MATH 467-3 **Fall 2005**

Dynamical Systems

Second Midterm Exam 30 November 2005

Name: ______________________________

Instructions: Show all your work for full credit, and indicate your answers clearly. There are four questions, for a total of 70 points.
You may use a page of handwritten notes; calculators are not permitted.
Throughout the test, $\dot{x}$ means dx/dt.

1. (24 points)
Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(y-a). \end{aligned} \tag{1}$$

(a) Show that the above system (1) is equivalent to the second-order ODE

$$\ddot{x} - x\dot{x} + ax = 0.$$

In the remaining questions (which are independent of (a)), we will study the dynamics of system (1) for different values of a:

(b) For $a \neq 0$, find the fixed point(s), and determine their linear stability.

(c) Find a conserved quantity for the system (1).

(d) Show that the system is reversible, and that the line $y = a$ is invariant (that is, an initial condition on this line remains on the line).

(e) Hence sketch the phase portrait of (1) for $a > 0$. What is the (nonlinear) type of the fixed point(s)?

(f) Sketch the phase portrait of (1) for $a < 0$.

(g) Lastly, consider $a = 0$, that is, consider the system

$$\dot{x} = y$$
$$\dot{y} = xy$$

Sketch the phase portrait and show that there is a family of bounded heteroclinic orbits for $y < 0$, while all orbits for $y > 0$ are unbounded.

2. (14 points)
Consider the system

$$\dot{x} = \frac{3}{2}\alpha - y,$$
$$\dot{y} = 4x - y - x^2.$$

(a) Sketch the nullclines and briefly describe how the fixed points move as α increases.

(b) Find the fixed point(s) as a function of α.

(c) Find the bifurcation value α_c of α, and give the type of bifurcation.

(d) What is the index of the fixed point(s) at $\alpha = \alpha_c$?

(e) Sketch the phase portrait for $\alpha = \alpha_c$.

3. (8 points)
Consider the system

$$\dot{x} = 2xy^2 - x^3,$$
$$\dot{y} = -\frac{2}{3}x^2 y - y^3.$$

Show that the origin $(0, 0)$ is the unique fixed point, and that linear stability analysis is inconclusive regarding its stability. Use a Lyapunov function of the form $V(x, y) = \frac{1}{2}\left(ax^2 + y^2\right)$ to show that the origin is, in fact, globally asymptotically stable.

4. (24 points)
Consider the system

$$\dot{x} = a(1-2b)x + y - ax(x^2+y^2),$$
$$\dot{y} = -x + ay - ay(x^2+y^2),$$

where a and b are parameters that can be, for now, positive, negative or zero.
Assume for now (for parts (a)–(h)) that $|ab| < 1$ and $b < \frac{1}{2}$.
We will consider the effect of varying a on the stability of the fixed point at the origin and the possibility of closed orbits:

(a) Perform a linear analysis of the fixed point at the origin $(0,0)$, to determine the (linear) type and stability of the origin for $a < 0$, $a = 0$ and $a > 0$.

(b) Convert the system to polar coordinates (r, θ), using

$$r\dot{r} = x\dot{x} + y\dot{y}, \qquad \dot{\theta} = (x\dot{y} - y\dot{x})/r^2.$$

(c) Prove that for $b = 0$, there is exactly one limit cycle for $a > 0$.

For parts (d)–(g), assume $a > 0$, $b < \frac{1}{2}$ and $|ab| < 1$.
Be sure to try part (h) on the next page, which can be answered independently of (d)–(g).

(d) Find a circle of maximum radius r_1, with centre at the origin, such that the vector field has a radially *outward* component everywhere along this circle, that is, $\dot{r} \geq 0$.

(e) Find a circle of minimum radius r_2, with centre at the origin, such that the vector field has a radially *inward* component everywhere along this circle.

(f) Hence prove that there is at least one limit cycle for $a > 0$, $b < \frac{1}{2}$, $|ab| < 1$.

(g) If there is exactly one limit cycle, determine its stability.
Show that if there are several limit cycles, they all have the same period T depending only on a and b.

(h) Show that a bifurcation occurs at $a = 0$. What type is it?

(i) *Extra...*
What behaviour do you expect for $a < 0$?
For further investigation of this model, discuss the (possible) implications of relaxing the conditions $|ab| < 1$ and/or $b < \frac{1}{2}$.