Balanced Weighing Matrices

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Summary

Preliminaries

Definition. Weighing Matrix

A $v \times v$ $(-1, 0, 1)$-matrix W such that

$$WW^t = kI_v.$$

Write $W(v, k)$.

- $W(v, v)$ is a *Hadamard matrix*
- $W(v, v-1)$ is a *conference matrix*

- A $W(19,9)$:

$$W = \begin{pmatrix}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{pmatrix}$$

- Verify W is a weighing matrix:

$$WW^t = \begin{pmatrix}
\mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{9}
\end{pmatrix}$$

Definition: Balanced Incomplete Block Design

- A binary $v \times b$ $(0,1)$-matrix A such that:
  1. $AA^t = rI_v + \lambda(J_v - I_v)$, and
  2. $J_vA = kJ_v$.

  Write 2-(v,k,λ)-*design*.
- The design is symmetric if $v = b$ (equiv. $k = r$).

- A symmetric 2-$(19, 9, 4)$-*design*:

$$A = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{pmatrix}$$

- Verify A is a symmetric design:

$$AA^t = \begin{pmatrix}
\mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9} & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{9}
\end{pmatrix}$$

- Verify A is a symmetric design:

$$JA = \begin{pmatrix}
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9
\end{pmatrix}$$

Definition. Balanced Weighing Matrices

- If W is a $W(v, k)$, then W is balanced if $|W|$ is the incidence matrix of a symmetric 2-(v, k, λ)-*design*, $\lambda = k(k-1)/(v-1)$.
- Write $BW(v, k)$.

- Our example $W(19,9)$ is a $BW(19,9)$:

$$W = \begin{pmatrix}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{pmatrix}$$

- The “classical” parameters from relative difference sets.
- The $BW(19,9)$ W is not constructable from a relative difference set. Computationally found by Gibbons and Mathon (1987).

- Previous state of the art:

Theorem. RDS construction of BWs

There is a BW with parameters

$$\left(\frac{q^{d+1}-1}{q-1}, q^d\right)$$

whenever (1) q odd and d arbitrary and (2) q and d even.

(1) Nonlinear hyperplanes of $GF(q^{d+1}) : GF(q)$ due to Bose (1942).

(2) Lifting of a "Waterloo decomposition" of classical difference sets due to Arasu, et al. (1995).

Novel Construction of Weighing Matrices

- Equivalencies of weighing matrices (and BWs):
  - permutations of rows
  - permutations of columns
  - negation of rows
  - negation of columns
- Every weighing matrix is equivalent to one of the following form

$$\begin{pmatrix} \mathbf{0} & R \\ \mathbf{1} & D \end{pmatrix}.$$

- R is the residual-part.
- D is the derived-part.

- Our example $BW(19,9)$:

$$W = \left(\begin{array}{c|cccccccccccccccccc}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
\hline
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{array}\right)$$

Definition. Simplex code

- q a prime power and $d > 0$.
- Form matrix G with columns given by reps. of 1-D subspaces of $GF(q^{d+1})$.
- The simplex code is $\mathcal{S}_{q,d} = row(G)$.

Proposition. Hamming weight

$wt(x) = q^d$ for all $x \in \mathcal{S}_{q,d}/\{\mathbf{0}\}$.

- Ingredients of construction:
  - A normalized $W(v, q)$ (seed matrix) with residual-part R and derived-part D.
  - A $W((q^{d+1} - 1)/(q - 1), q^d)$, say W.
  - A simplex code $\mathcal{S}_{q,d}$.

- Recipie of construction:
  - Form $A = W \otimes R$.
  - Form B by replacing elements of $\mathcal{S}_{q,d}$ by rows of D.
  - Then
$$\begin{pmatrix} \mathbf{0} & A \\ \mathbf{1} & B \end{pmatrix}$$
    is a $W((v-1)(q^{d+1}-1)/(q-1)+1, q^{d+1})$.

Theorem. (Kharaghani, et al., 2022b)

If there is a $W(v,q)$, then there is a weighing matrix with parameters

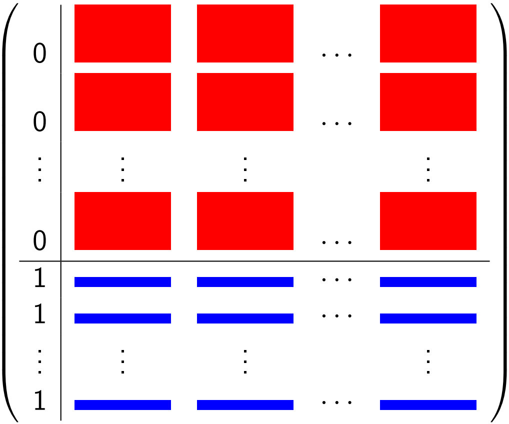
$$\left(\frac{(v-1)(q^{d+1}-1)}{q-1}+1, q^{d+1}\right)$$

whenever:

(1) q is odd and every $d > 0$, and

(2) q and d are both even.

Seed (v, k)	Succident (v', k')	Seed (v, k)	Succident (v', k')
$(6, 5)$:	$(31, 25)$, $(156, 125)$, $(781, 625)$	$(16, 3)$:	$(69, 9)$, $(196, 27)$, $(601, 81)$
$(8, 5)$:	$(43, 25)$, $(218, 125)$	$(16, 5)$:	$(91, 25)$, $(466, 125)$
$(8, 7)$:	$(57, 49)$, $(400, 343)$	$(16, 7)$:	$(121, 49)$, $(856, 343)$
$(10, 5)$:	$(55, 25)$, $(280, 125)$	$(16, 9)$:	$(151, 81)$
$(10, 9)$:	$(91, 81)$, $(820, 729)$	$(16, 11)$:	$(181, 121)$
$(12, 5)$:	$(67, 25)$, $(342, 125)$	$(16, 13)$:	$(211, 169)$
$(12, 7)$:	$(89, 49)$, $(628, 343)$	$(18, 13)$:	$(239, 169)$
$(12, 9)$:	$(111, 81)$	$(19, 9)$:	$(181, 81)$
$(13, 9)$:	$(121, 81)$	$(20, 7)$:	$(153, 49)$
$(14, 9)$:	$(131, 81)$	$(20, 13)$:	$(267, 169)$
$(14, 13)$:	$(183, 169)$		


0
0
0
1
1
1

A New Class of BWs

- Our example $BW(19,9)$:

$$
W = \left(\begin{array}{c|cccccccccccccccccc}
0 & + & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - \\
\hline
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{array}\right)
$$

$$R_1 = \begin{pmatrix}
+ & + & + & + & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 & 0 & 0 & + \\
- & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + & + & 0 & 0 \\
- & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 & 0 & + & 0 \\
0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + & 0 & + & 0 \\
+ & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + & 0 & 0 & + \\
0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & - & + & 0 & 0 \\
0 & + & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & - & + & + \\
0 & 0 & + & + & 0 & 0 & - & 0 & - & 0 & 0 & + & + & 0 & 0 & + & - & + \\
+ & 0 & 0 & 0 & + & 0 & - & - & 0 & + & 0 & 0 & 0 & + & 0 & + & + & -
\end{pmatrix}$$

$$|R_1| = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&1&1&0&1&0&0&0&1&1&1&1&0&1&0&0&0&1 \\ 1&0&1&0&0&1&1&0&0&1&1&1&0&0&1&1&0&0 \\ 1&1&0&1&0&0&0&1&0&1&1&1&1&0&0&0&1&0 \\ 0&0&1&0&1&1&0&1&0&0&0&1&1&1&1&0&1&0 \\ 1&0&0&1&0&1&0&0&1&1&0&0&1&1&1&0&0&1 \\ 0&1&0&1&1&0&1&0&0&0&1&0&1&1&1&1&0&0 \\ 0&1&0&0&0&1&0&1&1&0&1&0&0&0&1&1&1&1 \\ 0&0&1&1&0&0&1&0&1&0&0&1&1&0&0&1&1&1 \\ 1&0&0&0&1&0&1&1&0&1&0&0&0&1&0&1&1&1 \end{pmatrix}$$

$$|R_2| = J - |R_1| = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \\ 1&0&0&1&0&1&1&1&0&0&0&0&1&0&1&1&1&0 \\ 0&1&0&1&1&0&0&1&1&0&0&0&1&1&0&0&1&1 \\ 0&0&1&0&1&1&1&0&1&0&0&0&0&1&1&1&0&1 \\ 1&1&0&1&0&0&1&0&1&1&1&0&0&0&0&1&0&1 \\ 0&1&1&0&1&0&1&1&0&0&1&1&0&0&0&1&1&0 \\ 1&0&1&0&0&1&0&1&1&1&0&1&0&0&0&0&1&1 \\ 1&0&1&1&1&0&1&0&0&1&0&1&1&1&0&0&0&0 \\ 1&1&0&0&1&1&0&1&0&1&1&0&0&1&1&0&0&0 \\ 0&1&1&1&0&1&0&0&1&0&1&1&1&0&1&0&0&0 \end{pmatrix}$$

$$R_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & + & + & + & + & + & + \\
- & 0 & 0 & + & 0 & + & + & + & 0 & 0 & 0 & 0 & + & 0 & - & + & - & 0 \\
0 & - & 0 & + & + & 0 & 0 & + & + & 0 & 0 & 0 & - & + & 0 & 0 & + & - \\
0 & 0 & - & 0 & + & + & + & 0 & + & 0 & 0 & 0 & 0 & - & + & - & 0 & + \\
+ & + & 0 & - & 0 & 0 & + & 0 & + & + & - & 0 & 0 & 0 & 0 & + & 0 & - \\
0 & + & + & 0 & - & 0 & + & + & 0 & 0 & + & - & 0 & 0 & 0 & - & + & 0 \\
+ & 0 & + & 0 & 0 & - & 0 & + & + & - & 0 & + & 0 & 0 & 0 & 0 & - & + \\
+ & 0 & + & + & + & 0 & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & 0 & 0 \\
+ & + & 0 & 0 & + & + & 0 & - & 0 & - & + & 0 & 0 & + & - & 0 & 0 & 0 \\
0 & + & + & + & 0 & + & 0 & 0 & - & 0 & - & + & - & 0 & + & 0 & 0 & 0
\end{pmatrix}$$

$$
W' = \left(\begin{array}{c|cccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & + & + & + & + & + & + \\
0 & - & 0 & 0 & + & 0 & + & + & + & 0 & 0 & 0 & 0 & + & 0 & - & + & - & 0 \\
0 & 0 & - & 0 & + & + & 0 & 0 & + & + & 0 & 0 & 0 & - & + & 0 & 0 & + & - \\
0 & 0 & 0 & - & 0 & + & + & + & 0 & + & 0 & 0 & 0 & 0 & - & + & - & 0 & + \\
0 & + & + & 0 & - & 0 & 0 & + & 0 & + & + & - & 0 & 0 & 0 & 0 & + & 0 & - \\
0 & 0 & + & + & 0 & - & 0 & + & + & 0 & 0 & + & - & 0 & 0 & 0 & - & + & 0 \\
0 & + & 0 & + & 0 & 0 & - & 0 & + & + & - & 0 & + & 0 & 0 & 0 & 0 & - & + \\
0 & + & 0 & + & + & + & 0 & - & 0 & 0 & + & 0 & - & + & - & 0 & 0 & 0 & 0 \\
0 & + & + & 0 & 0 & + & + & 0 & - & 0 & - & + & 0 & 0 & + & - & 0 & 0 & 0 \\
0 & 0 & + & + & + & 0 & + & 0 & 0 & - & 0 & - & + & - & 0 & + & 0 & 0 & 0 \\
\hline
+ & 0 & 0 & 0 & + & 0 & - & + & - & 0 & 0 & - & - & 0 & + & 0 & 0 & 0 & + \\
+ & 0 & 0 & 0 & - & + & 0 & 0 & + & - & - & 0 & - & 0 & 0 & + & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0 & - & + & - & 0 & + & - & - & 0 & + & 0 & 0 & 0 & + & 0 \\
+ & + & - & 0 & 0 & 0 & 0 & + & 0 & - & 0 & 0 & + & 0 & - & - & 0 & + & 0 \\
+ & 0 & + & - & 0 & 0 & 0 & - & + & 0 & + & 0 & 0 & - & 0 & - & 0 & 0 & + \\
+ & - & 0 & + & 0 & 0 & 0 & 0 & - & + & 0 & + & 0 & - & - & 0 & + & 0 & 0 \\
+ & + & 0 & - & + & - & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - \\
+ & - & + & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & + & + & 0 & 0 & - & 0 & - \\
+ & 0 & - & + & - & 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 & - & - & 0
\end{array}\right)
$$

- Let G be a finite group not containing the symbol 0.
- For $A \subseteq G$, identify $A = \sum_{g \in A} g$ in $\mathbb{Z}[G]$.
- For $A \in \mathbb{Z}[G]$, write $A^{(h)} = \sum_{g \in G} a_g g^h$.

- Let Θ be a $v \times v$ $(0, G)$-matrix.
- We interpret Θ as a matrix over $\mathbb{Z}[G]$.
- Define $\Theta^{(h)}$ by $\Theta_{ij}^{(h)}$.
- Write $\Theta^* = (\Theta^{(-1)})^t$.

Definition. Balanced generalized weighing matrices

- G a finite group of order n.
- A $v \times v$ $(0, G)$-matrix Θ is a $BGW(v, k, \lambda; G)$ if

$$\Theta\Theta^* = (k \cdot e)I + \frac{\lambda G}{n}(J - I).$$

Theorem. Classical BGWs

- Let q be a prime power and $d > 0$ an integer.
- For each q and d there is a BGW with parameters

$$\left(\frac{q^{d+1}-1}{q-1}, q^d, q^d - q^{d-1}\right)$$

over C_{q-1}.

- A $BGW(10, 9, 8; C_4)$:

$$\begin{pmatrix} 1 & a & 1 & a^3 & a & 0 & 1 & a & a & a \\ a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a & a \\ a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a \\ a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 \\ a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 \\ 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a \\ a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 \\ 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 \\ a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a \\ a^2 & a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 \end{pmatrix}$$

- Decomposition matrices:

$$\begin{pmatrix}
1 & a & 1 & a^3 & a & 0 & 1 & a & a & a \\
a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a & a \\
a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 & a \\
a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 & 1 \\
a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a & 0 \\
0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 & a \\
a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 & a^3 \\
1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a & 1 \\
a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1 & a \\
a^2 & a & 1 & a^2 & 0 & a & a^2 & a^2 & a^2 & 1
\end{pmatrix}$$

- Decomposition matrices:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Let Θ be a BGW with parameters

$$\left(\frac{9^{d+1}-1}{8}, 9^d, 9^d - 9^{d-1}\right)$$

over the aroup $C_4 = \langle a : a^4 = 1 \rangle$.

- Decompose Θ as

$$\Theta = \Theta_1 + a\Theta_a + a^2\Theta_{a^2} + a^3\Theta_{a^3},$$

where the Θ_is are disjoint $(0,1)$-matrices.

- Apply $R_1 \mapsto -R_2 \mapsto -R_1 \mapsto R_2 \mapsto R_1$.
- Form:

$$\begin{aligned} \Theta \otimes R_1 &= \Theta_1 \otimes R_1 + \Theta_a \otimes {R_1}^a + \Theta_{a^2} \otimes {R_1}^{a^2} + \Theta_{a^3} \otimes {R_1}^{a^3} \\ &= \Theta_1 \otimes R_1 - \Theta_a \otimes R_2 - \Theta_{a^2} \otimes R_1 + \Theta_{a^3} \otimes R_2 \end{aligned}$$

- Form D by substituting for the elements of $S_{9,d}$ the rows of the derived part of W_{19}.

- The matrix

$$\begin{pmatrix} \mathbf{0} & \Theta \otimes R_1 \\ \mathbf{1} & D \end{pmatrix}$$

is a balanced weighing matrix.

Theorem. (Kharaghani, et al., 2022a)

For every $d > 0$, there is a balanced weighing matrix with parameters

$$\left(\frac{9^{d+2} - 9}{4} + 1, 9^{d+1} \right).$$

- These are signings of some of the Ionin-type symmetric designs (Ionin, 2001).

Done!