Introduction And
Overview Of Game Theory & Matrix Algebra

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Why study Networks?

Networks are everywhere:

- Many economic and political interactions are shaped by the structure of the relationships:
  - Trade of goods and services: many of markets are not centralized
  - Sharing of information about jobs, or asking favours and sharing risk
  - Transmission of viruses or opinions
  - Political or trade alliances
- They influence behaviour:
  - Crime, employment, voting, education, smoking, etc

Primary Questions

1. What do we know about network structure?
2. How do networks form?
3. Do the efficient networks form?
4. How do networks influence behaviour (and vice versa)?

Research Areas

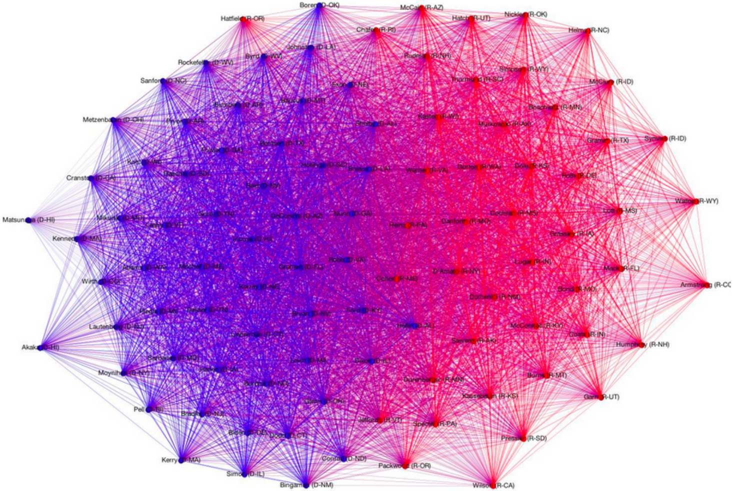
- Theoretical Research:
  - Network formation, network dynamics, design, ...
  - How does networks and individuals behaviour co-evolve?
- Empirical Research:
  - Observe networks and their patterns
  - Test theory and identify irregularities
- Methodological Research:
  - How to measure and analyze networks

Why Model Networks?

- Modelling provide insight into why we see certain phenomena:
  - Why do we observe "Small Worlds" phenomena?
- It also allows for comparative statics:
  - How does component structure change with density?
- Predict out of sample questions:
  - What will happen with vaccination policy?
- Allow for statistical estimation:
  - Is there significant clustering on a local level or did it appear at random?

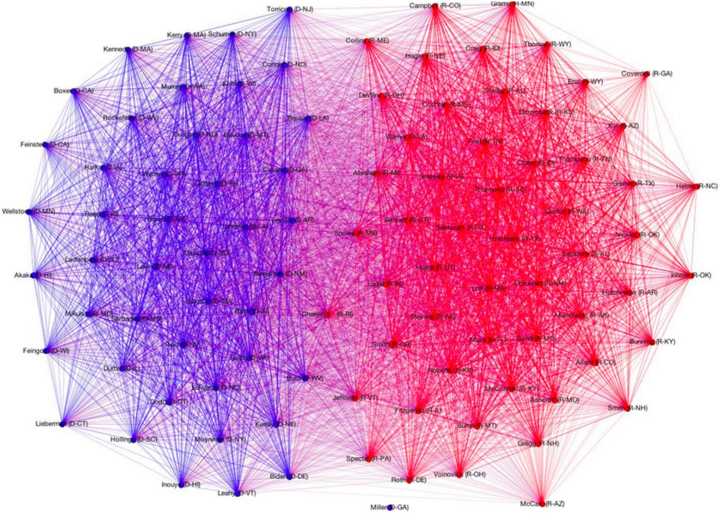
An Example: Lucioni 2013

Voting Relationships between Senators in 1990:



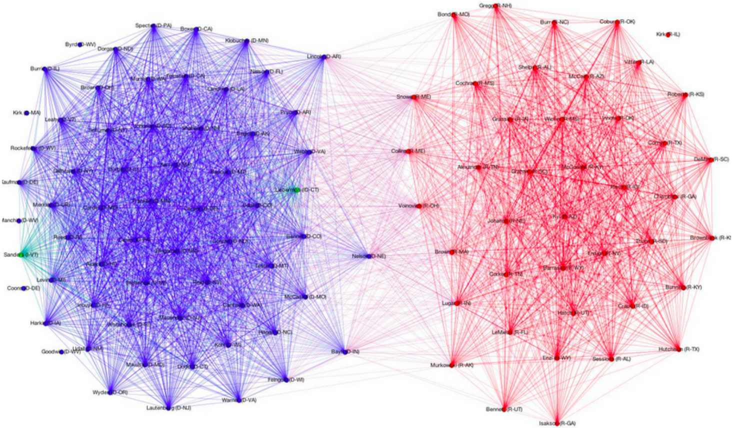
An Example: Lucioni 2013

Voting Relationships between Senators in 2000:



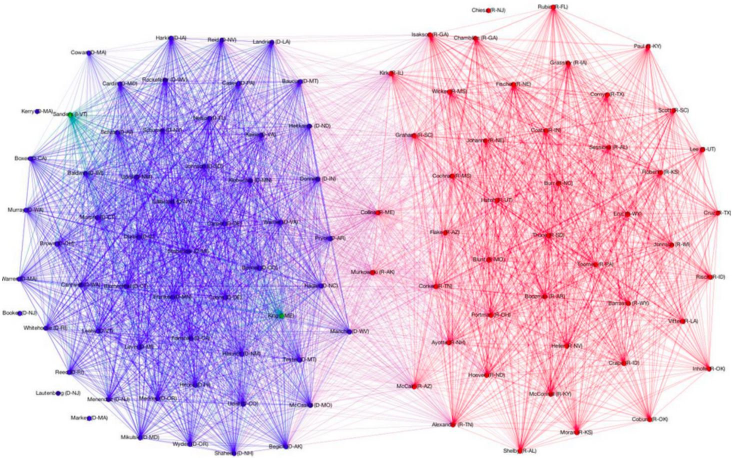
An Example: Lucioni 2013

Voting Relationships between Senators in 2010:



An Example: Lucioni 2013

Voting Relationships between Senators in 2013:



Course Outline

- Overview of Game Theory and Matrix Algebra (w1)
- Background & Fundamentals of Network Analysis
  - Why Model Networks? (w2)
  - Representing & Measuring Networks (w3)
  - Empirical Backgrounds on Networks (w4)
- Models of Network Formation
  - Random Graphs (w5)
  - Strategic Network Formation (w6)
- Implications of Network Structure
  - Diffusion & Learning (w8)
  - Behaviours & Games on Networks (w9)
  - Peer Effects (w10)
- Observing and Measuring Social Interactions (w11)
- Review & Finish Presentations (w12)

Course Evaluation

Your grade will be based on your performance in:

- Quiz from review material (i.e. this lecture): 5% of the final grade
- Midterm: 35% of the final grade
- Final: 40% of the final grade
- Class Presentations and participation: 20% of final grade

Midterm will be on March 2nd, 2015. Only if a student misses the midterm for documented medical reasons will the weight of that exam be transferred to the final.

Course Evaluation

- Required Text: Social and Economic Networks, Matthew O. Jackson, Princeton University Press, 2010.
- Practice assignments will be posted in the website. It is your responsibility to go through these practice questions and make sure that you could solve them. The midterm and final exam questions will be of similar difficulty level.
- Finally, there will be slides available on the course web site.

Presentations

I recommend a number of selected original papers to read on the topics we discuss. Each student is required to pick a *starred* paper to present at the class, starting on week 3.

- There will be 2 presentations per week. Each student has 45 min: 30 min for presentation and 15 min for Q&A.
- Each presenter should discuss:
  - Main questions studied in the paper,
  - Methodology (either empirical or theoretical),
  - Main findings of the paper,
  - Discussion & Summary.
- I request that all students review the two papers presented each week before the lecture.

Before We Begin

A few questions:

- Why are you taking this course? (Something other than you had to!)
- What was the most difficult thing about Game Theory for you? How about Interesting part?
- How many of you worked with "Matrices" before?

Review of Game Theory

Thanks to Anke Kessler & Shih En Lu

Suggested Textbooks:

- Eaton, Eaton and Allen: "Microeconomics"
- Varian: "Intermediate Microeconomics: A Modern Approach"

Game Theory

Game theory is a toolbox for studying strategic behaviour.

- Under perfect competition -as long as the price is known- one can maximize profit without worrying about other firms.
- But when there are a small number of firms that could impact the price (i.e. oligopoly), then each firm should take other firms' actions into account in order to maximize his own profit.

When Game Theory is Useful?

We see "strategic behaviour" whenever individual's own behaviour affects others' choices and that the outcome depends on others' actions as well as his own. Oligopoly, auctions, bargaining, public goods, etc are just a few examples of this.

Basic Concepts and Definitions

Basic Definitions

- A Game has players who choose actions available to them.
- An outcome is a collection of actions taken by each player.
- Each outcome generates a utility or payoff. A player cares about his expected payoff/utility.

A Strategy is:

- A complete contingent plan for every history of the game.
- A probability distribution over her actions:
  - A pure strategy is a strategy that puts probability 1 on a single action,
  - A mixed strategy is one where a player plays (some of) the available pure strategies with certain probabilities.

Simultaneous-Move Games & More

- In a Simultanous-move game, all players choose their strategies at the same time, without knowing others' strategies.
- Complete Information means that players know every player's payoff from each outcome (or action profile).
- Strictly Dominant Strategy is a strategy that results in the highest payoff for a player regardless of what other players do. It is the unique best choice no matter what others do.
- A player's strategy is Strictly Dominated if there is another strategy that gives him strictly better payoff for all combinations of actions by other players.

Dominance Solvability

Iterated Deletion of Strictly Dominated Strategies (IDSD)

- Take a player and delete his dominated strategies. Then look at the second player and delete his dominated strategies.
- Look at whether the other strategies become dominated with respect to the remaining strategies. If so, delete these newly dominated strategies.
- Repeat the process until there is no dominated strategy.
- A game is Dominance Solvable if IDSD leads to a unique predicted outcome, i.e. only one strategy for each player survives.
- The order of deleting dominated strategies does not matter.

Prisoner's Dilemma

- Prisoner's Dilemma is an example of Dominance Solvable games.

	Cooperate	Defeat
Cooperate	$-1, -1$	$-10, 0$
Defeat	$0, -10$	$-8, -8$

- Is the outcome Pareto Efficient?
- PD is a 2x2 game where:
  - Both players have a dominant strategy.
  - The outcome where both players play their dominated pure strategy is strictly better for both players than the outcome where they play their dominant strategy.

Best Response and Nash Equilibrium

- A Best Response Strategy is a strategy that results in the highest payoff for a player **given** other players' strategies.
- Nash Equilibrium is a strategy profile where every player is playing best response to other players' strategies.
- A Pure Strategy NE is where every player's strategy is pure.
- A Mixed Strategy NE is an equilibrium where at least one player randomizes over his or her actions.
- Some games only have mixed strategy NE (*Matching Pennies*); others have NE's in both pure and mixed strategies (*Chicken*).

Chicken I

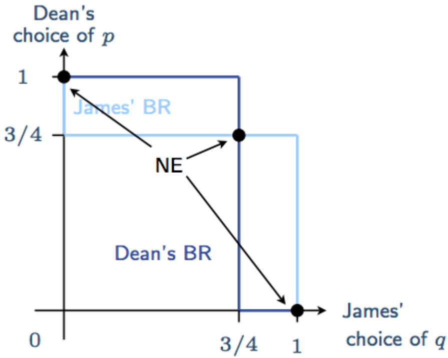
- Two teenagers drive toward each other at a high rate of speed, the driver that swerves first is deemed a chicken and loses face with the rest of the crowd.

	Swerve	Don't
Swerve	$0, 0$	$-1, 1$
Don't	$1, -1$	$-4, -4$

- Two NE in pure strategies: (Swerve,Don't) and (Don't, Swerve).
- There is a NE is mixed strategies as well.

	Swerve(prob p)	Don't(prob $1-p$)	1's exp payoff
Swerve (prob q)	$0, 0$	$-1, 1$	$p-1$
Don't (prob $1-q$)	$1, -1$	$-4, -4$	$5p-4$
2's exp payoff	$q-1$	$5q-4$	

Chicken II



James' best-response $q(p)$:
$p < 3/4 \rightarrow q = 1$, $p > 3/4 \rightarrow q = 1$
$p = 3/4 \rightarrow q \in (0, 1)$

Column's best-response $p(q)$:
$q < 3/4 \rightarrow p = 1$, $q > 3/4 \rightarrow q = 0$
$p = 3/4 \rightarrow p \in (0, 1)$

NE at intersection of
two best response functions
$p^* = 0, q^* = 1$, $p^* = 1, q^* = 0$,
$p^* = q^* = 3/4$

Matching Pennies I

- Matching Pennies is an example of Constant Sum games with no equilibrium in **pure strategies**.

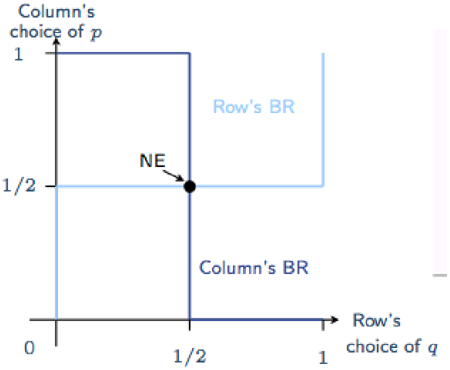
	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- Players randomize over their actions and play mixed strategies.

	Head (prob p)	Tail (prob $1-p$)	1's exp payoff
Head (prob q)	$1, -1$	$-1, 1$	$2p-1$
Tail (prob $1-q$)	$-1, 1$	$1, -1$	$1-2p$
2's exp payoff	$1-2q$	$2q-1$	

Matching Pennies II

- If $p > \frac{1}{2}$ player 1 is better off playing Head and if $p < \frac{1}{2}$, he is better off playing Tail.
- Player 1 would randomize **only if** player 2 plays Head with $p = \frac{1}{2}$.
- Randomization requires equality of expected payoffs.



Review of Matrix Algebra

Thanks to notes from Bertille Antoine

Suggested Textbooks:

- Sydsaeter and Hammond: "Essential mathematics for economic analysis".
- Hoy, Livernois, McKenna, Rees, Stengos: "Mathematical economics".

Basic Definitions

- A matrix is a rectangular array of real numbers, for example $\mathbf{A}$ is a matrix of order (or dimension) $m \times n$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

- A row vector is a matrix with only one row. A column vector is a matrix with only one column.

Addition and Scalar Multiplication

- If A and B are matrices of the same order, A + B is defined by:

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

- If p is a scalar (i.e. a real number) and $\mathbf{A}$ a matrix, then:

$$(pA)_{ij} = pA_{ij}$$

Matrix addition is commutative and associative:

- $A + B = B + A$
- $(A+B)+C = A+(B+C)$
- $p(A+B) = pA + pB$
- $(p+q)A = pA + qA$
- $p(qA) = (pq)A$
- $A(pB) = (pA)B = p(AB)$

Matrix Multiplication

If $\mathbf{A}$ is a $m \times n$ matrix, and if $\mathbf{B}$ is a $n \times p$ matrix with columns $\mathbf{b_1}, ..\mathbf{b_p}$, then the product $\mathbf{AB}$ is a $m \times p$ matrix whose columns are:

$$AB = A[b_1, ..., b_p] = [Ab_1, ..., Ab_p]$$

Row-Column rule for calculating AB

If $(AB)_{ij}$ denotes the (i,j)-entry in AB, then:

$$\begin{aligned}(AB)_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj}\end{aligned}$$

In General, $AB \neq BA$, and $A(BC) = (AB)C$ and $A(B + C) = AB + AC$.

Matrix Transpose

The transpose of a $k \times l$ matrix A, denoted A' , is the $l \times k$ matrix:

$$(A')_{ij} = A_{ji}$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \quad \Rightarrow \quad \mathbf{A}' = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$$

Assume that A, B denote matrices whose size is appropriate for the following sums and products:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Square matrices

- A matrix is square if it has as many rows as columns.
- A square matrix A is symmetric if $A = A'$, and diagonal if $A_{ij} = 0$ for all $i \neq j$.
- The identity matrix, I_n, is the diagonal matrix which has all diagonal elements equal to 1.
- The identity matrix is the neutral element of matrix multiplication: $IB = BI = B$

Square matrices: Inversion

- An $n \times n$ matrix A is said to be Inversable if there is an $n \times n$ matrix C such that $CA = I$ and $AC = I$.
- A matrix that is not invertible is called Singular Matrix.

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$, t hen A is invertable:

$$\mathbf{A^{-1}} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

.

- $(A^{-1})' = (A')^{-1}$ and $(AB)^{-1} = B^{-1}A^{-1}$.
- A nonsingular matrix A is orthogonal if $A^{-1} = A'$.