Math 178 Notes on Symbolic Dynamics

References: Text Sections 10.4 - 10.6

$\mathcal{B}$ = the set of all binary sequences;

$$\mathcal{B} \;=\; \{\vec{a} = [.a_1a_2a_3\ldots] \text{ where } a_i \text{ is 0 or 1}\}$$

If $x \in [0,1]$, then $[x]_2 = .a_1a_2a_3\ldots$ is the binary (base 2) representation of x. Since

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots$$

we see that if x and y have binary representations that agree for the first k places (i.e., $[x]_2 = .a_1a_2\ldots a_kb_{k+1}b_{k+2}\ldots,\quad [y]_2 = .a_1a_2\ldots a_kc_{k+1}c_{k+2}\ldots$), then $|x-y| \le (1/2)^k$. Note also that if $[x]_2 = .a_1a_2a_3\ldots$, then $[2x]_2 = a_1.a_2a_3\ldots$, $[2^2x]_2 = a_1a_2.a_3a_4\ldots$, etc., and $[\frac{1}{2}x]_2 = .0a_1a_2a_3\ldots$ etc. (multiplying by 2 shifts the binary representation to the left, dividing by 2 shifts it to the right).

The saw tooth transformation;

$$S(x) \;=\; \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

The representation $\tilde{S}$ of S on $\mathcal{B}$;

$$\tilde{S}(.a_1a_2a_2\ldots) \;=.a_2a_3a_4\ldots \qquad \text{shift left and drop } a_1$$

So, the saw tooth transformation acts on numbers by shifting their binary sequences to the left (and dropping a_1).

The tent transformation;

$$T(x) \;=\; \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2-2x & \frac{1}{2} < x \le 1 \end{cases}$$

The representation $\tilde{T}$ of T on $\mathcal{B}$;

$$\tilde{T}(.a_1a_2a_2\ldots) \;=\begin{cases} .a_2a_3a_4\ldots & a_1 = 0 \qquad \text{shift} \\ .a_2^\star a_3^\star a_4^\star\ldots & a_1 = 1 \qquad \text{shift and conjugate} \end{cases}$$

Here,

$$a_i^\star \;=\; \begin{cases} 1 & \text{if } a_i = 0 \\ 0 & \text{if } a_i = 1 \end{cases}$$

The coordinate transformation $h(x)$;

$$h(x) \;=\sin^2\left(\frac{\pi x}{2}\right)$$

Over $\longrightarrow$

Fixed points for Logistic

Steps 1 and 2: Fixed points for $\tilde{T}$ and T

Suppose $\tilde{T}(\vec{a}) = \vec{a}$. Then,

$$\tilde{T}(a_1 a_2 a_3 \ldots) \;=\; a_1 a_2 a_3 \ldots$$

Case 1: $a_1 = 0$. Then $\tilde{T}(a_1 a_2 a_3 \ldots) = a_2 a_3 a_4 \ldots$. Comparing the left and right sides, $a_1 = a_2 \rightarrow a_2 = 0$ (since $a_1 = 0$). So then since $a_2 = a_3$, we must have that $a_3 = 0$. Continuing in this way we see that all $a_i = 0$. Thus, if $\tilde{T}(\vec{a}) = \vec{a}$ and $a_1 = 0$, then $\vec{a} = 000\ldots$. So the number x with $[x]_2 = 000\ldots$ is a fixed point of $T(x)$. This x is clearly 0. Check; $T(0) = 2(0) = 0$. Also, $h(0) = 0$ so 0 is a fixed point of $f(x) = 4x(1-x)$.

Case 2: $a_1 = 1$. Then $\tilde{T}(a_1 a_2 a_3 \ldots) = a_2^\star a_3^\star a_4^\star \ldots$. Comparing the left and right sides, $a_1 = a_2^\star \rightarrow a_2 = 1$ (since $a_1 = 0$). So then since $a_2 = a_3$, we must have that $a_3^\star = 1 \rightarrow a_3 = 0$. Continuing in this way we see that $\vec{a} = \overline{1010}$. Thus, if $\tilde{T}(\vec{a}) = \vec{a}$ and $a_1 = 1$, then $\vec{a} = \overline{1010}$. So the number x with $[x]_2 = \overline{1010}$ is a fixed point of $T(x)$.

Now, if x is such that $[x]_2 = \overline{1010}$, then $x - \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = 2(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots) = 2(\frac{1/4}{1\,-\,1/4}) = 2/3$. (Note how we found this by summing a geometric series; whenever the binary representation (or in any base)of a number repeats, we can determine that number by summing a geometric series.) Thus, 2/3 is a fixed point of $T(x)$;

$$T(2/3) \;=\; 2 - 2(2/3) \;= 2 - 4/3 \;=\; 2/3$$

Step 3: Convert to fixed point for Logistic

And so $\tilde{x} = h(2/3) = \sin^2(\frac{1}{2}\frac{2}{3}\pi) = \sin^2(\frac{\pi}{3}) = \frac{3}{4}$ is a fixed point of $f(x) = 4x(1-x)$;

$$f(3/4) \;=\; 4(3/4)(1 \;-\; 3/4) \;= 3/4$$