Computation of atomic fibers

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Generalizing the relation $\sqsubseteq$

For $u, v \in \mathbb{R}^n$ let $u \sqsubseteq v$ iff $u^{(i)} v^{(i)} \geq 0$ and $|u^{(i)}| \leq |v^{(i)}|$ for $i = 1, \ldots, n$.

Let $S, T, U \subseteq \mathbb{R}^n$. We say that $S = T \oplus U$ if for every $z \in S$ there are $z_1 \in T$ and $z_2 \in U$ with $z = z_1 + z_2$ and $z_1, z_2 \sqsubseteq z$.

We are interested in all indecomposable sets $\{z \in \mathbb{Z}^n : Az = 0\}$ or $\{z \in \mathbb{Z}^n : Az = 0, z \geq 0\}$.

These indecomposable sets generalize the notions of Graver bases and Hilbert bases.

What are atomic fibers?

$A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$

$P^I_{A,b} := \{z : Az = b, z \in \mathbb{Z}^n_+\}$ — b-fiber of A

$Q^I_{A,b} := \{z : Az = b, z \in \mathbb{Z}^n\}$ — extended b-fiber of A

We call $P^I_{A,b}$ an atomic, if $P^I_{A,b}$ cannot be written as $P^I_{A,b_1+b_2} = P^I_{A,b_1} \oplus P^I_{A,b_2}$ with two other fibers P^I_{A,b_1} and P^I_{A,b_2}.

Analogously, we define $Q^I_{A,b}$ to be an extended atomic fiber.

By $F(A)$ and by $E(A)$ denote the atomic and the extended atomic fibers of A, respectively.

Twisted cubic $A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

$(0,3)$	$\{(0,0,0,1)\}$
$(1,2)$	$\{(0,0,1,0)\}$
$(2,1)$	$\{(0,1,0,0)\}$
$(3,0)$	$\{(1,0,0,0)\}$
$(2,4)$	$\{(0,1,0,1),(0,0,2,0)\}$
$(3,3)$	$\{(1,0,0,1),(0,1,1,0)\}$
$(4,2)$	$\{(0,2,0,0),(1,0,1,0)\}$
$(3,6)$	$\{(1,0,0,2),(0,1,1,1),(0,0,3,0)\}$
$(4,5)$	$\{(0,2,0,1),(0,1,2,0),(1,0,1,1)\}$
$(5,4)$	$\{(1,1,0,1),(0,2,1,0),(1,0,2,0)\}$
$(6,3)$	$\{(2,0,0,1),(1,1,1,0),(0,3,0,0)\}$
$(4,8)$	$\{(0,2,0,2),(1,0,1,2),(0,1,2,1),(0,0,4,0)\}$
$(6,6)$	$\{(2,0,0,2),(0,3,0,1),(1,1,1,1),(1,0,3,0),(0,2,2,0)\}$
$(8,4)$	$\{(2,1,0,1),(0,4,0,0),(1,2,1,0),(2,0,2,0)\}$
$(6,9)$	$\{(2,0,0,3),(0,3,0,2),(1,1,1,2),(1,0,3,1),(0,2,2,1),(0,1,4,0)\}$
$(9,6)$	$\{(3,0,0,2),(1,3,0,1),(2,1,1,1),(2,0,3,0),(1,2,2,0),(0,4,1,0)\}$
$(6,12)$	$\{(2,0,0,4),(0,3,0,3),(1,1,1,3),(1,0,3,2),(0,2,2,2),(0,1,4,1),(0,0,6,0)\}$
$(12,6)$	$\{(4,0,0,2),(2,3,0,1),(3,1,1,1),(3,0,3,0),(2,2,2,0),(0,6,0,0),(1,4,1,0)\}$

Why are $F(A)$ and $E(A)$ finite?

Theorem. (Maclagan, 1999) Any infinite family of monomial ideals in $k[x_1,\ldots,x_n]$ contains two ideals I,J with $I\subseteq J$.

Corollary. Any family of monomial ideals in $k[x_1,\ldots,x_n]$ contains only finitely many inclusion-maximal ideals.

Corollary. Any sequence $\{I_1,I_2,\ldots\}$ of monomial ideals in $k[x_1,\ldots,x_n]$ with $I_i\supsetneq I_j$ whenever $i<j$ is finite.

And consequently...?!

$$\mathcal{I}_{A,b} := \langle x^u : u \in P^I_{A,b} \rangle$$

Then

$$P^I_{A,b_1+b_2} = P^I_{A,b_1} \oplus P^I_{A,b_2}$$

if and only if

$$\mathcal{I}_{A,b_1+b_2} \subseteq \mathcal{I}_{A,b_1}.$$

Consequently, there are only finitely many inclusion-maximal ideals $\mathcal{I}_{A,b}$ corresponding to the finitely many atomic fibers of A.

Algorithm to compute $E(A)$

Input: $F := \{\pm Ae_1, \ldots, \pm Ae_n\}$ Output: $E(A)$

$G := F$ $C := \bigcup_{f,g \in G} \{f + g\}$

while $C \neq \emptyset$ do

  $s :=$ an element in C $C := C \setminus \{s\}$

  $f := \mathrm{normalForm}(s, G)$

  if $f \neq 0$ then

   $G := G \cup \{f\}$ $C := C \cup \bigcup_{g \in G} \{f + g\}$

return $G \cup \{0\}$.

Normal form algorithm

Input: s, G

Output: a normal form of s with respect to G

while there is some $g \in G$ such that $Q^I_{A,s} = Q^I_{A,g} \oplus Q^I_{A,s-g}$ do

$s := s - g$

return s

Idea of proof

$M^I_{A,\bar{b}} = \{z_1, \ldots, z_k\}$ $\sqsubseteq$-minimal elements in $Q^I_{A,\bar{b}}$

From all representations

$$Q^I_{A,\bar{b}} = \sum_{j \in J} Q^I_{A,b_j} \quad \text{with } Q^I_{A,b_j} \in G$$

choose one such that the sum

$$\sum_{i=1}^{k} \sum_{j \in J} \|v_{i,j}\|_1,$$

where $z_i = \sum_{j \in J} v_{i,j}$ and $v_{i,j} \in Q^I_{A,b_j}$, $i = 1, \ldots, k$, is minimal.

$$Q^I_{A,s} = Q^I_{A,g} \oplus Q^I_{A,s-g}?$$

Theorem. Let P be a polyhedron. Then there exist a polytope Q and a cone C such that

$$P \cap \mathbb{Z}^n = Q \cap \mathbb{Z}^n + C \cap \mathbb{Z}^n.$$

$$\begin{aligned} Q^I_{A,s} &= Q^I_{A,g} \oplus Q^I_{A,s-g} \\ M^I_{A,s} + M^I_{A,0} &= M^I_{A,g} + M^I_{A,0} \oplus M^I_{A,s-g} + M^I_{A,0} \end{aligned}$$

Thus, we only need to test whether for all $u \in M^I_{A,s}$ there exists a $v \in M^I_{A,g}$ with $v \sqsubseteq u$.

Let's talk about applications

Atomic integer programs

$\min\{c^\intercal z : Az = b, z \in \mathbb{Z}_+^n\}$ atomic integer program associated to $P_{A,b}^I$.

If

$$P_{A,b}^I = \bigoplus_{i=1}^k \alpha_i P_{A,b_i}^I, \quad \alpha_i \in \mathbb{Z}_+,$$

then

$$\bar{z} := \sum_{i=1}^k \alpha_i z_i^{\text{opt}}$$

is an optimal solution to

$$\min\{c^\intercal z : Az = b, z \in \mathbb{Z}^n\}.$$

Stochastic programming

$\longrightarrow$ optimization under uncertainty

$\longrightarrow$ optimization of immediate plus expected costs

$$A_N = \begin{pmatrix} T & W & & \\ \vdots & & \ddots & \\ T & & & W \end{pmatrix}$$

Graver basis of A_N

Question: Does the Graver basis of A_N become arbitrarily complicated as N increases?

$$A_N = \begin{pmatrix} T & W & & \\ \vdots & & \ddots & \\ T & & & W \end{pmatrix}$$

$$(u, v_1, \ldots, v_N) \in \ker(A_N) \Leftrightarrow (u, v_i) \in \ker(T|W) \quad \forall i$$

Refined question: Is there a N_0 such that there are no new "building blocks" u or v_i in the Graver bases of A_N, $N \geq N_0$?

Answer: Yes! (H. & Schultz, 2003)

Aschenbrenner & H. (2004): The same holds true also in the multi-stage situation.

Computation of building blocks

Extended atomic fibers of

$$A = \begin{pmatrix} I & 0 \\ T & W \end{pmatrix}$$

encode exactly all necessary building blocks.

Open Problem

$$b = (b_1, \ldots, b_N)^\intercal \in E(A_N)$$

The b_i are building blocks of extended atomic fibers.

Question: Do the extended atomic fibers of A_N become arbitrarily complicated as N increases?

Refined question: Is there a N_0 such that there are no new "building blocks" b_i in the right-hand side vectors defining extended atomic fibers of A_N, $N \geq N_0$?

Conjecture: Yes.

Conjecture: The same holds true also in the multi-stage situation.