Solutions 5

8.7.1 (Page 295)

Note that $\frac{1}{r(1-r^2)} = \frac{1}{r} - \frac{1}{2(1+r)} - \frac{1}{2(1-r)}$. Now

$$\begin{aligned}\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} &= \log\frac{r_1}{r_0} - \frac{1}{2}\log\frac{1+r_1}{1+r_0} - \frac{1}{2}\log\frac{1-r_1}{1-r_0} \\ &= \frac{1}{2}\log\frac{r_1^2(1-r_0^2)}{r_0^2(1+r_1^2)} \\ &= 2\pi,\end{aligned}$$

and therefore $r_1^2(1-r_0^2) = e^{4\pi}(r_0^2(1+r_1^2))$. Rearrange terms and we get $r_1 = [1-e^{-4\pi}(r_0^{-2} - 1)]^{-1/2}$. Note that $P'(r) = [1-e^{-4\pi}(r^{-2}-1)]^{-3/2}(\frac{e^{-4\pi}}{r})$ which equal to $e^{-4\pi}$ for $r^* = 1$.

8.7.2 (Page 295)

Let y_0 be an initial condition on S, which can be chosen to be any vertical line on the cylinder. Since $\dot{\theta} = 1$, the first return to S occurs after a time of flight 2π, then $y_1 - P(y_0)$, where y_1 satisfies $\int_{y_0}^{y_1} \frac{dy}{ay} = \int_0^{2\pi} dt = 2\pi$. This yields $y_1 = y_0 e^{2a\pi}$. In this case, $P(y) = ye^{2a\pi}$. It has a fixed point at $y^* = 0$. which corresponds to a periodic orbit in the dynamical system. This fixed point is stable when $|P'(0)| < 1$ or $a < 0$. Note that for $a = 0$ the dynamics is trivial.

9.2.1 (Page 342)

a) Recall that the fixed points C^+ and C^- are

$$(x^*, y^*, z^*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1),$$

where $r > 1$. In the following we'll write (x, y, z) instead of (x^*, y^*, z^*).

The Jacobian $A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$ has characteristic polynomial

$$\begin{aligned}\det(\lambda I - A) = \lambda^3 + (\sigma + 1 + b)\lambda^2 + \\ [b(\sigma+1) + x^2 + \sigma(z+1-r)]\lambda + [b\sigma(z+1-r) + \sigma(x^2+xy)].\end{aligned}$$

Notice that at C^+ and C^-, we have $x^2 = xy = b(r-1)$, $z = r-1$, therefore the eigenvalues satisfy

$$\lambda^3 + (\sigma+1+b)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1) = 0.$$

b) Hopf bifurcation occurs when two eigenvalues are pure imaginary (cf. Figure 8.2.4). In this case $\lambda = i\omega$, where ω is real and nonzero. Thus the characteristic equation becomes

$$-i\omega^3 - (\sigma + 1 + b)\omega^2 + ib(r + \sigma)\omega + 2\sigma b(r - 1) = 0.$$

We then seperate the real part and the imaginary part of the above and obtain that

$$\omega^2 = b(r + \sigma) = \frac{2\sigma b(r - 1)}{\sigma + 1 + b}.$$

Solving for r, we get

$$r = r_H = \sigma(\frac{\sigma + b + 3}{\sigma - b - 1}).$$

It is required that r must be positive, so Hopf bifurcation can only occur if r_H is positive, i.e. $\sigma > b + 1$.

c) If $r = r_H$, then the two imaginary roots are

$$\lambda_{1,2} = \pm i\sqrt{b(r_H + \sigma)}.$$

It's well known that all 3 roots should add up to $-(\sigma + b + 1)$, since the two imaginary ones cancel, we have $\lambda_3 = -(\sigma + b + 1)$.

9.2.2 (Page 343)

Let $C(t) = rx(t)^2 + \sigma y(t)^2 + \sigma(z(t) - 2r)^2$ be the value of C at time t. Then

$$\begin{aligned} C'(t) &= 2rxx' + 2\sigma yy' + 2\sigma(z - 2r)z' \\ &= 2rx\sigma(y - x) + 2\sigma y(rx - y - xz) + 2\sigma(z - 2r)(xy - bz) \\ &= -2\sigma(rx^2 + y^2 + bz^2 - 2brz) \\ &= -2\sigma(rx^2 + y^2 + b(z - r)^2 - r^2 b) \\ &= -\frac{2\sigma}{r^2 b}(\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z - r)^2}{r^2} - 1). \end{aligned}$$

It is then clear that if at time t, (x, y, z) is outside the ellipsoid

$$K: \quad \frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z - r)^2}{r^2} \leq 1,$$

then $C(t)$ will decrease. We can pick C so large that the above ellipsoid is contained in

$$E: \quad rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C,$$

which is another ellipsoid, then eventually all trajectories will enter E.

The smallest possible value of C is obtained when the ellipsoids E and K are tangent. The picture is that one shrinks E by decreasing C until the surface of E touches K. This is equivalent to the following problem:

Given condition

$$\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} = 1,$$

find the maximum of

$$rx^2 + \sigma y^2 + \sigma(z-2r)^2.$$

The maximum is the smallest possible C. This problem can be solved using Langrange multipliers. In practice, given a pair of parameters r and σ, one can use a computer to handle the extreme value problem.

9.2.6 (Page 343)

a) Let $\mathbf{f} = (-\nu x + zy, -\nu y + (z-a)x, 1 - xy)$ be the instantaneous velocity, then

$$\begin{aligned} \nabla \cdot \mathbf{f} &= \frac{\partial}{\partial x}(-\nu x + zy) + \frac{\partial}{\partial y}(-\nu y + (z-a)x) + \frac{\partial}{\partial z}(1 - xy) \\ &= -\nu - \nu + 0 = -2\nu < 0, \end{aligned}$$

therefore the system is dissipative (cf. Figure 9.2.1).

b) A fixed points (x, y, z) satisfies that

$$zy = \nu x \tag{1}$$

$$(z-a)x = \nu y \tag{2}$$

$$1 = xy. \tag{3}$$

From (1) and (3) we have $z = \nu x^2$, then use (2)

$$(\nu x^2 - a)x^2 = \nu xy = \nu.$$

Since $xy = 1$, $x \neq 0$, so

$$a = \nu x^2 - \nu x^{-2} = \nu(k^2 - k^{-2}),$$

where $x^2 = k^2$, $xy = 1$, and $z = \nu x^2$. This is exactly the parametric form described in the problem.

c) The Jacobian $A = \begin{pmatrix} -\nu & z & y \\ z-a & -\nu & x \\ -y & -x & 0 \end{pmatrix}$ evaluates to $\begin{pmatrix} -\nu & \nu k^2 & k^{-1} \\ \nu k^2 - a & -\nu & k \\ -k^{-1} & -k & 0 \end{pmatrix}$ at fixed point $(k, k^{-1}, \nu k^2)$. The characteristic polynomial is

$$f(\lambda) = \lambda^3 + 2\nu\lambda^2 + [\nu^2 - \nu k^2(\nu k^2 - a) + k^2 + k^{-2})\lambda + 2\nu(k^2 + k^{-2}).$$

Using the relation $a = \nu(k^2 - k^{-2})$ to eliminate a, it follows that

$$f(\lambda) = \lambda^3 + 2\nu\lambda^2 + (k^2 + k^{-2})\lambda + 2\nu(k^2 + k^{-2}) = (\lambda + 2\nu)(\lambda^2 + k^2 + k^{-2}).$$

Therefore the eigenvalues are -2ν and $\pm\sqrt{k^2+k^{-2}}\,i$. Since there are two imaginary eigenvalues, the fixed points are centers.

9.3.8 (Page 344)

a) Yes. D is the set $r \leq 1$. It's clear from $\dot{r} = r(1-r^2)$ that r either stays at 0 or approaches 1 as $t \to \infty$. So trajectories starting from D will never leave the region.

b) Yes. Any open subset of D can be an open set of initial conditions that are attracted to D (in fact they never leave D). The basin of attraction (we can speak of that even we don't know D is an attractor) of D is the whole plane.

c) No. The circle $x^2+y^2=1$ is a proper subset of D. It's also invariant and attracts an open set of initial conditions (cf. part d).

d) Yes. First $r^*=1$ is a stable fixed point of $\dot{r}=r(1-r^2)$, so $x^2+y^2=1$ is invariant and any initial condition with $r>0$ will be attracted to $r=1$. Secondly notice that that $\dot{\sigma}=1$, trajectories starting on the circle will wind along it and never stop. So there can be no invariant proper subset of $x^2+y^2=1$.

9.4.2 (Page 344)

a) Because the graph of $f(x_n)$ looks like a tent.

b) The fixed points satisfy $x^*=f(x^*)$, Hence $x^*=0$ or $x^*=\frac{2}{3}$. Since the multiplier is $\lambda=f'(x^*)=\pm 2$, both these fixed points are unstable.

c) We can solve for $p=f(q), q=f(p)$ where $[p,q]$ constitutes a 2-cycle. Let $p<\frac{1}{2}$ and $q>\frac{1}{2}$, then $2p=q, p=2-2q$ gives $p=\frac{2}{5}, q=\frac{4}{5}$. Since the multiplier is $\lambda=[f(f(x))]'|_{x=p}=f'(p)f'(f(p))=f'(p)f'(q)=-4$, the 2-cycle is unstable.

d) Similarily, we can solve for the period-3 and period-4 points by noting that only one of these points in the cycle is greater than $\frac{1}{2}$. Hence $[\frac{2}{9},\frac{4}{9},\frac{8}{9}]$ and $[\frac{2}{17},\frac{4}{17},\frac{8}{17},\frac{16}{17}]$ are the period-3 and period-4 solutions. (In fact, the general period-n solutions has the form $[\frac{2^1}{2^n+1},\frac{2^2}{2^n+1},\ldots,\frac{2^n}{2^n+1}]$.) Since the multiplier for all a period-n orbit equal to $\prod_1^n f'(p_i)$, where p_i are the points on the orbit, the multiplier is always greater than 1 and all the periodic orbit are unstable.

10.1.2 (Page 388)

The fixed points satisfy $x^*=(x^*)^3$. Hence $x^*=0$ or $x^*=\pm 1$. The multiplier is $\lambda=f'(x^*)=3(x^*)^2$. The fixed point $x^*=0$ is stable since $|\lambda|=0<1$, and $x^*=\pm 1$ is unstable since $|\lambda|=3>1$. If we keep pressing the appropriate function key of a pocket calculater, unless we start at ± 1, eventually we get very close to 0.

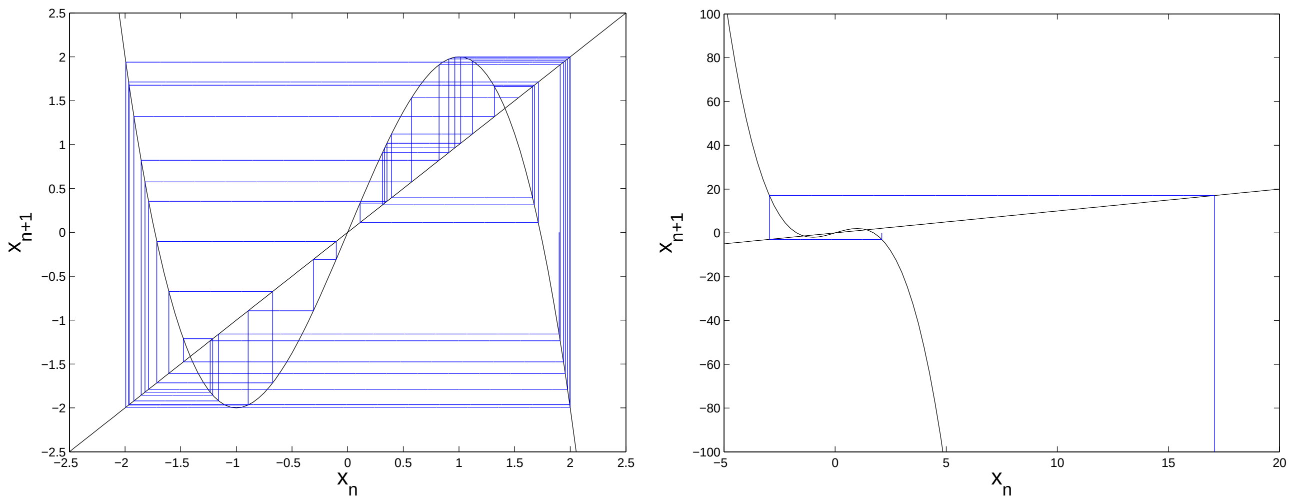
10.1.6 (Page 388)

Line $y = x$ intersects $y = \tan x$ infinitely many times, so there are infinitely many fixed points. For any fixed point x^* where $x^* = \tan x^*$, the multiplier $\lambda = f'(x^*) = \sec^2(x^*)$ is always greater than 1 unless $x^* = 0$. So they're all unstable. Even in the marginal case $x^* = 0$, it's clear from the cobweb that x^* is also unstable since the positive part of the curve $y = \tan x$ containing $(0, 0)$ is always above $y = x$ and the negative part below it. Therefore unless we start with a fixed point, we are not likely to get a pattern no matter how many times we press the button.

10.1.11 (Page 388)

a) Fixed points satisfy $x^* = f(x^*) = 3x^* - (x^*)^3$, therefore $x^* = 0, \pm\sqrt{2}$. The multiplier $\lambda = 3(1 - (x^*)^2)$ is always greater than 1, so all these fixed point are unstable.

b,c) Here are the cobweb graphs for $x_0 = 1.9$ (left) and $x_0 = 2.1$ (right).



d) Note that $f(x)$ has local extrema equal to ± 2 at $x = \pm 1$. From the graphs above, we can see that if we start with an initial values x_0 where $|x_0| < 2$, the cobweb will stay inside the square with corners at $(\pm 2, \pm 2)$. However, if we start an initial value x_0 where $|x_0| > 2$, then after the cobweb misses the first peak/valley of $f(x)$, x_n will get larger and larger.

10.3.4 (Page 390)

a) Fixed points satisfy $x^* = (x^*)^2 + c$, therefore

$$x^* = \frac{1 \pm \sqrt{1 - 4c}}{2},$$

where $c \le \frac{1}{4}$. The multiplier $\lambda = 2x^* = 1 \pm \sqrt{1-4c}$ is always greater than 1 at the greater fixed point, denoted by x_1. So x_1 is unstable. At the other fixed point x_2 we have $-1 < \lambda < 1$ when $-\frac{3}{4} < c < \frac{1}{4}$, so x_2 is stable when $c > -\frac{3}{4}$, and unstable when $c < -\frac{3}{4}$.

b) It's clear from part a) that a saddle-node bifurcation occurs at $c = \frac{1}{4}$, where two fixed points are created, and a flip bifurcation occurs at $c = -\frac{3}{4}$, where x_2 loses its stability.

c) To get the 2-cycles we apply $f(x) = x^2 + c$ to itself and obtain the equation

$$x = f(f(x)) = (x^2 + c)^2 + c.$$

This is a quartic equation, but recall that all the fixed points should satisfy this equation and the fixed points are also roots of $x = x^2 + c$. So we write $x = f(f(x))$ as

$$(x^2 - x + c)(x^2 + x + c + 1) = 0,$$

and get the other two roots

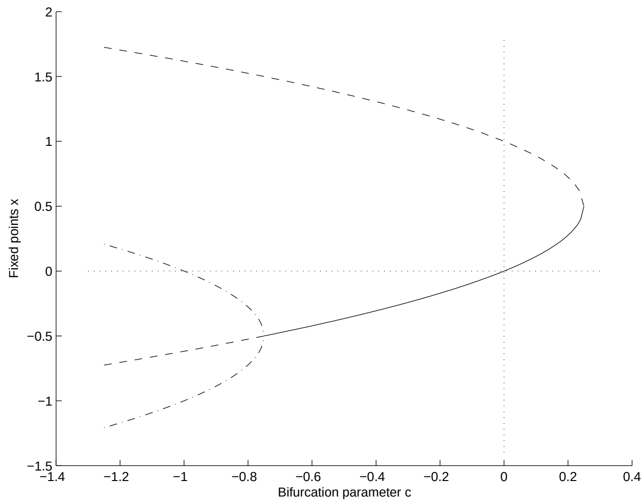
$$p, q = \frac{-1 \pm \sqrt{-3-4c}}{2},$$

which are real for $c < -\frac{3}{4}$, thus a 2-cycle exists for all $c < -\frac{3}{4}$. The multiplier of the 2-cycle is

$$\lambda = \frac{d}{dx}(f(f(x))_{x=p} = f'(p)f'(q) = 4pq.$$

Notice that p, q are roots of $x^2 + x + c + 1$, so $pq = c + 1$. Then $|\lambda| < 1$ if $-\frac{5}{4} < c < -\frac{3}{4}$, those are the values of c for which the 2-cycle is stable. The 2-cycle is superstable when $\lambda = 0$, i.e. $c = -1$.

d) In the bifurcation diagram below, solid lines indicate stable fixed points, dashed lines indicate unstable fixed points, the pair of dashed-dot lines indicate stable 2-cycles. The diagram is drawn according to the results in shown part a, b and c. Notice that the stable part is also part of the orbit diagram.



10.3.6 (Page 390)

a) Fixed points satisfy $x^* = rx^* - (x^*)^3$, therefore

$$x^* = 0, \pm\sqrt{r-1},$$

where $r \geq 1$. The multiplier $\lambda = r - 3(x^*)^2$ which equal to r for $x^* = 0$ and $3 - 2r$ for $x^* = \pm\sqrt{r-1}$. For $|r| < 1$, zero fixed point is stable, while for $1 < r < 2$ the fixed point $x^* = \pm\sqrt{r-1}$ is stable.

b) Suppose $f(p) = q$ and $f(q) = p$ and let $s = q^2 - r$ then we have

$$\begin{aligned} r(rq - q^3) - (rq - q^3)^3 &= q \\ q[r(r - q^2) - q^2(r - q^2)^3 - 1] &= 0 \\ q[-rs + (s + r)s^3 - 1] &= 0 \\ q[s^4 + rs^3 - rs - 1] &= 0 \\ q(s-1)(s+1)(s^2 + rs + 1) &= 0 \\ q(q^2 - r - 1)(q^2 - r + 1)(q^4 - rq^2 + 1) &= 0 \end{aligned}$$

Solve for q and we have $q = 0, \pm\sqrt{r-1}, \pm\sqrt{r+1}, \pm\sqrt{\frac{r\pm\sqrt{r^2-4}}{2}}$. Note that the first three solutions are the fixed point. Therefore the 2-cycles are $[+\sqrt{r+1}, -\sqrt{r+1}]$and $[\pm\sqrt{\frac{r+\sqrt{r^2-4}}{2}}, \pm\sqrt{\frac{r-\sqrt{r^2-4}}{2}}$ for $r > 2$.

c) For the 2-cycle $[+\sqrt{r+1}, -\sqrt{r+1}]$, the multipler $\lambda = f'(p)f'(q) = (r - 3(r+1))^2 = (3+2r)^2$ which is always larger than 1 for $r > -1$. Therefore this 2-cycle is always unstable.

For $[\pm\sqrt{\frac{r+\sqrt{r^2-4}}{2}}, \pm\sqrt{\frac{r-\sqrt{r^2-4}}{2}}$, the multipler $\lambda = f'(p)f'(q) = 9 - 2r^2$ which sits between -1 and 1 for $2 < r < \sqrt{5}$. Therefore these 2-cycles are stable for this range of values of r.

d) In the bifurcation diagram below, solid lines indicate stable fixed points, the pair ofdashed-dot lines indicate stable 2-cycles.