1. (15 points)

(a) Find an equation $\dot{x} = f(x)$ for a flow on the line such that there are precisely three points: an unstable fixed point at $x = -2$, a stable fixed point at $x = 0$, and an unstable fixed point at $x = 3$. Justify your answer.

(b) Consider the following two flows on the line:

$$\dot{x} = \frac{x}{x-1}, \tag{1}$$

and

$$\dot{x} = \frac{x}{1-x}. \tag{2}$$

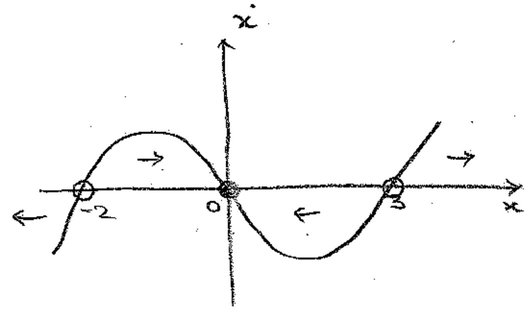
For each of the two equations, sketch the vector field on the real line, identify the fixed point(s) and classify their stability, and describe the behaviour in time of the solutions.

Parts c) and d) are on the next page.

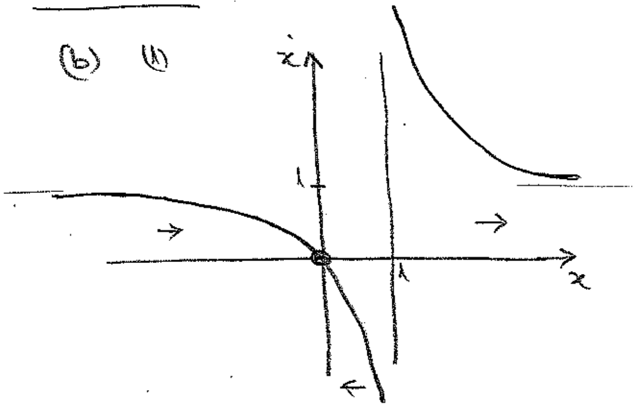
(a) Look for a cubic polynomial with roots at $-2, 0$, and 3

$f(x) = x(x+2)(x-3)$

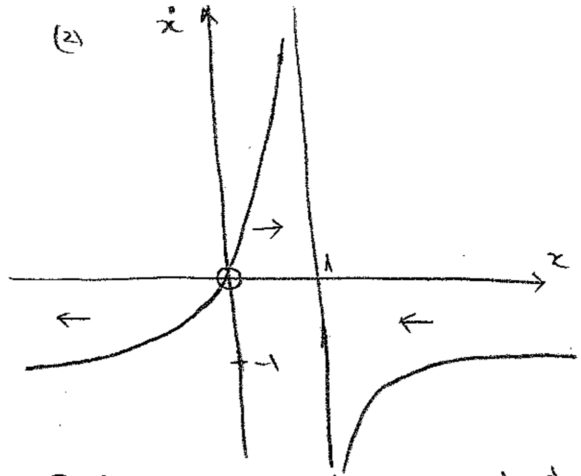
Flow indicated by arrows; stability as desired



(b) (1)



(2)



Eqn (1): stable fixed point at origin.

Origin attracts trajectories that start at $x_0 < 1$

Trajectories that start at $x_0 > 1$ approach ∞ as $t \to \infty$

Eqn (2): unstable fixed point at origin.

Solutions that start at $x_0 < 0$ approach $-\infty$ as $t \to \infty$

Solutions that start at $x_0 > 1$ or $0 < x_0 < 1$ reach the singular value $x = 1$ in finite time.

Here are a few more details on Equ (2) in part (b). These details were not required by the test.

Attempt to integrate exactly $\dot{x} = \frac{x}{1-x}$

$\Rightarrow \quad \frac{1-x}{x}\,dx = dt \quad \Rightarrow \quad \left(\frac{1}{x}-1\right) dx = dt$

Integrate on both sides: $\int\left(\frac{1}{x}-1\right) dx = \int dt \quad \Rightarrow$

$\ln|x| - x = t + \ln|x_0| - x_0. \quad (*)$

Start at $x_0 > 0$, $x_0 \neq 1$. We have $x_0 - \ln x_0 > 1$

Claim: $x=1$ is reached in <u>finite</u> time.

Indeed, make $x=1$ in $(*)$ $\Rightarrow$ $\underbrace{\ln 1}_{0} - 1 = t_1 + \ln x_0 - x_0.$

$\Rightarrow \quad t_1 = x_0 - \ln x_0 - 1 > 0.$

The singular point $x=1$ is reached at time $t_1 > 0$.

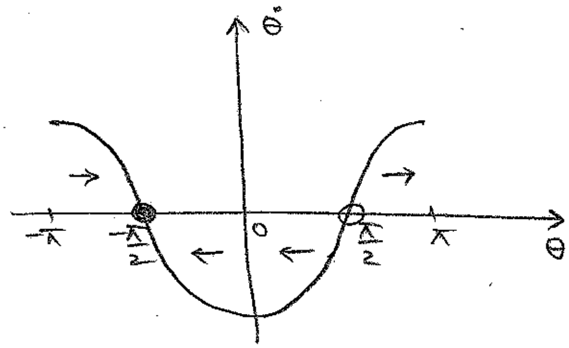
(c) Find an equation $\dot{\theta} = f(\theta)$ for a flow on the circle such that there are exactly two fixed points: a stable fixed point at $\theta = -\pi/2$ and an unstable fixed point at $\theta = \pi/2$. Justify your answer.

(d) Find an equation of the form $\dot{\theta} = f(\theta, \mu)$ for a flow on the circle so that all solutions are oscillatory for $\mu > 0$, and the period of oscillation blows up like $\mu^{-1/2}$ as $\mu \to 0^+$. Describe briefly the bottleneck phenomenon that occurs as $\mu \to 0^+$.

(c) Look for a trigonometric function with zeros at $\pm\frac{\pi}{2}$

$f(\theta) = -\cos\theta$.

Flow indicated by arrows; stability as desired



(d) Equation studied in class:

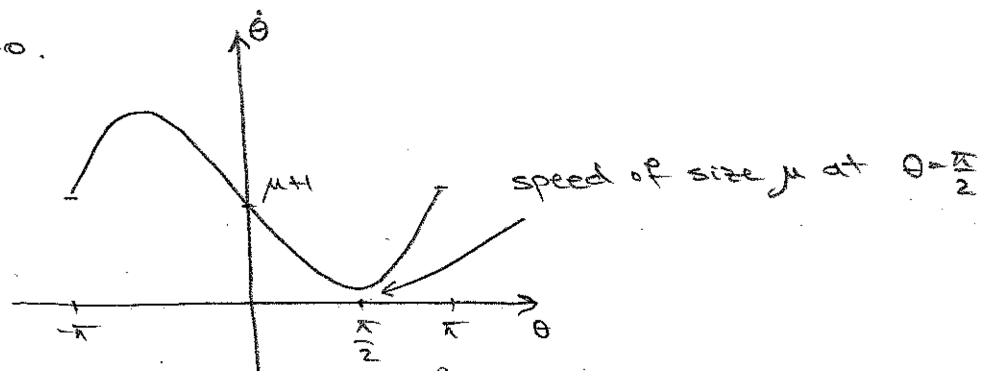
$\dot{\theta} = \omega - a\sin\theta$; oscillations for $\omega > a$ of period $T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$

For $\omega \to a^+$, T scales as $\frac{1}{\sqrt{\omega - a}}$

To solve the problem in (d): take $a = 1$, $\omega = \mu + 1$; $\boxed{\dot{\theta} = \mu + 1 - \sin\theta}$

Bottleneck phenomenon:

$\mu + 1 > 1 \iff \mu > 0$.



As $\mu \to 0^+$, the speed approaches 0 near $\theta = \frac{\pi}{2}$; the solution spends a long time to pass through $\theta = \frac{\pi}{2}$

scales with $\frac{1}{\sqrt{\mu}}$

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2. (10 points) Consider the equation

$$\dot{x} = x(x-1)(1-r+x^2).$$

Find all the fixed points as functions of r. Determine any value(s) of r at which a bifurcation occurs, and classify the type(s) of bifurcation. Sketch the bifurcation diagram.

Fixed points: $x(x-1)(1-r+x^2) = 0.$

$\to x_* = 0, \; x_* = 1$

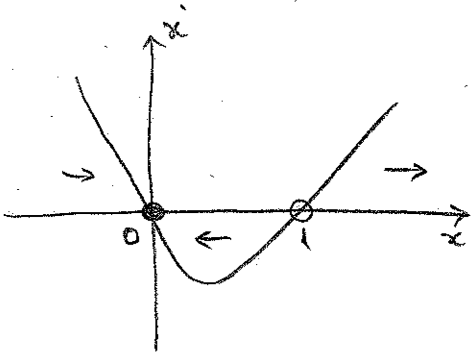
$\to 1 - r + x_*^2 = 0$ (possible only for $r \geq 1$)

$x_* = \pm\sqrt{r-1}.$

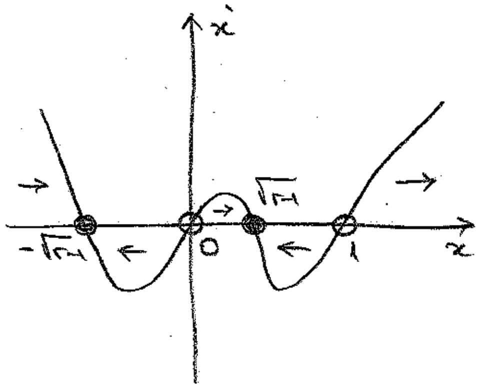
Two fixed points are created at $r=1$, $x=0$.

At $r=2$, the fixed point $\sqrt{r-1}$ meets the fixed point at 1.

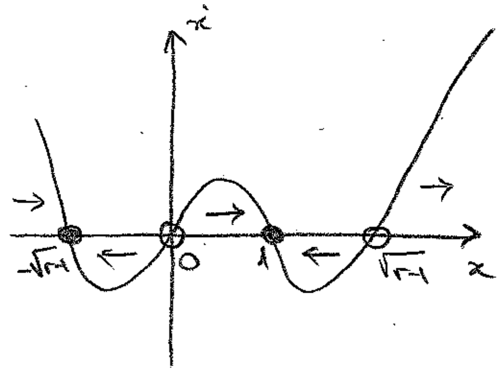
We distinguish the cases:



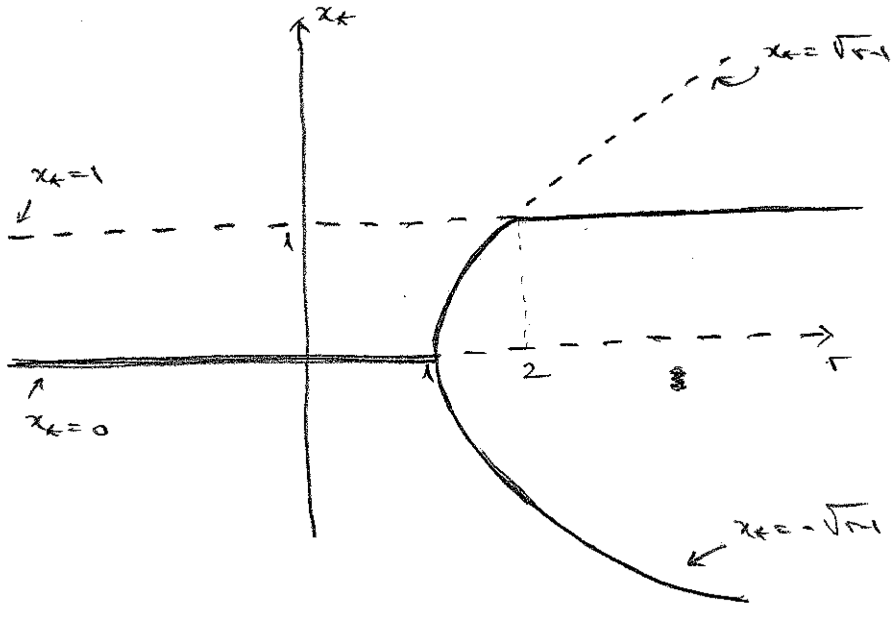
$r<1$



$1<r<2$



$r>2$



Supercritical pitchfork bifurcation at $r=1$, $x=0$.

transcritical bifurcation at $r=2$, $x=1$

3. (15 points) Consider the linear system

$$\dot{x} = \alpha x - y$$
$$\dot{y} = x + y,$$

where α is a real parameter.

(a) Classify the fixed point at the origin for all real values of the parameter α.

(b) Consider the case $\alpha = -\frac{3}{2}$. Calculate the eigenvalues and eigendirections and sketch the phase portrait. Explain the behaviour of the solutions as t approaches ∞ and $-\infty$, respectively.

Parts c) and d) are on the next page.

(a) $\begin{bmatrix} \alpha & -1 \\ 1 & 1 \end{bmatrix}$ $\tau = \alpha+1$, $\Delta = \alpha+1$

ch. polyn: $\lambda^2 - (\alpha+1)\lambda + (\alpha+1) = 0.$

$\tau^2 - 4\Delta = (\alpha+1)^2 - 4(\alpha+1) = (\alpha+1)(\alpha-3)$

<u>case i.</u> $(\alpha+1)(\alpha-3) > 0 \iff \alpha < -1$ or $\alpha > 3$: real e-values

$\alpha < -1 \Rightarrow \Delta < 0$ $\lambda_1 < 0, \lambda_2 > 0$ saddle

$\alpha > 3 \Rightarrow \Delta > 0, \tau > 0 \Rightarrow \lambda_1 > 0, \lambda_2 > 0$ unstable node

<u>case ii</u> $(\alpha+1)(\alpha-3) < 0 \iff -1 < \alpha < 3$ imaginary e-values

$\tau = \alpha+1 > 0$ positive real part unstable spiral

<u>case iii</u> $\alpha = -1$ ch. polyn: $\lambda^2 = 0.$

The origin is a non-isolated fixed point (see part d)

<u>case iv:</u> $\alpha = 3$ ch. polyn: $(\lambda-2)^2 = 0.$ degenerate (unstable) node

b) $\alpha = -\frac{3}{2}$

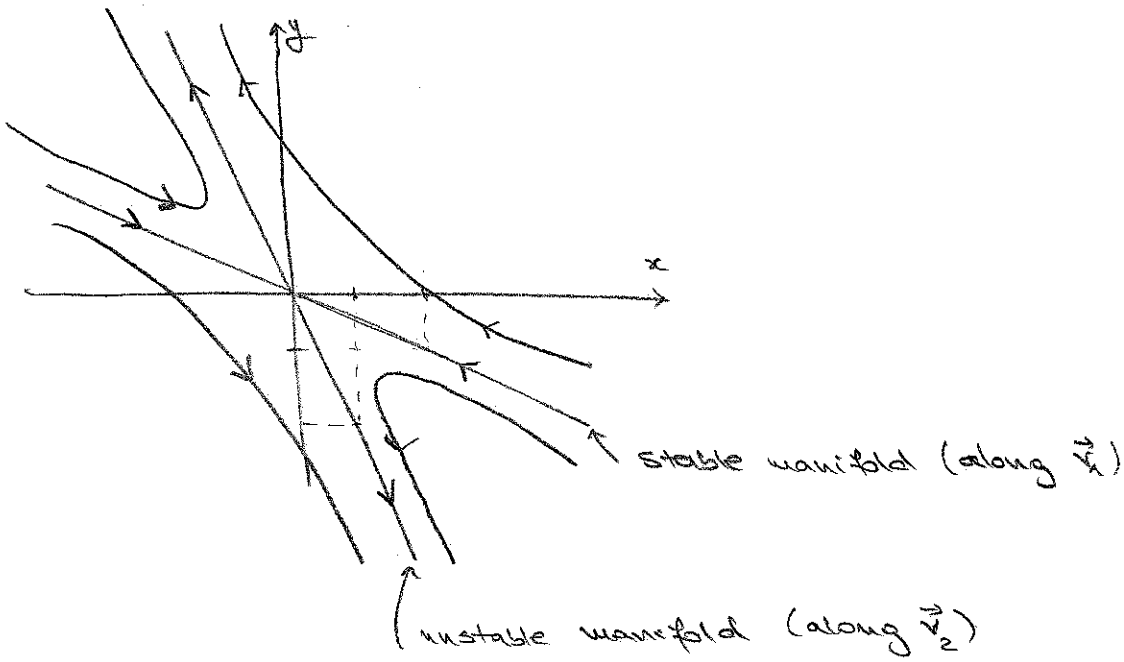
$$\begin{bmatrix} -\frac{3}{2} & -1 \\ 1 & 1 \end{bmatrix} \qquad \tau = -\frac{1}{2}, \quad \Delta = -\frac{1}{2} \qquad \lambda^2 + \frac{1}{2}\lambda - \frac{1}{2} = 0.$$

$$\lambda_1 = -1, \quad \lambda_2 = \frac{1}{2}$$

e-vector $\lambda_1 = -1$: $\begin{bmatrix} -3/2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow -\frac{1}{2}\alpha - \beta = 0.$ $\quad \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

e-vector $\lambda_2 = \frac{1}{2}$ $\begin{bmatrix} -3/2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow -2\alpha - \beta = 0.$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

solution $\quad c_1 e^{-t}\vec{v}_1 + c_2 e^{\frac{1}{2}t}\vec{v}_2$



Solutions asymptote to the unstable manifold as $t \to \infty$, and to the stable manifold as $t \to -\infty$

Special cases: solutions which start on the stable manifold approach the origin as $t \to \infty$

solutions which start on the unstable manifold approach the origin as $t \to -\infty$

(c) Consider the case $\alpha = \frac{7}{2}$. Calculate the eigenvalues and eigendirections and sketch the phase portrait. Explain the behaviour of the solutions as t approaches ∞ and $-\infty$, respectively.

(d) Consider the case $\alpha = -1$. Sketch the phase portrait. Explain the behaviour of the solutions.

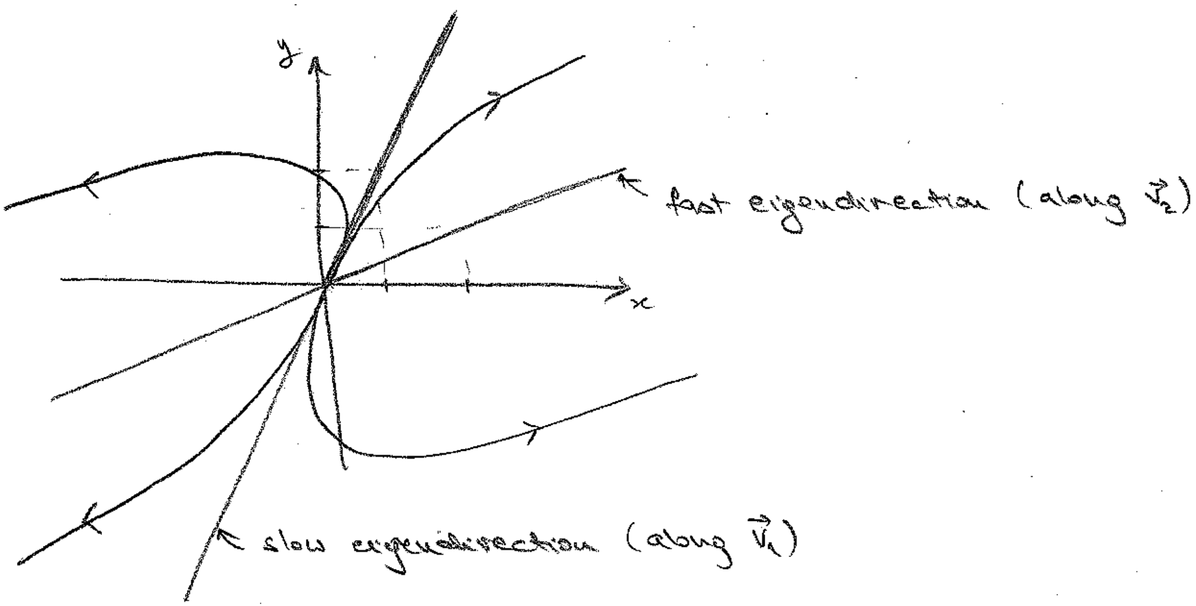
Hint for d). This is a degenerate case that needs to be treated carefully. Try to find a conserved quantity $E(x, y)$ (that is, $E(x(t), y(t)) = const.$ along trajectories).

(c) $\begin{bmatrix} \frac{7}{2} & -1 \\ 1 & 1 \end{bmatrix}$ $\qquad \tau = \frac{9}{2}, \ \Delta = \frac{9}{2} \qquad \lambda^2 - \frac{9}{2}\lambda + \frac{9}{2} = 0.$

$\lambda_1 = \frac{3}{2}, \ \lambda_2 = 3$

e-vector $\lambda_1 = \frac{3}{2}$ $\qquad \begin{bmatrix} \frac{7}{2} & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{3}{2}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow 2\alpha - \beta = 0 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

e-vector $\lambda_2 = 3$ $\qquad \begin{bmatrix} \frac{7}{2} & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 3\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow \frac{\alpha}{2} - \beta = 0. \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



Solutions become parallel to the fast eigendirection as $t \to \infty$

Solutions -"- -"- slow -"- $t \to -\infty$ (or equivalently, solutions leave the origin tangent to the slow e-direction).

d) $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ $\qquad \tau = \Delta = 0. \qquad \lambda^2 = 0.$

$\lambda = 0$ evalue, with e-vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

fixed points (x_*, y_*) satisfy $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_* \\ y_* \end{bmatrix} = 0 \Rightarrow x_* + y_* = 0.$

All points on the line $x+y=0$ are fixed points!

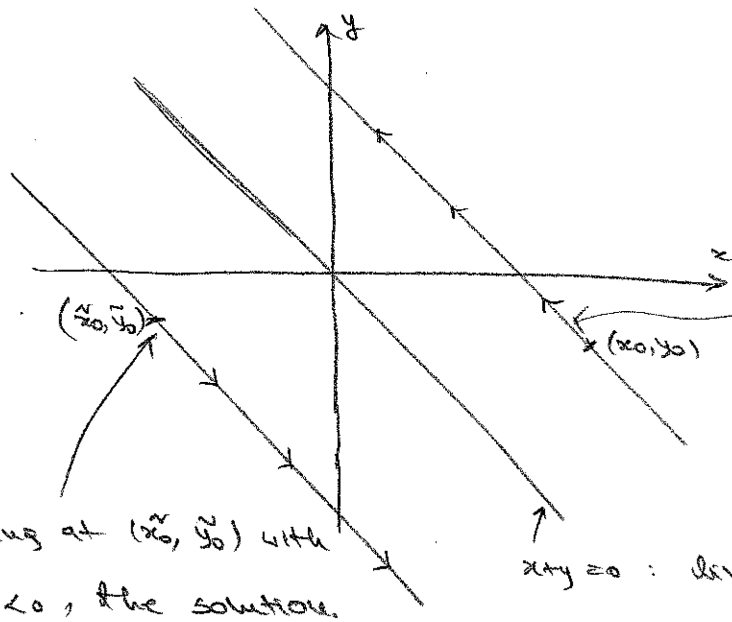
Look for a conserved quantity: $\begin{cases} \dot{x} = -x-y \\ \dot{y} = x+y \end{cases} \Rightarrow \dot{x}+\dot{y} = 0.$

$\Rightarrow \frac{d}{dt}(x+y) = 0 \Rightarrow x(t) + y(t) = \text{const}$

The trajectory that starts from (x_0, y_0) lies on the line $x+y = x_0+y_0.$

We can now go back to the system and solve exactly:

$\begin{cases} \dot{x} = -(x_0+y_0) \\ \dot{y} = x_0+y_0 \end{cases} \Rightarrow x(t) = -(x_0+y_0)t + x_0, \quad y(t) = (x_0+y_0)t + y_0. \qquad (*)$



starting at (x_0, y_0) with $x_0+y_0 > 0$, the solution approaches $(-\infty, \infty)$ along the line $(*)$.

starting at $(\tilde{x}_0, \tilde{y}_0)$ with $\tilde{x}_0 + \tilde{y}_0 < 0$, the solution approaches $(\infty, -\infty)$ along the line $(*)$, where $\tilde{x}_0, \tilde{y}_0$ replaces x_0, y_0.

$x+y=0$: line of fixed points.