MATH 467-3 **Fall 2005**

Dynamical Systems

First Midterm Exam 24 October 2005

Name: ____________________

Instructions: Show all your work for full credit, and indicate your answers clearly. There are five questions *on two pages*, for a total of 50 points.
You may use a page of handwritten notes; calculators are not permitted.
Throughout the test, $\dot{x}$ means dx/dt.

1. (10 points)

   (a) The equation $\dot{x} = f(x)$ has precisely 5 fixed points, m of which are stable. What are the possible values of m?

   (b) Does there exist an equation of the form $\dot{x} = f(x)$ for which the unique extremum of the potential function $V(x)$ is a minimum at $x = 3$? If so, give an example.

   (c) Give an example (or if none exists, explain why not) of an equation $\dot{x} = f(x)$ which has the following properties:
   (i) If $x(0) = 1$, then $x(t) = 1$ for all $t \geq 0$.
   (ii) If $x(0) = 0$, then $x(t) = 1$ for all $t \geq T$ for some *finite* value T.

   (d) Find an equation of the form $\dot{\theta} = f(\theta, r)$ so that all solutions are oscillatory for $r > 0$, and the period of oscillation blows up like $r^{-1/2}$ as $r \to 0^+$.

2. (8 points)
   Consider the equation
   $$\dot{x} = -x(1-x)(1-r+x^2).$$
   Find all the fixed points as functions of r. Determine any value(s) of r at which a bifurcation occurs, and classify the type(s) of bifurcation. Sketch the bifurcation diagram.

3. (12 points)
   Consider the dynamical system
   $$\dot{x} = x + \frac{rx}{1+x^2} \quad .$$

   (a) Find all fixed points as functions of the parameter r, and determine the stability of the fixed point at the origin.

   (b) Determine r_c, the critical value(s) of r at which a bifurcation occurs.

   (c) Determine the type of bifurcation.

   (d) Sketch the bifurcation diagram.

   (e) Find the change of variables which transforms the dynamical system into the normal form for this type of bifurcation.

4. (12 points)
Consider the quadratic map $x_{n+1} = f(x_n) = x_n^2 + c$.

   (a) By drawing a cobweb diagram or otherwise, describe the behaviour of the iterates x_n beginning at $x_0 = 0$, if (i) $c = 2$, (ii) $c = 0$, (iii) $c = -1$.

   (b) Find all the fixed points of the quadratic map as a function of the parameter c.

   (c) Determine the stability of the fixed points you found in (b). For which c are they superstable? Find any values of c at which the fixed points bifurcate, and classify those bifurcations. What behaviour do you expect after the bifurcations?

5. (8 points + bonus for (c))
A simple model of calcium-induced calcium release (CICR) is given by

$$\frac{dc}{dt} = L + \frac{k_1 c^2}{k_2^2 + c^2} - k_3 c \ ,$$

where c is the concentration of Ca^{2+}, L is a constant leak of calcium, and the k_i are positive constants.
[A mathematically equivalent model describes a biochemical switch.]

   (a) Show that the system can be rewritten in nondimensional form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2},$$

   for suitable defined dimensionless quantities $r > 0$ and $s \geq 0$.

   (b) Show that when $s = 0$, there are two positive steady states if $r > r_c$, where r_c is to be determined. What is r_c, and what bifurcation occurs at r_c?

   (c) (Bonus) Now consider what happens as s is increased from 0. Assuming that initially $x(0) = 0$, what happens to $x(t)$ as L is slowly increased? Show that there is a critical value s_c of s so that for $s > s_c$, only one positive solution exists. What happens to $x(t)$ as s is then decreased below s_c again?