q)$ exists then $n(1/r, k+1, q)$ exists. Similar statements holds for $n'(\varepsilon, k, q)$ and $n''(\varepsilon, k, t)$.*

Proof. We give the proof in the vector space case $n(\varepsilon, k, q)$. The proofs for $n'(\varepsilon, k, q)$ and $n''(\varepsilon, k, t)$ are entirely analogous. In the lattice case $n'(\varepsilon, k, q)$ it is merely necessary to replace "k-space" with "$(3, k)$-lattice" and "line" with "$(3, 1)$-lattice" throughout. In the combinatorial case $n''(\varepsilon, k, t)$ it is necessary to replace "affine k-space" with "combinatorial k-space" and "affine line" with "combinatorial line" throughout.

Let $n_0 = n(1/(r+1), k, q)$. Let e be the number of distinct k-dimensional *vector* subspaces of any n_0-dimensional vector space over $\mathbb{F}_q$. Let $\delta = (q^{n_0} e r^2)^{-1}$ and let $s = n(\delta, 1, q)$. We claim that

$$n(1/r, k+1, q) \leq n_0 + s.$$

To prove this we must start with a vector space V over $\mathbb{F}_q$ of dimension at least $n_0 + s$. Let A be a subset of V with

$$|A| > (1/r)|V| \geq (1/r) q^{n_0 + s}.$$

Let W_0 be a n_0-dimensional subspace of V and let

$$V = \bigcup W_\alpha$$

be the decomposition of V into a union of the pairwise disjoint translates (cosets) of W_0. For the proof to work in the combinatorial case it is necessary at this point to choose W_0 to be the subspace consisting of the vectors whose last s components are 0.

Let t be the number of cosets W_α such that

$$|A \cap W_\alpha| \leq \frac{1}{r+1}|W_\alpha| = \frac{1}{r+1} q^{n_0}.$$

There are q^s cosets altogether, so

$$\frac{1}{r}|V| < |A| = \sum |A \cap W_\alpha| \leq \frac{t}{r+1}|W_\alpha| + (q^s - t)|W_\alpha|.$$

This gives

$$q^s - t > q^s / r^2.$$

Hence there are $d = q^s - t > q^s/r^2$ cosets W_α such that

$$|A \cap W_\alpha| > \frac{1}{r+1}|W_\alpha|,$$

and since the dimension of W_0 is $n_0 = n(1/(r+1), k, q)$ each such $A \cap W_\alpha$ must contain an affine k-space

$$a_\alpha + U_\alpha,$$

Where U_α is a k-dimensional vector subspace of W_o.

Since there are exactly e distinct k-dimensional vector subspaces of W_0 at least d/e of the k-spaces $a_\alpha + U_\alpha$ must have the form $a_\alpha + U$ for a fixed U. Let these be

$$a_1 + U, \ldots, a_h + U,$$

where $h \geq d/e$.

Let $A' = \{a_1, \ldots, a_h\}$. Then

$$|A'| = h \geq d/e > \frac{q^s}{er^2} = \frac{1}{q^{n_0}er^2}q^{n_0+s} = \delta|V|.$$

Since the dimension of V is $n_0 + s > s = n(\delta, 1, q)$, there must be an affine line in A'. By renumbering if necessary we can assume that this line is $\{a_1, \ldots, a_q\}$.

It is now easy to check that

$$U' = (a_1 + U) \cup \cdots \cup (a_q + U)$$

is an affine $(k+1)$-space contained in A. Since A was an arbitrary subset of V with $|A| > (1/r)|V|$ this shows that

$$n(1/r, k+1, q) \leq n_0 + s = \dim_{\mathbb{F}_q}(V)$$

as claimed. This finishes the proof of the lemma. □

Theorem 1 now follows immediately from the lemma by induction. Indeed, we are given in the hypotheses of the theorem that $n(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$. If $n(\varepsilon, k, q)$ exists for all ε, then it exists for $\varepsilon = 1/r$. By the lemma, $n(\varepsilon, k+1, q)$ exists for all $\varepsilon > 0$. Theorem 1 now follows by induction on k.

The proof of Theorem 3 is identical; we merely replace $n(\varepsilon, k, q)$ with $n''(\varepsilon, k, t)$.

To prove Theorem 2 for odd q we first observe that $n'(\varepsilon, 1, q)$ exists for all $\varepsilon > 0$ as a consequence of the main result in [2]. For this case Theorem 2 follows from the lemma and induction as above.

To prove Theorem 2 for even q we observe that a $(3,k)$-lattice is just a $(2,k)$-lattice since $2 = 0$ in $\mathbb{F}_q$. It then follows that $n'(\varepsilon, 1, q)$ exists since any two elements of an abelian group form a $(2,1)$-lattice. The rest of the proof is as above. (An upper bound for $n'(\varepsilon, k, q)$ for even q can also be deduced from Lemma 1 in [2].)

Note added in proof. The lemma can be easily improve to show that $n(1/r, k+1, q) \leq n(1/(r+1), k, q) + n(1/(er^2), 1, q)$.

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