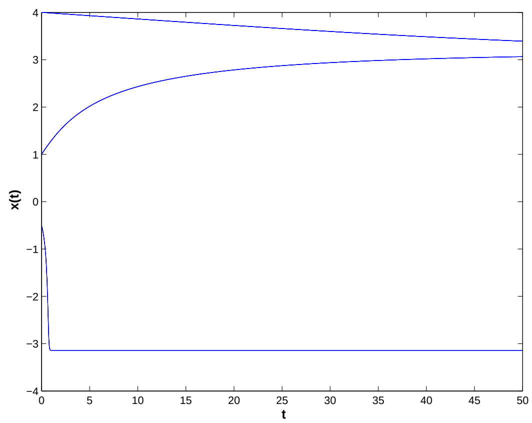
Solutions 1

2.2.4 (Page 37)

The fixed points for $\dot{x} = \exp(-x)\sin(x)$ are $x^* = n\pi$, where n is an integer. When $n = \ldots -4, -2, 0, 2, 4, \ldots$, x^* is unstable, while when $n = \ldots -3, -1, 1, 3, \ldots$, x^* is stable. The graph below shows the solution $x(t)$ for $x(0) = -0.5, 1$ and 4.



2.2.13 (Page 38)

Part a) Write $\dot{x} = \frac{dx}{dt}$, then we need to solve

$$\frac{mdv}{mg - kv^2} = dt.$$

Notice that

$$\frac{m}{mg - kv^2} = \frac{1}{2g}\left(\frac{1}{1 - \sqrt{\frac{k}{mg}}v} + \frac{1}{1 + \sqrt{\frac{k}{mg}}v}\right), \qquad \text{(a little tricky)}$$

we get solution

$$\sqrt{\frac{m}{4gk}}\left(\log \frac{1 + \sqrt{\frac{k}{mg}}v}{1 - \sqrt{\frac{k}{mg}}v}\right) = t + C.$$

Putting in the condition $v(0) = 0$ we get $C = 0$, therefore the analytical solution is

$$v = \frac{rm}{k}\left(\frac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}}\right), \qquad \text{where} \qquad r = \sqrt{gk/m}.$$

As this course does not emphasize on solving ODE, you can just solve it using some math software such as Matlab, Mathematica, etc.

Part b) When $t \to \infty$, both e^{-rt} terms in the above vanish and the big fraction becomes 1. The limit is whatever remained which turns out to be $(rm)/k = \sqrt{mg/k}$.

Part c) Now we solve it geometrically, the equation can be written as

$$\dot{v} = g - (k/m)v^2,$$

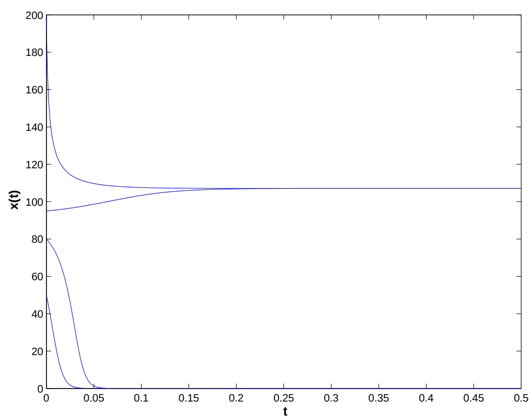
and we set it equal to 0. The graph of $\dot{v}$ versus v is a parabola crossing the x-axis from the above. The terminal velocity is the stable fixed point $v = \sqrt{mg/k}$.

2.3.4 (Page 39)

Part a) The effect growth rate is at its highest ($= r$) when $N = b$. If N is either too high or too low, then the effect growth rate will be negative.

Part b) Fixed points are $N_+^* = b + \sqrt{\frac{r}{a}}, N_-^* = b - \sqrt{\frac{r}{a}}$ (provided that $b > \sqrt{\frac{r}{a}}$) and $N_0 = 0$. Here N_+^* and N_0 are stable while N_-^* is unstable.

Part c) The graph below shows the solution $N(t)$ for $N(0) = 50, 80, 95$ and 200 for $r = 1, a = 0.02, b = 100$.



Part d) Note that when $N(0) > N_-^*$ the behaviour of $N(t)$ will be the same as the the solution of the logistic equation (approaches a non-zero fixed point). The different here is that when $N(0) < N_-^*$, tehn $N(t) \to 0$.

2.4.2 (Page 40)

The fixed points are 0, 1 and 2. Since $f'(x) = x(x-1) + x(x-2) + (x-1)(x-2)$, we have

$$\begin{aligned} f'(0) &= 2, && 0 \text{ is unstable,} \\ f'(1) &= -1, && 1 \text{ is stable,} \\ f'(2) &= 2, && 2 \text{ is unstable.} \end{aligned}$$

2.4.8 (Page 40)

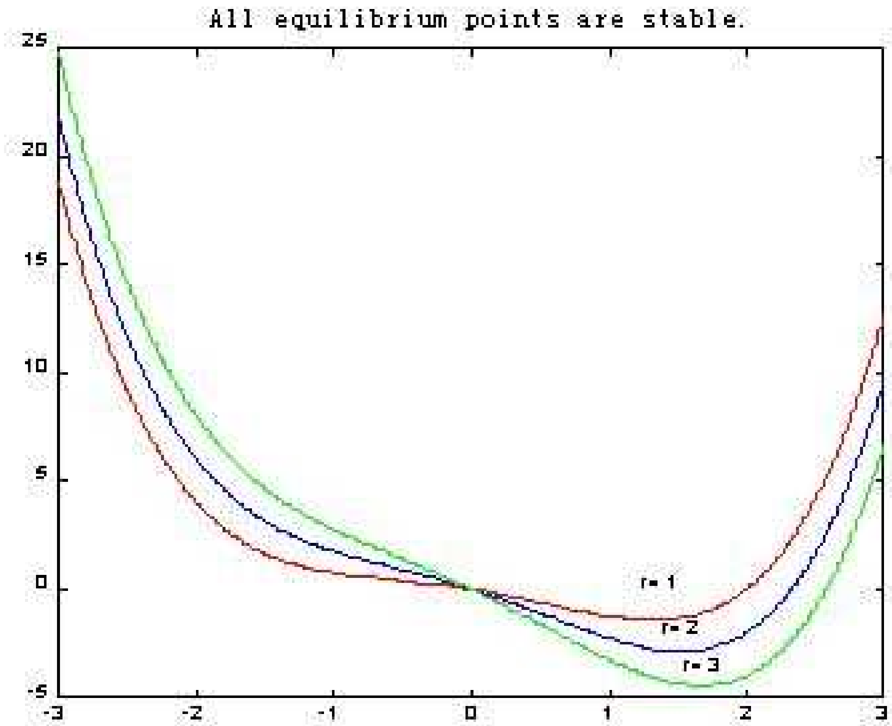
Letting $\dot{N} = 0$ we get $N = 1/b$. Taking derivative:

$$f'(N) = -a\ln(bN) - \frac{a}{b},$$

then $f'(1/b) = -\frac{a}{b} < 0$, $1/b$ is stable.

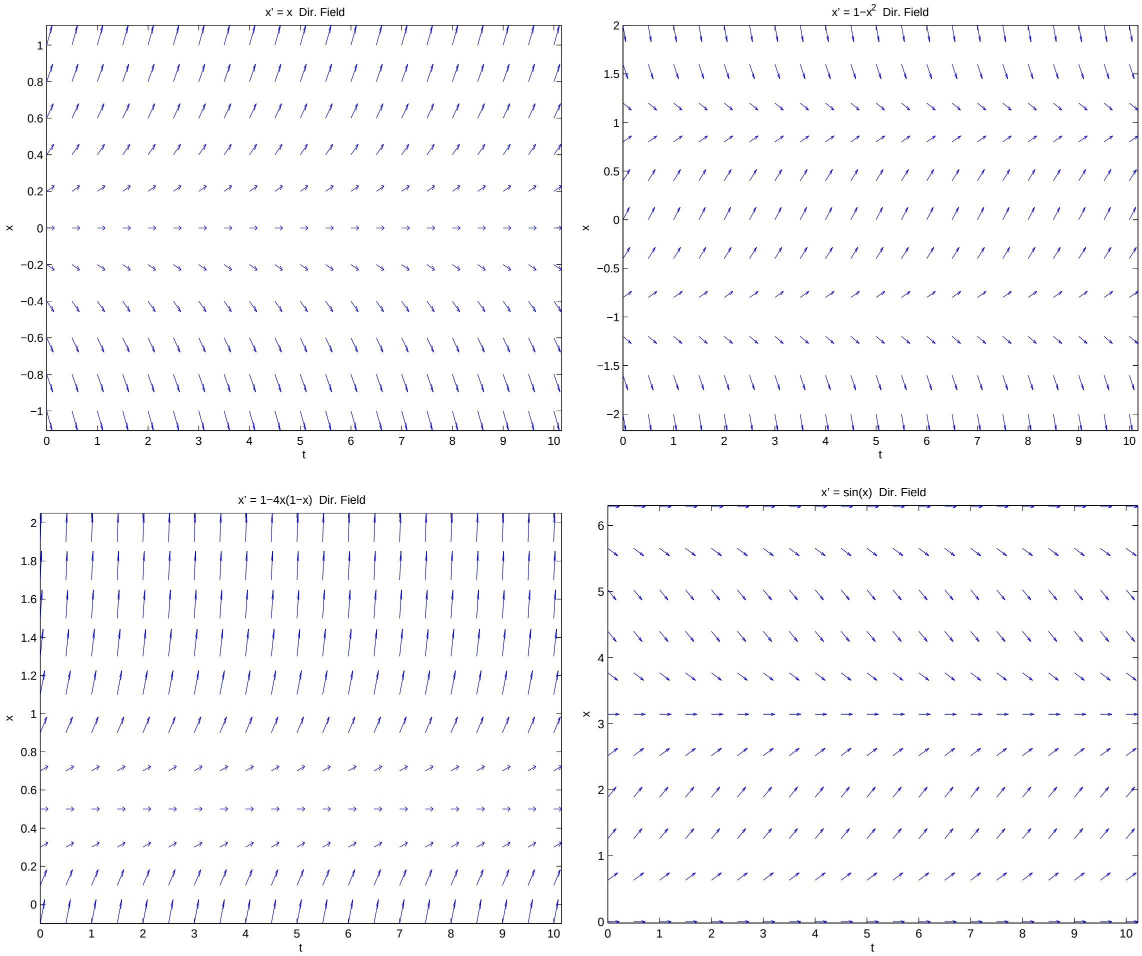
2.7.6 (Page 42)

Similar to the above $V(x) = -rx - \frac{1}{2}x^2 + \frac{1}{4}x^4$. Graphs of $V(x)$ for some r values are shown in the figure. The equilibrium points are the local minima.



2.8.2 (Page 42)

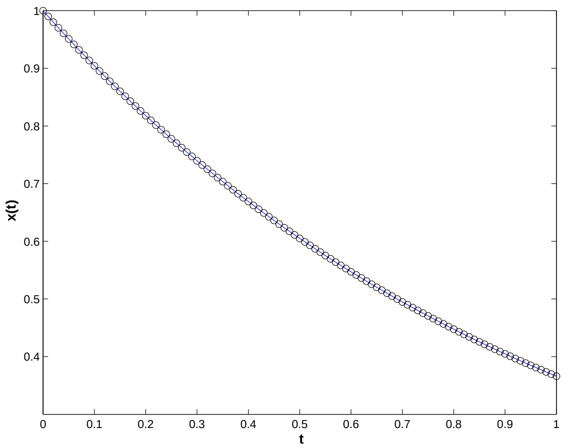
Plots of the slope fields for a) $\dot{x} = x$ (top left), a) $\dot{x} = 1-x^2$ (top right), a) $\dot{x} = 1-4x(1-x)$ (bottom right) and a) $\dot{x} = \sin(x)$ (bottom right).

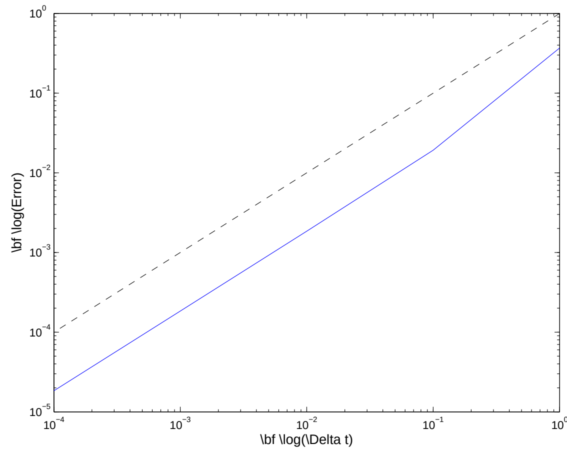


2.8.3 (Page 42)

Part a) The solution for $\dot{x} = -x, x(0) = 1$ is $x(t) = \exp(-t)$ and the exact value for $x(1)$ is e^{-1}.

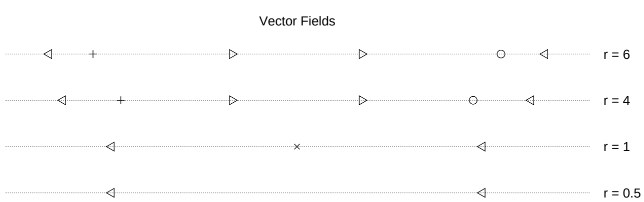
Part b & c) Left: The solution found using Euler method with step szie $\Delta t = 0.01$. Right: Log-log plot of the error E is a function of Δt (solid lines). Note that the dotted line represents the plot of Δt^{-1} due to the fact that the rate of convergence for Euler method is first order.

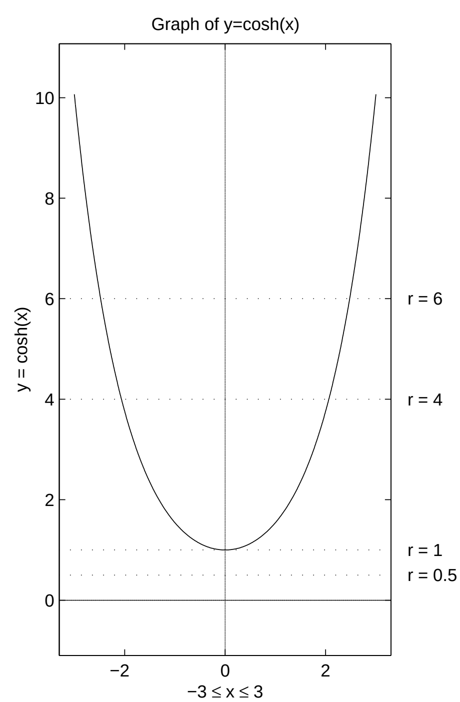




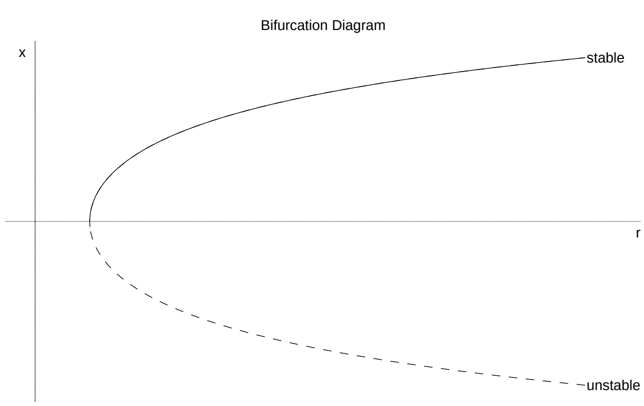
3.1.2 (Page 79)

The graph of the function $y = \cosh x$ is shown below on the right, with dotted lines indicating the values of r. It's then clear that x moves to the right when y is below r and vice versa. The vector fields are sketched as follows

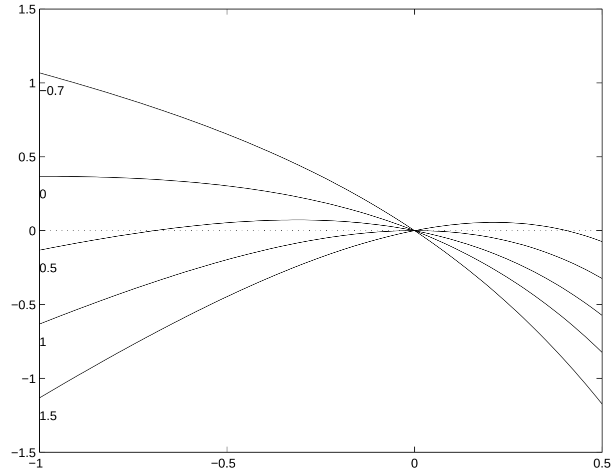




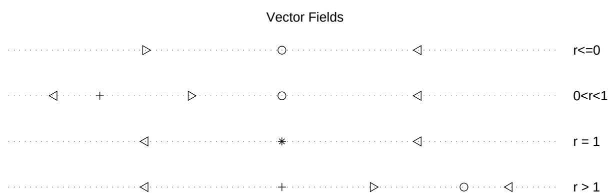
A bifurcation occurs at $r = 1$.



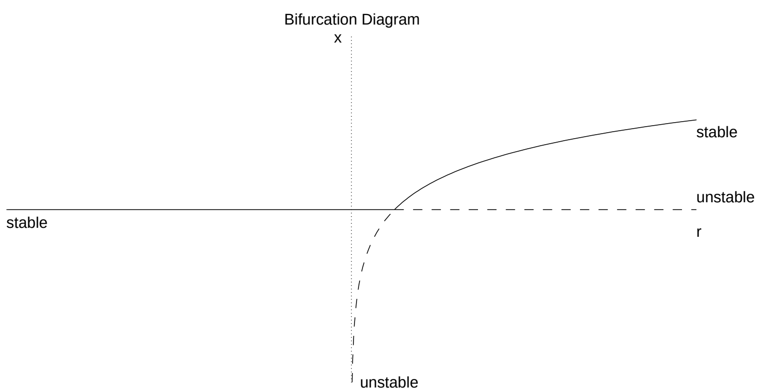
3.2.4 (Page 80) First we plot the graph of $\dot{x}$ versus x for various r and get the following picture:



It's then clear that the vector fields can be described qualitatively as follows:



Now we can draw the bifurcation diagram

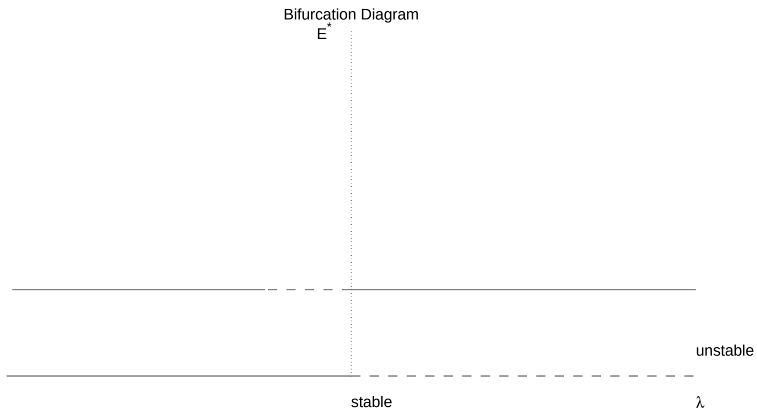


3.3.2 (Page 82)

Part a) Assume that $\dot{P} \approx 0, \dot{D} \approx 0$ then, to first order, $ED = P$ and $\lambda + 1 - \lambda EP = D$. Substitute the $D = \frac{E}{P}$ into the second equation we will have $P = \frac{(\lambda+1)E}{1+\lambda E^2}$. Since $\dot{E} = \kappa(P - E)$, the evolution equation of E is thus $E = \frac{\lambda\kappa(1-E^2)E}{1+\lambda E^2}$.

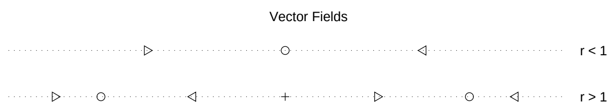
Part b) The fixed points of E are $E^* = 0$ and $E^* = 1$.

Part c) Bifurcation diagram

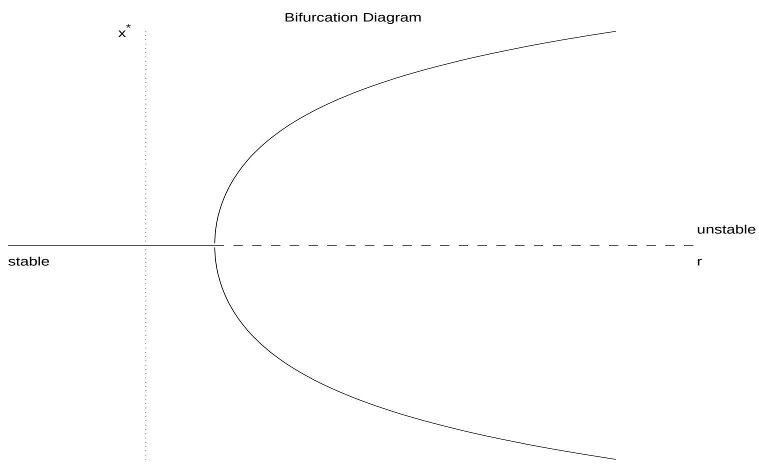


3.4.2 (Page 82)

Fixed points of x and r are related by $r = \frac{\sinh x}{x}$, its graph shows that the critical value is $r = 1$. There are two qualitatively different vector fields:

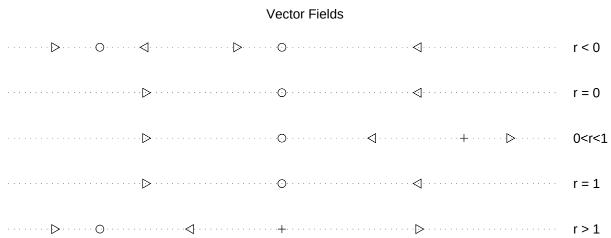


It's readily seen that 0 changes from stable to unstable after r passes critical value 1 and two more stable fixed points are created. This is a pitchfork bifurcation, it's supercritical.



3.4.6 (Page 83)

We start by solving equation $rx = x/(1+x)$, 0 is always a solution and another solution is given by $x = 1/r - 1$ as long as r is nonzero. The critical values for r are 0 and 1, representing the cases in which there is only one fixed point. The vector fields can be sketched as follows



At $r = 1$ two fixed points changed their types, this is a transcritical bifurcation.