Math 467 Midterm exam - Solutions

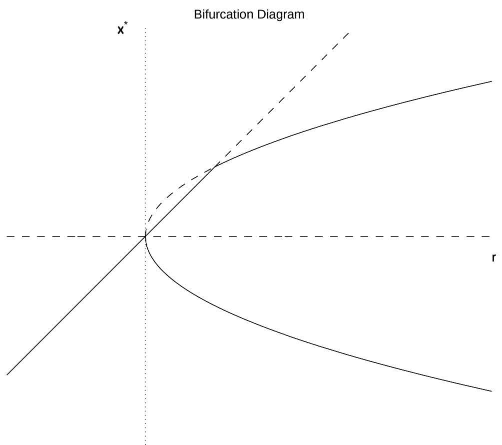
Problem 1.

(a) The fixed points are $x = 0$, $x = r$ and $x = \pm\sqrt{r}$ (the latter only for $r > 0$). By plotting the function $f(x) = (r - x^2)(rx - x^2)$ or by linear analysis, one can conclude:

- Case $r < 0$: $x = 0$ is unstable, $x = r$ is stable
- Case $0 < r < 1$, for which $-\sqrt{r} < 0 < r < \sqrt{r}$: $x = -\sqrt{r}$ and $x = r$ are stable, $x = 0$ and $x = \sqrt{r}$ are unstable.
- Case $r > 1$, for which $-\sqrt{r} < 0 < \sqrt{r} < r$: $x = -\sqrt{r}$ and $x = \sqrt{r}$ are stable, $x = 0$ and $x = r$ are unstable.

There are two bifurcations, one at $r = 0$, $x = 0$ and the other at $r = 1$, $x = 1$.

(b) The bifurcation diagram is:



(c) The bifurcation at $(1, 1)$ is clearly transcritical - there is an exchange of stability between $x = r$ and $x = \sqrt{r}$ as they pass through each other at $r = 1$. To analyze the bifurcation at $(0, 0)$ we need to make the following essential observation. At $r = 0$ the fixed points $x = 0$ and $x = r$ do not change stability, they do not "feel" the bifurcation at $r = 0$. It is a pair of fixed points, one stable ($x = -\sqrt{r}$) and one unstable ($x = \sqrt{r}$) that is being created at $r = 0$. Therefore, the bifurcation is of saddle-node type.

Problem 2.

(a) We basically need to discuss what the signs of the various terms in the right-hand-sides of the ODE system mean for the evolution of the variables R and J.

Firstly, in the right-hand-side of the R equation we have the term R and in the right-hand-side of the J equation we have the term J. This means that both Romeo and Juliet respond to their own feelings toward each other. For instance, if Romeo loves Juliet ($R > 0$), he will tend to love her even more.

Secondly, we have the term $-J$ in the equation for R. This means that Romeo responds in the following way to Juliet's feelings: the more Juliet hates him ($J < 0$), the more he feels attracted to her. The term R in the equation for J means that Juliet's response is: the more Romeo loves her, the more she loves him back.

(b) A simple calculation leads to $\tau = 2 > 0$, $\Delta = 2 > 0$, $\tau^2 - 4\Delta = -4 < 0$. Therefore the origin is an unstable spiral. A trajectory will spiral outward and the couple's feelings will pass repeatedly through the stages: love/love, hate/love, hate/hate, love/hate with increasing intensity of their affection. This can be seen also from part (c).

(c) Compute

$$\begin{aligned}\frac{dE}{dt} &= 2R\dot{R} + 2J\dot{J} \\ &= 2R(R - J) + 2J(R + J) \\ &= 2(R^2 + J^2) \\ &= 2E\end{aligned}$$

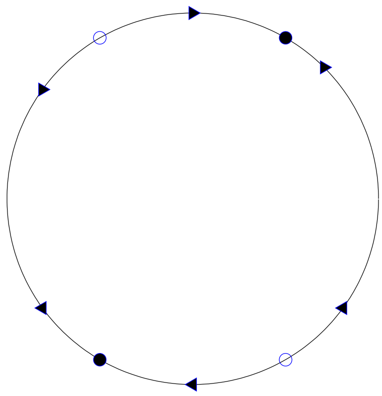
The ODE can be integrated easily. The solution is:

$$E(t) = E(0)e^{2t}.$$

This means that the "strength" of their combined feelings increases exponentially fast!

Problem 3.

The fixed points are $\theta = n\pi \pm \frac{\pi}{3}$, where n is an integer. By plotting the phase portrait, we can see that for $\theta = n\pi + \frac{\pi}{3}$, they are stable fixed points and for $\theta = n\pi - \frac{\pi}{3}$, they are unstable fixed points.



Problem 4.

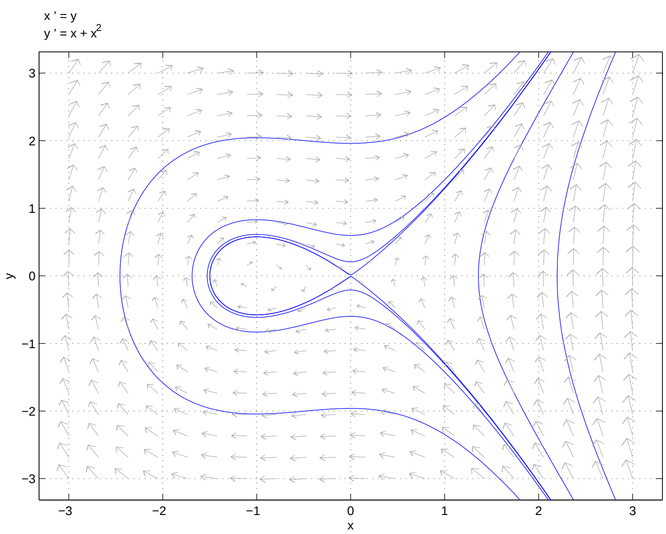
(a) The equilibrium points are $(x,y)=(0,0)$ and $(-1,0)$. The Jacobian of the system is $A=\begin{pmatrix}0 & 1\\ 1+2x & 0\end{pmatrix}$. At $(0,0)$, we find $A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, so $(0,0)$ is a saddle point. At $(-1,0)$, $A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, so $(-1,0)$ is a centre.

(b) Note that $\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{x+x^2}{y}$. Separate the variables and integrate on both sides, we have $E=\frac{y^2}{2}-\frac{x^2}{2}-\frac{x^3}{3}$ as a conserved quantity for the system.

(c) The unstable manifold of the saddle point at $(0,0)$ leaves the origin along the vector $\binom{-1}{-1}$. Hence, close to the origin, part of the unstable manifold lies in the third quadrant, where $\dot{x}<0, \dot{y}<0$ for $-1<x<0$. Therefore a phase point near the origin will first move down and to the left. It will reach the line $x=-1$, where the vertical velocity vanishes, then it will start moving up and to the left, eventually reaching $y=0$. Since the system is reversible under the transformation $t\to -t, y\to -y$, there must be a twin trajectory with the same end points but with the arrow reversed. Together the two trajectories form a homoclinic orbit.

(d) Since E is conserved and the homoclinic orbit passes through $(0,0)$, $E=0$ on the homoclinic orbit and its equation would be $\frac{y^2}{2}=\frac{x^2}{2}+\frac{x^3}{3}$.

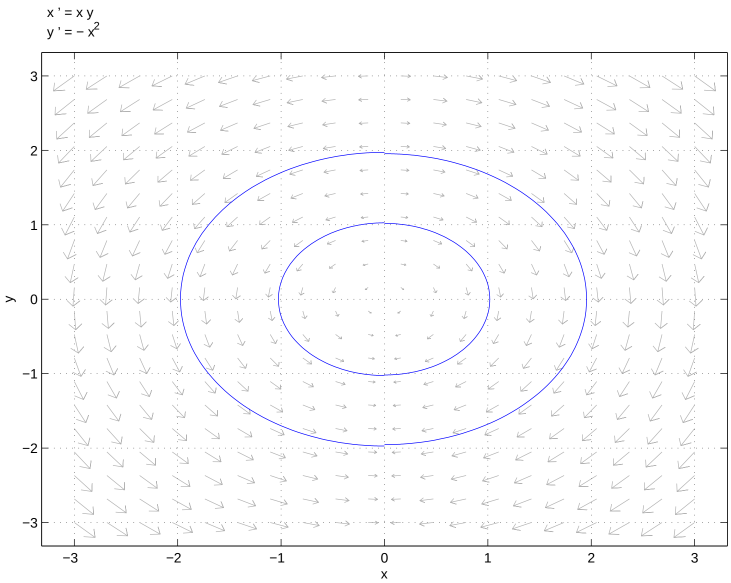
(e)



Problem 5.

(a) Note that $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-x^2}{xy} = \frac{-x}{y}$. Separate the variables and integrate on both sides, we have $E = y^2 + x^2$ as a conserved quantity for the system.

(b) The fixed points are the y-axis.



(c) By inspecting the signs of $\dot{x}$ and $\dot{y}$, an argument similar to the one used in Problem 4, part (c) might be made here as well. However, we can also use the fact that the trajectories lie on circles and also that they are not supposed to cross the line of fixed points $x = 0$ (otherwise the uniqueness would be violated). Therefore it becomes evident from the vector field that "starting" (as $t \to -\infty$) at the fixed point $(0, k)$, with $k > 0$, along either $(1, 0)$ or $(-1, 0)$, the trajectory will follow a semicircle and enter the fixed point $(0, -k)$ as $t \to \infty$. There exists a pair of such heteroclinic orbits for each $k > 0$.

(d) The fixed points are not isolated, violating one of the hypotheses of that theorem.