Mr. Madoff's Amazing Returns: *An Analysis of the Split-Strike Conversion Strategy*

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It is now known that the very impressive investment returns generated by Bernie Madoff were based on a sophisticated Ponzi scheme. Madoff claimed to use a split-strike conversion strategy. This strategy consists of a long equity position plus a long put and a short call. In this article we examine Madoff's returns and compare his investment performance with what could have been obtained using a split-strike conversion strategy based on the historical data. We also analyze the split-strike strategy in general and derive expressions for the expected return, standard deviation, Sharpe ratio and correlation with the market of this strategy. We find that Madoff's returns lie well outside their theoretical bounds and should have raised suspicions about Madoff's performance.

In December 2008, the investment operation of Bernie Madoff was exposed as a giant Ponzi scheme.[1] Madoff had attracted a wide following because he delivered consistently high returns with very low volatility over a long period. He claimed to use a split-strike conversion strategy to obtain these low-risk returns. This strategy involves taking a long position in equities together with a short call and a long put on an equity index to lower the volatility of the position. We know now that these returns were fictitious. The Madoff affair raises the obvious questions as to why it was not discovered earlier and why investors and regulators missed the various red flags. A number of these points are discussed in Markopolos [2009] and Gregoriou and Lhabitant [2009]. The paper of Clauss, Roncalli, and Weisang [2009] complements our work. It focuses on risk management implications of Madoff's fraud, in particular on the lack of regulation, and proposes to improve capital requirements for operational risk.

This article discusses certain aspects of the split-strike strategy and analyzes the reported performance of Madoff's funds. We analyze the Fairfield Sentry Ltd hedge fund, which was one of Madoff's feeder funds. Fairfield Sentry describes its strategy as follows.

> The Fund seeks to obtain capital appreciation of its assets principally through the utilization of a nontraditional options strategy described as a split-strike conversion to which the Fund allocates the predominant portion of its assets. The investment strategy has defined risk and reward parameters. The establishment of a typical position entails i) the purchase of a group or basket of securities that are intended to highly correlate to the S&P 100 Index, ii) the purchase of out-of-the-money S&P 100 Index put options with a notional value approximately equal to the market value of the basket of equity securities and iii) the sale of out-of-the-money S&P 100 Index call options with a

notional value approximately equal to the market value of the basket of equity securities. The basket typically consists of 40–50 stocks in the S&P 100 Index. The primary purpose of the long put options is to limit the market risk of the stock basket at the strike price of the long puts. The primary purpose of the short call options is to largely finance the cost of the put hedge and increase the stand-still rate of return. The "split-strike conversion" strategy[2] is implemented by Bernie L. Madoff Investment Securities LLC("BLM"), a broker dealer registered with the Securities and Exchange Commission through accounts maintained by the Fund in that firm. The services of BLM and its personnel are essential to the continued operation of the Fund and its profitability.

In the next section we analyze the performance of Fairfield Sentry for the period December 1990 to October 2008. The most dramatic aspect of the performance is the very low volatility of the returns. This in turn leads to an unusually high Sharpe ratio. We compare these returns with what could have been obtained by following a split-strike conversion strategy in real time using the actual historical returns and find that while the expected return is plausible, the volatility of the strategy in practice is much higher. We then analyze some theoretical properties of the distribution of returns of the split-strike strategy. In particular we develop closed-form expressions for the expected return, standard deviation, correlation and Sharpe ratio of this strategy. We also illustrate numerically the properties of the split-strike strategy.

ANALYSIS OF THE EMPIRICAL RESULTS

In this section we analyze the Fairfield Sentry return performance and contrast the reported returns with those that could be achieved using a split-strike conversion strategy. The reported monthly returns for the period December 1990 to October 2008 are given in Exhibit 1. These returns are amazingly consistent with an exceptionally low volatility. The monthly volatility is 71 basis points corresponding to an annual volatility of 2.45%. The average monthly return for the strategy is 84 bps corresponding to an annual average return of 10.59%.

EXHIBIT 1
Fairfield Sentry Monthly Returns from December 1990 to October 2008

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2008	0.63	0.06	0.18	0.93	0.81	−0.06	0.72	0.71	0.50	−0.06	na	na
2007	0.29	−0.11	1.64	0.98	0.81	0.34	0.17	0.31	0.97	0.46	1.04	0.23
2006	0.70	0.20	1.31	0.94	0.70	0.51	1.06	0.77	0.68	0.42	0.86	0.86
2005	0.51	0.37	0.85	0.14	0.63	0.46	0.13	0.16	0.89	1.61	0.75	0.54
2004	0.88	0.44	−0.01	0.37	0.59	1.21	0.02	1.26	0.46	0.03	0.79	0.24
2003	−0.35	−0.05	1.85	0.03	0.9	0.93	1.37	0.16	0.86	1.26	−0.14	0.25
2002	−0.04	0.53	0.39	1.09	2.05	0.19	3.29	−0.13	0.06	0.66	0.09	0.00
2001	2.14	0.08	1.07	1.26	0.26	0.17	0.38	0.94	0.66	1.22	1.14	0.12
2000	2.14	0.13	1.77	0.27	1.30	0.73	0.58	1.26	0.18	0.86	0.62	0.36
1999	1.99	0.11	2.22	0.29	1.45	1.70	0.36	0.87	0.66	1.05	1.54	0.32
1998	0.85	1.23	1.68	0.36	1.69	1.22	0.76	0.21	0.98	1.86	0.78	0.26
1997	2.38	0.67	0.80	1.10	0.57	1.28	0.68	0.28	2.32	0.49	1.49	0.36
1996	1.42	0.66	1.16	0.57	1.34	0.15	1.86	0.20	1.16	1.03	1.51	0.41
1995	0.85	0.69	0.78	1.62	1.65	0.43	1.02	−0.24	1.63	1.53	0.44	1.03
1994	2.11	−0.44	1.45	1.75	0.44	0.23	1.71	0.35	0.75	1.81	−0.64	0.60
1993	−0.09	1.86	1.79	−0.01	1.65	0.79	0.02	1.71	0.28	1.71	0.19	0.39
1992	0.42	2.72	0.94	2.79	−0.27	1.22	−0.09	0.85	0.33	1.33	1.35	1.36
1991	3.01	1.40	0.52	1.32	1.82	0.30	1.98	1.00	0.73	2.75	0.01	1.56
1990												2.77

Investors clearly put a very high value on this combination of high returns and low volatility.

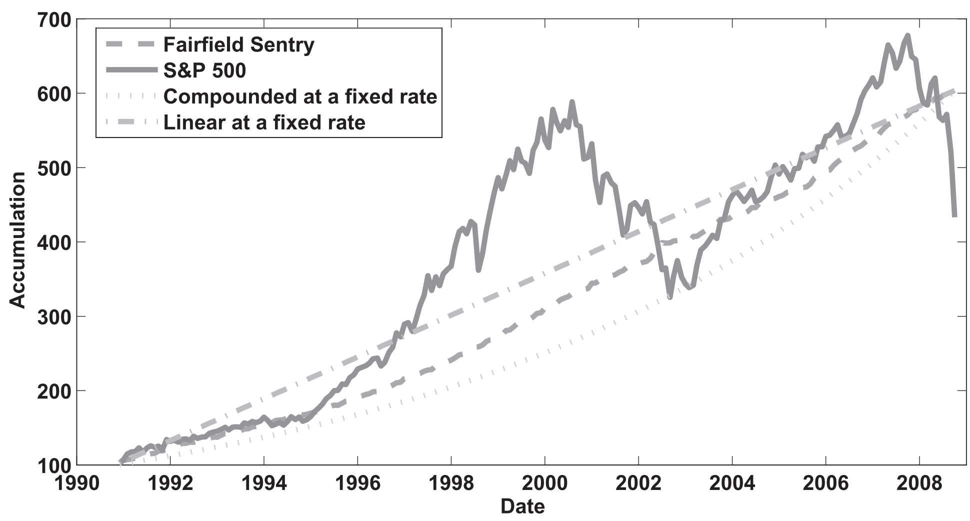
If these returns were in fact achievable they would dominate those obtained from investing directly in the S&P 500 Index, for instance. In fact investing in the index offers comparable returns with much higher volatility. Exhibit 2 compares the Fairfield Sentry (FS) performance with the strategy of investing directly in the S&P 500 with dividends reinvested over the period December 1990 to October 2008. It shows how an initial investment of 100 would have grown under both assumptions. One hundred invested in the FS fund would have accumulated to 603.8 by October 2008 whereas one hundred invested in the S&P would have accumulated to 433.03 by October 2008 reflecting a lower growth rate. The annual return from investing in the S&P has been 9.64% with a standard deviation of 14.28% over the 17 years and 11 months. We note that we assume no expenses in the S&P investment and it is not clear whether the Fairfield Sentry returns are net of expenses.[3]

We note that the growth of an investment in Fairfield Sentry is almost linear. We illustrate this aspect of the returns in Exhibit 2 by showing that the Fairfield Sentry growth rate lies inside two bounds. The top bound corresponds to a constant arithmetic growth rate of 2.343 per month starting at 100 in December 1990 to October 2008. The lower bound corresponds to a constant monthly compound rate of 83.98 bps per month. Note that $(1.008398)^{215} = 6.0377$. The actual performance of the FS Fund lies inside these two bounds.

Given the consistently high returns and the incredibly low volatility of the FS returns it is of interest to examine what sort of returns could be expected under a split-strike strategy during this period. To do so, we assume that a hypothetical investor takes a long position in the S&P 500 Index starting in December 1990. At the same time he buys a put option on the index and sells short a call option on the S&P 500 Index.[4] For the purpose of illustration, we assume that the strike price of the put is 5% below the initial spot price of the index and that the strike price of the call is 5% above the initial spot price of the index.[5] Both options are assumed to be European and have a one-month maturity. Typically the call price will exceed the put price and the proceeds are invested for one month at the risk-free rate. At the end of the month the option positions are settled in cash. If there is a loss under the option strategy it is financed in the first place from the risk-free investment of the net option premiums and, if that is not enough, by selling enough shares of the index.

EXHIBIT 2

Accumulated Investment Proceeds Using the Fairfield Sentry Returns Given in Exhibit 1 and Investment in the S&P 500 with Dividends Reinvested



Notes: The top line shows how the initial 100 accumulates to 603.78 in October 2008 if the dollar increase is constant and equal to 2.343 per month. The bottom line shows how the initial 100 in December 1990 accumulates to 603.78 in October 2008 under a constant monthly compound rate of 83.98 bps per month.

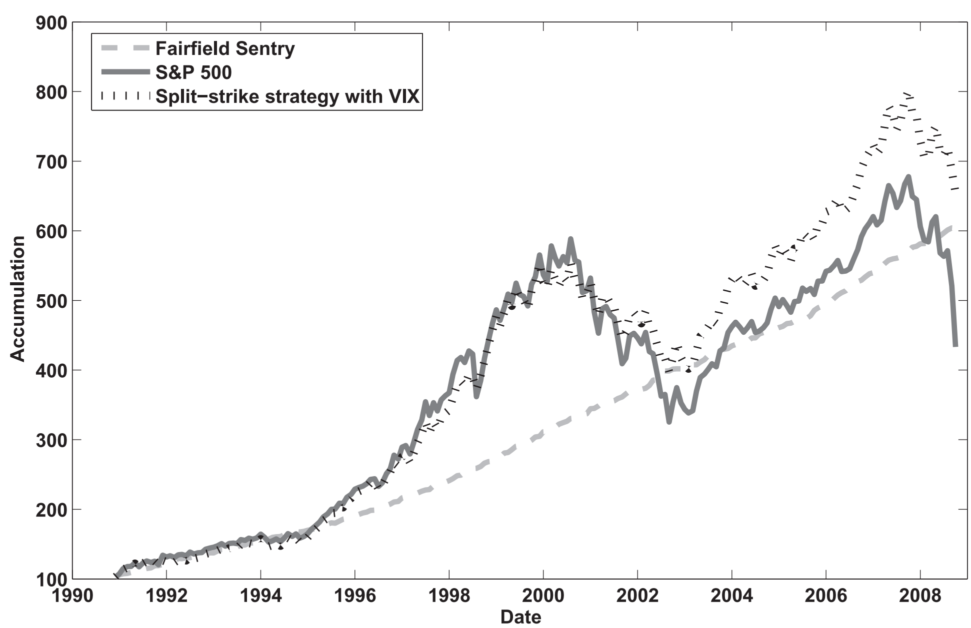
It is assumed that all available monies are invested in the index at the start of the second month and the same option strategy is implemented. This procedure is repeated every month for 215 months. We ignore transaction costs and any price impact of the trades. In order to price the options we use the Black-Scholes formula. As a proxy for the implied volatility we use the VIX to price both the call and put options.[6] The average level of the VIX over this period is 19.24%. We use the prevailing one-month U.S. T-bill rates to proxy the risk-free rates in pricing the options. We take the dividend yield on the S&P 500 into account when pricing the options. We assume the investor receives the monthly dividend yield on the Index at the end of each month from his long position in the index.

The results are summarized in Exhibit 3 and Exhibit 4. Exhibit 3 shows that the split-strike conversion strategy appears to do quite well as compared to direct investing in the S&P 500 Index. Exhibit 4 gives the performance statistics of both strategies. The split-strike conversion strategy has a higher expected return than the FS strategy (11.68% versus 10.59%). However the returns are more highly correlated with the S&P Index and have much higher volatility than the FS strategy.

As shown in Exhibit 4 the expected return for the split-strike strategy is 11.68% per annum with a standard deviation of 10.72% leading to an annual Sharpe ratio of 0.657 (see Sharpe [1964, 1966, 1994] for more discussion about the Sharpe ratio). While this Sharpe ratio is much less than the FS Sharpe ratio of 2.47 it is almost twice the Sharpe ratio of investing in the index over this period. Our theoretical analysis in the next section will show that the split-strike conversion cannot produce a Sharpe ratio twice as large as direct investing in the index. As we will discuss later, there is a maximum possible Sharpe ratio that could be achieved (Goetzmann et al. [2002]). So this result is very surprising. Alternatively, we could compare the variance and the expected return of Madoff to other stocks and hedge funds. Clauss et al. [2009] show that the returns of all the hedge funds that had a significant investment in Madoff's fund lie outside the efficiency frontier of the CAPM (see Figure 2 of Clauss et al. [2009]).

EXHIBIT 3

Accumulated Investment Proceeds Using the Fairfield Sentry Returns Given in Exhibit 1 and Investment in the S&P 500 with Dividends Reinvested



Notes: The Fairfield Sentry performance and the S&P 500 accumulation are compared with the results of investing in the split-strike strategy when the options are priced using the prevailing value of the VIX.

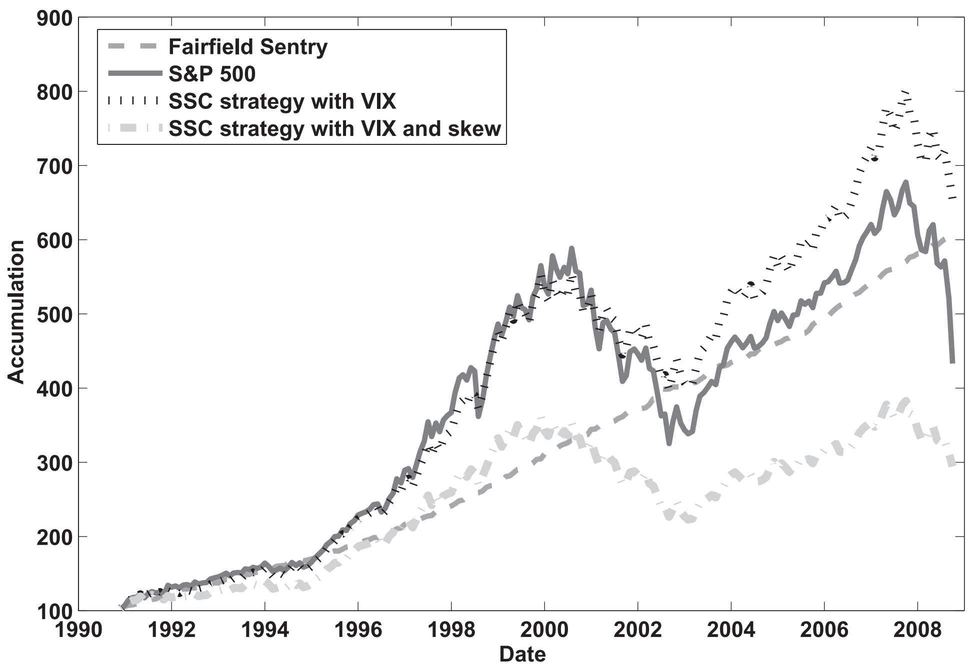
EXHIBIT 4

Summary Performance Statistics for Four Strategies for the Period December 1990 to October 2008

Strategy	Investment in S&P	Split-Strike No Volatility Skew	Split-Srike with Volatility Skew	Fairfield Sentry
Average Return (Monthly)	0.77	0.93	0.55	0.84
Average Return (Annual)	9.64	11.68	6.83	10.59
St. Deviation (Monthly)	4.12%	3.09%	3.13%	0.71%
St. Deviation (Annual)	14.28%	10.72%	10.85%	2.45%
Sharpe Ratio (Monthly)	0.105	0.190	0.069	0.712
Sharpe Ratio (Annual)	0.363	0.657	0.237	2.466
Max Monthly Return	11.44	5.37	5.15	3.29
Min Monthly Return	−16.79	−4.92	−5.76	−0.64
Percent Positive	64.65%	64.65%	63.72%	92.09%
Correlation with S&P	1.00	0.95	0.95	0.32

EXHIBIT 5

Accumulated Investment Proceeds Using the Fairfield Sentry Returns Given in Exhibit 1 and Investment in the S&P 500 with Dividends Reinvested



Notes: The Fairfield Sentry performance and the S&P 500 accumulation are compared with the results of investing in the split-strike strategy when the options are priced using the prevailing value of the VIX and more realistically with the results of investing in the split-strike conversion strategy when the options are priced using a volatility skew assumption based on data kindly supplied by Prof. Gurdip Bakshi.

The option prices that were used to implement the strategy and construct the returns in Exhibit 3 or 5 assume that both the call and put options were priced using the prevailing value of the VIX. As we shall now explain this assumption overestimates the returns on the split-strike strategy because of the existence of the volatility skew in equity index options.

The phenomenon whereby the implied volatility at a fixed maturity is a decreasing function of the strike price is known as the volatility skew. It is well known that the volatility skew is much more severe for index options than for individual stock options. One can see that a severe volatility skew will reduce the attractiveness of the split-strike strategy in practice. As compared to a constant volatility assumption the volatility skew implies that put options are relatively more expensive and call options are relatively cheaper. Since the investor is going long puts and short calls the returns to the strategy will be diminished.

To incorporate more realistic option prices by including the volatility skew we used a data set moneyness.[7] Specifically we used implied volatilities for one-month call options that were 5% out of the money and for one-month put options that were 5% out of the money. These implied volatilities were obtained from market option prices and the Black-Scholes formula using the approach described in Bakshi, Kapadia, and Madan [2003]. The average of the monthly implied volatilities for the 5% out-of-the-money puts was 20.34%. In contrast the average of the monthly implied volatilities for the 5% out-of-the-money calls was 14.92%.

We can see from Exhibit 4 that the volatility skew dramatically reduces the returns on the split-strike strategy by about 5% per annum. Exhibit 5 shows how the performance of the split-strike conversion strategy deteriorates when we include the impact of the volatility skew.

Our analysis of the impact of the skew complements the study of Clauss et al. [2009]. These authors show that it is possible to construct a split-strike strategy with a very low volatility but then it also has a very low return (to do so, one needs to buy put options almost at the money). They argue that the only way to obtain returns comparable to Madoff's returns is to assume that Madoff was an outstanding stock-picker. Ignoring the skew but including an 8.5% extra return per year, Clauss et al. [2009] construct a split-strike strategy that gives similar returns to those of Madoff (see Figure 5 of Clauss et al. [2009]). Including the impact of the skew on the strategy's cost in their study would lead to a much higher required extra return than 8.5%. Note that if Madoff was indeed able to generate an 8.5% additional return by picking the right stock, then buying as much protection as he did to protect the downside would have been unnecessary.

Manipulation-Proof Performance Measures

Recently Goetzmann, Ingersoll, Spiegel, and Welch [2007] (GISW) have developed a manipulation-free portfolio performance measure. GISW demonstrate that their measure is robust to various manipulation strategies. Even though their measure was not designed to detect outright fraud it can provide valuable insights on the nature of the split-strike strategy and the Fairfield Sentry returns. The formula for the GISW measure $\hat{\Theta}$ for a series of N monthly returns is defined as follows

$$\hat{\Theta} = \frac{1}{(1-\rho)h} \ln\left(\frac{1}{N} \sum_{i=1}^{N} \left[\frac{1+r_{pi}}{1+r_{fi}} \right]^{1-\rho} \right) \tag{1}$$

where

- r_{pi} is the rate of return on the portfolio for month i,
- r_{fi} is the risk-free rate for month i,
- h is the time interval in years. Here $h = \frac{1}{12}$.
- ρ corresponds to the relative risk aversion of the investor.

GISW note that this measure $\hat{\Theta}$ has an intuitive economic interpretation. It measures the portfolio's implied excess return after adjusting for risk. Thus for the risk-free portfolio, $\hat{\Theta} = 0$. If one had a portfolio that earned exactly 50 bps above the risk-free rate every month with no variation this portfolio would have $\hat{\Theta} = 0.06$. If the portfolio is risky then $\hat{\Theta}$ decreases if a more risk-averse investor is considered. Exhibit 6 shows the values of the GISW measure for direct investment in the S&P, the split-strike strategy (incorporating the volatility skew) and the Fairfield Sentry returns for different levels of ρ.

Exhibit 6 shows that the Fairfield Sentry portfolio outperforms the other strategies based on this measure. If the investor becomes more risk averse the value of $\hat{\Theta}$ declines rapidly for the investment in the S&P and the split-strike strategy. However $\hat{\Theta}$ hardly changes at all for the Fairfield Sentry returns. The Fairfield Sentry returns correspond to an extra 6% per year above the risk-free rate

EXHIBIT 6

Values of $\hat{\Theta}$ Corresponding to Different Levels of ρ for Three Investment Strategies

Value of ρ	Investment in S&P	Split-Strike with Volatility Skew	Fairfield Sentry
2	0.0309	0.0237	0.0597
3	0.0201	0.0177	0.0594
4	0.0090	0.0117	0.0592
5	−0.0025	0.0056	0.0589
10	−0.0664	−0.0254	−0.0575

for all investors, even the most risk averse. Mr. Madoff's returns were cleverly designed to appeal to even the most risk-averse investors.

This section showed that returns on Fairfield Sentry portfolio looked very suspicious. There is empirical evidence that volatility was too low, that the Sharpe ratios were too high to be true. However these returns were designed such that any investor would be thrilled with them, especially the most risk averse. Investors chose to invest with Madoff because they had full confidence in him. He was a former chairman of the NASDAQ.

> His solid and consistent track record generated a mixture of amazement, fascination, and curiosity. Investing with him was an exclusive privilege... All Madoff investors should in retrospect kick themselves for not asking more questions before investing. As many of them have learned there is no substitute for due diligence... (T)here were a number of red flags in Madoff's investment advisory business that should have been identified as serious concerns and warded off potential clients (Gregoriou and Lhabitant [2009]).

The next section supplements our empirical study by providing some formal analysis of what the returns under a split-strike conversion should be in a Black-Scholes setting. This theoretical study confirms the previous analysis and shows also that even a simple model could help to identify fraud and detect unrealistic returns.

THEORETICAL ANALYSIS OF THE SPLIT-STRIKE STRATEGY

We analyze the returns on the split-strike strategy in the standard Black-Scholes framework. The goal of this section is to provide some theoretical support to the suspicion that Fairfield Sentry portfolio was committing a fraud that could have been detected much earlier. We give some of the main formulae in the appendices and the detailed derivations are available on the *Journal of Derivatives* website.

Let S_0 be the price of the underlying index at current time zero. Assume the index pays no dividends and follows a geometric Brownian motion under the real world measure P so that

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2}$$

where W_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with respect to the filtration $\{\mathcal{F}_t\}$, and μ and σ are constants. We do not specifically include dividends in the formulae but one can estimate the impact of including a dividend yield δ by replacing μ by $\mu + \delta$.

The risk-free rate r is constant and continuously compounded. The index value at time h is $S_h = S_0 \exp((\mu - \frac{\sigma^2}{2})h + \sigma W_h)$. It follows a lognormal distribution. Its first two moments are

$$\begin{cases} \mathbb{E}_P[S_h] = S_0 e^{\mu h} \\ \mathbb{V}\mathrm{ar}_P[S_h] = S_0^2 e^{2\mu h}(e^{\sigma^2 h} - 1) \end{cases} \tag{3}$$

We first describe the split-strike strategy, then derive the first two moments of the standard call and put options, and finally obtain some theoretical results about the return distribution of the split-strike strategy.

Split-Strike Strategy

Suppose the time horizon is h. At time zero, the portfolio manager buys one share of the index and simultaneously sells a call option at the premium c_0 and buys a put option at the premium p_0. The call option has a strike price K_c and the put option has a strike price K_p, where

$$K_c > S_0 > K_p$$

Both options have the same time to maturity $T \geq h$ and are priced by the Black-Scholes formula. The time-zero value of the portfolio V_0 is

$$V_0 = S_0 = S_0 + (c_0 - p_0) + (\text{long put} + \text{short call}) \tag{4}$$

We assume that the amount $(c_0 - p_0)$ accumulates to the end of the period at the risk-free rate. The value of the call option (respectively, the put option) at time h if the stock price is S_h is denoted by $\mathcal{C}_h$ (respectively, $\mathcal{P}_h$). The value of the portfolio at time h is

$$V_h = S_h + (c_0 - p_0)e^{rh} + (\mathcal{P}_h - \mathcal{C}_h) \tag{5}$$

Remark 1 When $K_c = K_p = K$, the payoff of the strategy is deterministic and equal to

$$V_h = S_0 e^{rh} \tag{6}$$

This is a straightforward consequence of the call-put parity. Indeed, the call-put parity relationship applied at time h implies $V_h = Ke^{-r(T-h)} + (c_0 - p_0)e^{rh}$. But at time 0, the call-put parity can be written as $c_0 - p_0 = S_0 - Ke^{-rT}$. Thus Equation (6) is proved.

We are interested in the general case when the strike K_p of the put option is different from the strike K_c of the call option. The Sharpe ratio of this portfolio is

$$\mathcal{SR}(V_h) = \frac{\frac{\mathbb{E}_P[V_h]}{V_0} - e^{rh}}{\sqrt{\frac{\mathbb{V}\mathrm{ar}_P[V_h]}{V_0^2}}} = \frac{\mathbb{E}_P[V_h] - S_0 e^{rh}}{\sqrt{\mathbb{V}\mathrm{ar}_P[V_h]}} \tag{7}$$

since $V_0 = S_0$. The expected value of the portfolio and its standard deviation are respectively calculated as follows

$$\begin{cases} \mathbb{E}_P[V_h] = \mathbb{E}_P[S_h] + \mathbb{E}_P[\mathcal{P}_h] - \mathbb{E}_P[\mathcal{C}_h] + (c_0 - p_0)e^{rh} \\ \mathbb{V}\mathrm{ar}_P[V_h] = \mathbb{V}\mathrm{ar}_P[S_h] + \mathbb{V}\mathrm{ar}_P[\mathcal{P}_h] + \mathbb{V}\mathrm{ar}_P[\mathcal{C}_h] \\ \qquad + 2(\mathbb{C}\mathrm{ov}_P(S_h, \mathcal{P}_h) - \mathbb{C}\mathrm{ov}_P(S_h, \mathcal{C}_h) \\ \qquad - \mathbb{C}\mathrm{ov}_P(\mathcal{C}_h, \mathcal{P}_h)) \end{cases}$$

To derive expressions of the Sharpe ratio, the expected return, and the variance of the split-strike strategy, we need to know the moments of standard options. The next paragraph gives the expressions of the first moment of standard options. Second moments are provided in Appendix A.

First Moments of Standard Options Under the Physical Measure

The price dynamics of the underlying asset S under the P measure are given by Equation (2). Denote by X_T the payoff of the option (in the case of the call option $X_T = \max(S_T - K_c, 0)$ and in the case of the put $X_T = \max(K_p - S_T, 0)$). Let h be such that $0 < h < T$. Denote by X_h the value of the derivative at time h.

Let us denote by $\mathcal{C}_h$ and $\mathcal{P}_h$ the value at time h of, respectively, the call option and the put option in the Black-Scholes framework. The price is expressed at time h with current asset price S_h at time h, with respective exercise prices K_c and K_p and maturity T.

$$\mathcal{C}_h := \mathcal{C}[S_h, h, K_c, T], \quad \mathcal{P}_h := \mathcal{P}[S_h, h, K_p, T]$$

Proposition 3.1 *First moments of standard options: The first moments of this call option and this put option are respectively given as follows*

$$\begin{aligned} \mathbb{E}_P[\mathcal{C}_h] &= \mathcal{C}[S_0 e^{\mu h}, 0, K_c e^{rh}, T] \\ &= S_0 e^{\mu h}\, \Phi(\tilde{d}_1(K_c)) - K_c e^{rh} e^{-rT} \Phi(\tilde{d}_2(K_c)) \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbb{E}_P[\mathcal{P}_h] &= \mathcal{P}[S_0 e^{\mu h}, 0, K_p e^{rh}, T] \\ &= K_p e^{rh} e^{-rT} \Phi(-\tilde{d}_2(K_p)) - S_0 e^{\mu h}\, \Phi(-\tilde{d}_1(K_p)) \end{aligned} \tag{9}$$

where

$$\tilde{d}_1(K) = \frac{\ln(\frac{S_0 e^{\mu h}}{K e^{rh}}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}; \quad \tilde{d}_2(K) = \tilde{d}_1(K) - \sigma\sqrt{T}$$

and Φ is the cdf of a standard normal distribution $\mathcal{N}(0,1)$.

It turns out that in the case of standard call and put options, explicit formulae for their first and second moments are available in Cox and Rubinstein [1985].[8] A full proof of the proposition is provided on the *Journal of Derivatives* website.

The first moment of the distribution of the call price has a simple and intuitive form. Note that the resulting expression is equal to a Black-Scholes call option with the same time to maturity as the initial call and the same volatility and interest rate. However it has a higher input asset price and a higher input strike price. The new input asset price is equal to the expected value (under P) of the asset price at time h, $\mathbb{E}_p[S_h] = S_0 e^{\mu h}$. The new input strike price is equal to the original strike price, respectively K_c or K_p, accumulated at the risk-free rate.

The limit cases when $h = 0$ or $h = T$ are easily verified. When $h = 0$, the result is well known. When $h = T$, the result can be found in the appendix of Goetzmann et al. [2002, 2007].

Properties of the Split-Strike Strategy

In this subsection, we present a number of useful results concerning the split-strike conversion strategy. To derive these results, we use formulae for the moments of the option prices given in the previous section or in the appendix. The first proposition is related to the expected return under the split-strike conversion strategy.

Proposition 3.2 *Expectation of the strategy: The expected payoff of the strategy is equal to*

$$\begin{aligned} \mathbb{E}_P[V_h] = S_0 e^{\mu h} &- \mathcal{C}[S_0 e^{\mu h}, 0, K_c e^{rh}, T] \\ &+ \mathcal{P}[S_0 e^{\mu h}, 0, K_p e^{rh}, T] + (c_0 - p_0)e^{rh} \end{aligned}$$

when $K_c = K_p = K$, V_h is deterministic and its expectation is equal to $\mathbb{E}_p[V_h] = S_0 e^{rh}$.

The result is a consequence of Proposition 3.1. The special case when $K_c = K_p = K$ is established earlier. See expression (6).

Remark 2 The expected return from the strategy is an increasing function of K_c and a decreasing function of K_p.

Remark 3 The split-strike strategy has a lower expected return and a lower variance than a direct investment in the index,

$$\mathbb{E}_P[V_h] \le \mathbb{E}_P[S_h] \quad \text{and } \mathbb{V}\text{ar}_P[V_h] \le \mathbb{V}\text{ar}_P[S_h]$$

The proof of this remark can be found on the *Journal of Derivatives* website.

Proposition 3.3 *Variance and Sharpe ratio of the strategy:*

The variance of the strategy at time h is equal to

$$\mathbb{V}\text{ar}_P[V_h] = S_0^2 e^{2\mu h}(e^{\sigma^2 h} - 1) + \mathbb{V}_C + \mathbb{V}_P - 2\mathbb{C}\text{ov}_{\mathcal{C},\mathcal{P}} - 2\mathbb{C}\text{ov}_{\mathcal{C},\mathcal{S}} + 2\mathbb{C}\text{ov}_{\mathcal{S},\mathcal{P}}$$

where

$$\begin{cases} \mathbb{V}_C = \mathbb{E}_P[\mathcal{C}_h^2] - \mathbb{E}_P[\mathcal{C}_h]^2 \\ \mathbb{V}_P = \mathbb{E}_P[\mathcal{P}_h^2] - \mathbb{E}_P[\mathcal{P}_h]^2 \\ \mathbb{C}\text{ov}_{\mathcal{C},\mathcal{P}} = \mathbb{E}_P[\mathcal{C}_h\mathcal{P}_h] - \mathbb{E}_P[\mathcal{C}_h]E_P[\mathcal{C}_h] \\ \mathbb{C}\text{ov}_{\mathcal{C},\mathcal{S}} = \mathbb{E}_P[\mathcal{C}_h S_h] - \mathbb{E}_P[\mathcal{C}_h]S_0 e^{\mu h} \\ \mathbb{C}\text{ov}_{\mathcal{P},\mathcal{S}} = \mathbb{E}_P[\mathcal{P}_h S_h] - \mathbb{E}_P[\mathcal{P}_h]S_0 e^{\mu h} \end{cases}$$

First moments can be found in Proposition 3.1 *and formulae for second moments are in Appendix A* (*see Equations* (A-1) *and* (A-2)) *and the three cross products:* $\mathbb{E}_P[\mathcal{C}_h\mathcal{P}_h]$, $\mathbb{E}_P[\mathcal{C}_h S_h]$, $\mathbb{E}_P[\mathcal{P}_h S_h]$ *can be found in Appendix B.*

The Sharpe ratio of the strategy is defined by Equation (7):

$$\mathcal{SR}(V_h) = \frac{\mathbb{E}_P[V_h] - S_0 e^{rh}}{\sqrt{\mathbb{V}\text{ar}_P[V_h]}}$$

where the expectation is given in Proposition 3.2.

The proof of Proposition 3.3 is available on the JOD website. The Sharpe ratio is not defined when $K_p = K_c$ because both the numerator and the denominator are equal to zero. This fact will be explained in the numerical analysis in the following section.

Proposition 3.4 *Correlation of the strategy with the index S:*

The correlation can be written as

$$\text{Corr}(V_h, S_h) = \frac{\mathbb{V}\text{ar}_P[S_h] - \mathbb{C}ov_{\mathcal{C},\mathcal{S}} + \mathbb{C}ov_{\mathcal{P},\mathcal{S}}}{\sqrt{\mathbb{V}\text{ar}_P[V_h]}\, S_0 e^{\mu h}\sqrt{e^{\sigma^2 h} - 1}}$$

where all terms are given in Proposition 3.3.

The correlation of the strategy with the index is of course equal to zero when $K_c = K_p$, but similar to the Sharpe ratio, it is not defined at zero.

NUMERICAL ANALYSIS

This section contains some numerical results concerning the return distribution of the split-strike strategy. This will provide additional insight into the properties of the strategy. First we consider a split-strike strategy with maturity $T = h$. Similar results hold when $h < T$. We will examine two cases. First, we assume the strikes of the call option and the put option are given by

$$K_c = S_0 + b \quad K_p = S_0 - b \tag{10}$$

where $b \in [0, S_0]$. Second, we study the case when K_p and K_c are chosen independently. We will finally discuss optimal choices of the two strikes.

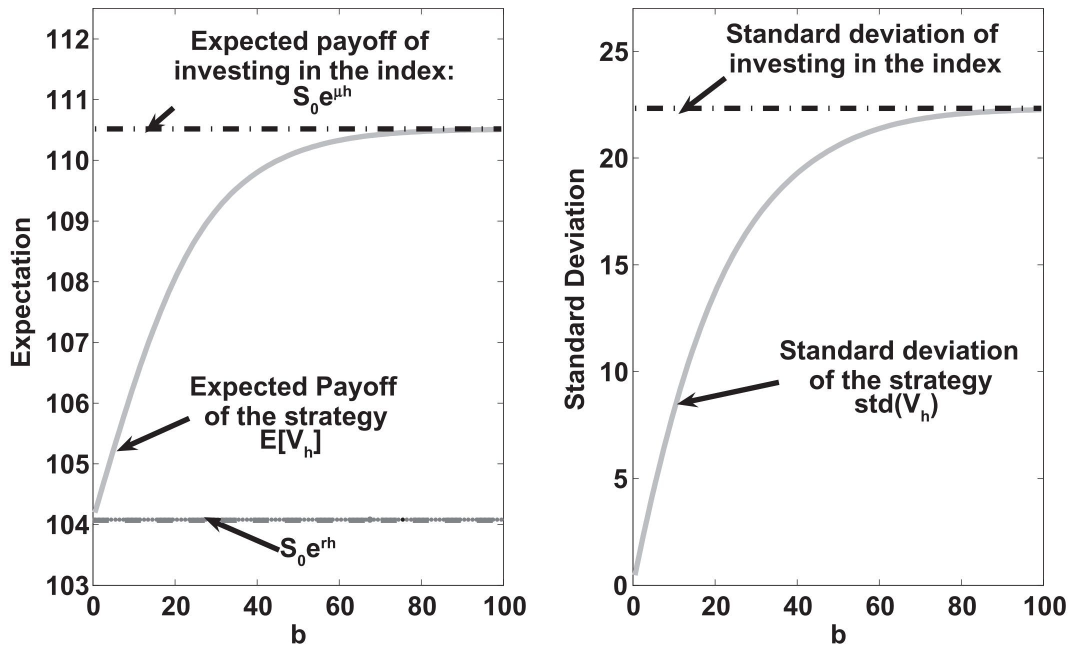
Case When the Strikes Are $K_c = S_0 + b$, $K_p = S_0 - b$ with $b \in [0, S_0]$

We plot the expected payoff of the strategy and the standard deviation when b varies in Exhibit 7. We assume plausible values for the parameters of the financial market. The conclusions hold for other choices of the volatility σ, the interest rate r, the instantaneous expected return μ and maturity of the strategy T. Note that it is not necessary that $h = T$.

Consistent with our theoretical findings, Exhibit 7 shows that the expected return of a split-strike strategy is always lower than the expected return of investing in the index. A lower expected return is compensated by a lower standard deviation. Note that as b goes to 0, the expected payoff of the strategy converges to $S_0 e^{rh}$ which means that the return of the strategy is the risk-free rate. This is not a surprise because when $K_c = K_p = S_0$, $V(h) = S_0 e^{rh}$. In this case $V(h)$ is deterministic and its variance is equal to 0.

EXHIBIT 7

Expectation and Standard Deviation of the Split-Strike Strategy



Notes: Assume $S_0 = 100$, $\sigma = 20\%$, $\mu = 0.1$, and $r = 0.04$. Assume $h = 1$, $T = 1$. The strikes of the call and the put are $K_c = S_0 + b$ and $K_p = S_0 + b$. In the left panel, we display the expected payoff of the strategy and of an investment in the index ($\mathbb{E}_P[V(h)]$ and $\mathbb{E}_P[S_h]$). The right panel represents the standard deviations $std[V(h)]$ and $std[S_h]$. The range of b is $[0,100]$.

Exhibit 8 displays the Sharpe ratio of the strategy against the Sharpe ratio of investing in the index with a horizon $h = T = 1$ year. We observe that the positions in options can enhance the Sharpe ratio. However the enhancement is bounded from below as well as from above. Goetzmann et al. [2002] show that there is a maximum possible Sharpe ratio attainable in the complete market of Black-Scholes. The formula for this maximum Sharpe ratio when $h = T$ is

$$\sqrt{e^{\frac{(\mu-r)^2 T}{\sigma^2}}-1}$$

In addition the limit of the Sharpe ratio when $b \to 0$,[9]

$$\lim_{b\to 0^+} \mathcal{SR}(V_T) = \frac{\Phi(a_2)-\Phi(\hat{a}_2)}{\sqrt{\Phi(a_2)(1-\Phi(a_2))}}$$

where $a_2 = \frac{\mu\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}$, $\hat{a}_2 = \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}$. When $b = 0$, the strategy is equivalent to investing in bonds and the Sharpe ratio is not defined.

Exhibit 8 assumes $\sigma = 20\%$, $\mu = 0.1$, $r = 0.04$ and $h = T = 1$ year. In this case, the minimum Sharpe ratio is 0.2432 and the maximum Sharpe ratio is equal to 0.3069. Note also that Exhibit 8 is consistent with Figure 3 of Lhabitant [1998].

Finally, in Exhibit 9 we investigate the correlation between the strategy and the index and the beta of the strategy. We assume the underlying index S is a good proxy for the financial market. Then the beta is defined as follows:

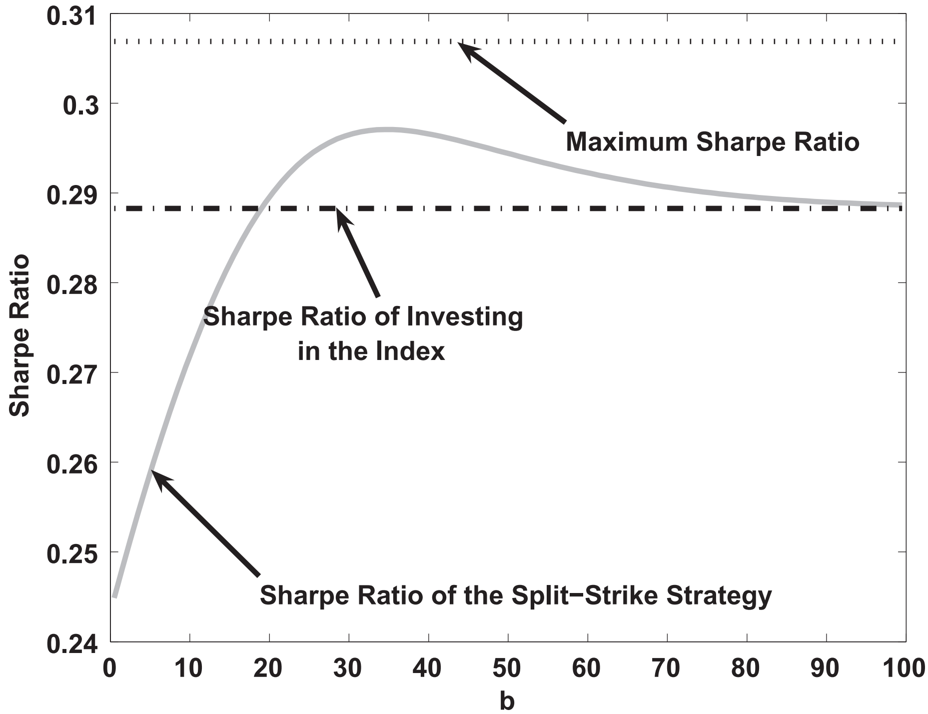
$$\beta = \frac{\mathbb{C}\mathrm{ov}_P\left(\frac{V_h}{V_0}, \frac{S_h}{S_0}\right)}{\mathbb{V}\mathrm{ar}_P\left(\frac{S_h}{S_0}\right)} = \frac{\mathbb{C}\mathrm{ov}_P(V_h, S_h)}{\mathbb{V}\mathrm{ar}_P(S_h)}$$

Formulas for the covariance of V_h and S_h are established in Proposition 3.3 and explicitly given in Appendix B.

Exhibit 9 shows that the strategy is highly correlated with the index and that the beta lies between zero and one. When $b = 0$, the strategy is deterministic and the correlation is not defined but the beta is defined and equal to 0. Similar to the Sharpe ratio, the limit of the correlation when $b \to 0^+$ is positive

EXHIBIT 8

Sharpe Ratio of the Split-Strike Conversion Strategy Compared to the Sharpe Ratio of Investing in the Index



Notes: Here $\sigma = 20\%$, $\mu = 0.1$, and $r = 0.04$ and $h = 1$, $T = 1$. The strikes of the call and the put are respectively equal to $K_c = S_0 + b$ and $K_p = S_0 - b$, $b \in [0,100]$.

$$\lim_{b \to 0^+} \text{Corr}(V_T, S_T) = \frac{\Phi(a_1) - \Phi(a_2)}{\sqrt{\Phi(a_2)(1 - \Phi(a_2))(e^{\sigma^2 T} - 1)}}$$

where[10] $a_1 = \frac{\mu\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}$, $a_2 = \frac{\mu\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}$. If $b > 0$, the correlation is always greater than 0.732 (which is its limit calculated with the same parameters as before). However, as reported to the SEC in 2005 by H. Markopolos, the beta of the strategy was 0.06 and the correlation with the index was only 0.3 (see attachment 1 on Fairfield Sentry Performance Data in 2005 in SEC [2005]). The formal analysis in this section confirms that the returns claimed by Madoff are theoretically impossible.

General Case

Consider now the case of the split-strike strategy where the strike prices of the call and the put, respectively K_c and K_p, vary independently. Similar results as before can be obtained for the expectation, the variance, the correlation, and the beta of the strategy. We present only the Sharpe ratio of the strategy in Exhibit 10.

Exhibit 10 shows that there are choices of strikes for the call and for the put that maximize the Sharpe ratio. The optimal strikes do not necessarily correspond to the symmetric case with respect to S_0 as we will see.

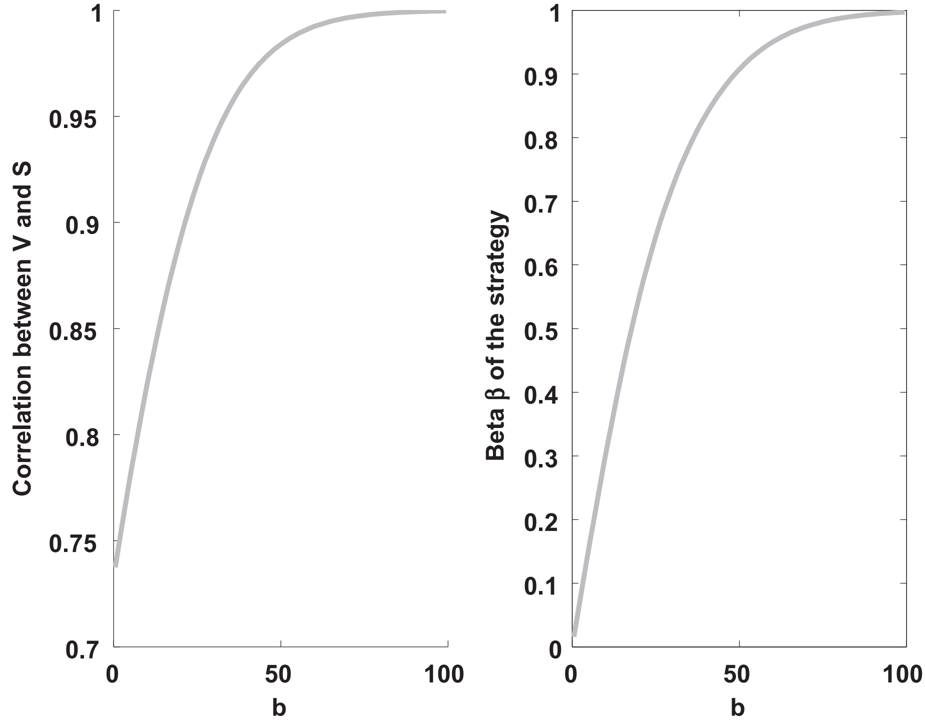
Optimal Choice of the Parameters K_p and K_c

In this last subsection, we numerically derive the optimal strikes of the put and of the call in a split-strike strategy for different choices of the horizon $h = T$. The objective is to maximize the Sharpe ratio over the given maturity.

As mentioned in the literature by Goetzmann et al. [2007] and by Cvitanic, Lazrak, and Wang [2009], the choice of the horizon can dramatically change the results. With a longer horizon, it is optimal to buy a call with a higher strike and a put with a lower strike. Note also that

EXHIBIT 9

Correlation between S_h and V_h and Beta of the Strategy



Notes: Here $\sigma = 20\%$, $\mu = 0.1$, and $r = 0.04$ and $h = T = 1$. The strikes of the call and the put are respectively equal to $K_c = S_0 + b$ and $K_p = S_0 - b$, $b \in [0, 100]$.

one can see from Exhibit 11 that it is optimal not to have a perfectly symmetric split-strike strategy. It is optimal to buy a put option more deeply out-of-the-money than the call option.

CONCLUSIONS

This article analyzed certain features of Bernie Madoff's investment performance. It is now known that these results were based on a giant Ponzi scheme which flourished for a long time despite several red flags and the highly suspicious nature of the returns. Indeed were it not for the current financial crisis it seems likely that the Madoff investment scheme would still be in operation.

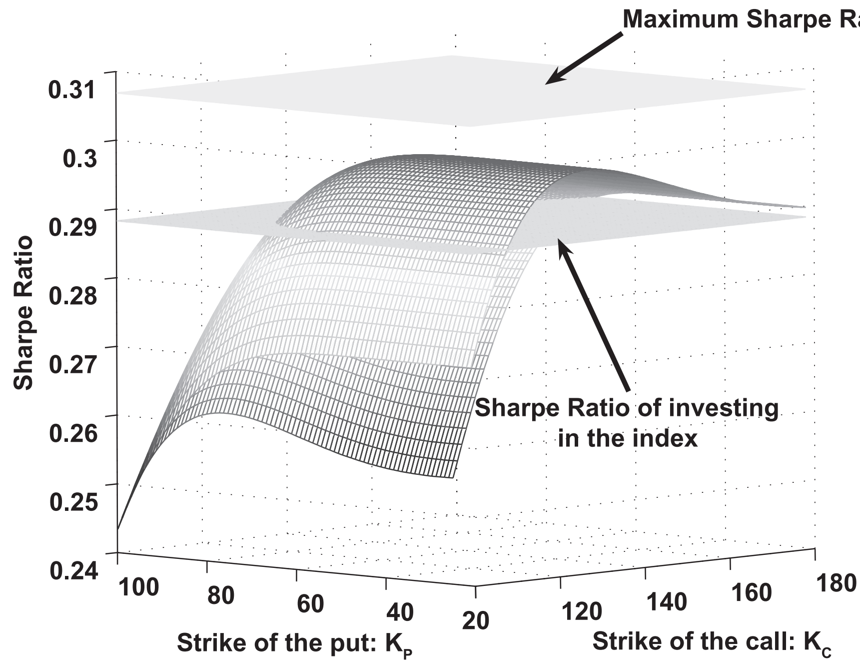
We implemented a version of the split-strike strategy similar to the one allegedly used by Madoff and compared the results with those reported by Fairfield Sentry one of Madoff's feeder funds. The Sharpe ratio based on our version of the split-strike strategy was very much lower than Fairfield Sentry's Sharpe ratio over the same period. In addition the correlation between the split-strike strategy and the market was more than twice the corresponding correlation for the Madoff strategy. One of the most unbelievable statistics of Madoff's performance is the very low volatility. This makes the Madoff's returns very attractive even to the most risk averse investors. These returns were concocted in a very clever way.

Our theoretical analysis reaches the same conclusions. There are closed-form expressions for the moments of the split-strike strategy and its correlation with the market. In addition there is a theoretical maximum Sharpe ratio that can be obtained using options. We find that the performance statistics reported by Fairfield Sentry lie well outside their theoretical bounds. These results are incredible in the most literal sense.

In summary there are some simple quantitative diagnostics that should have raised suspicions about Madoff's performance.

Exhibit 10

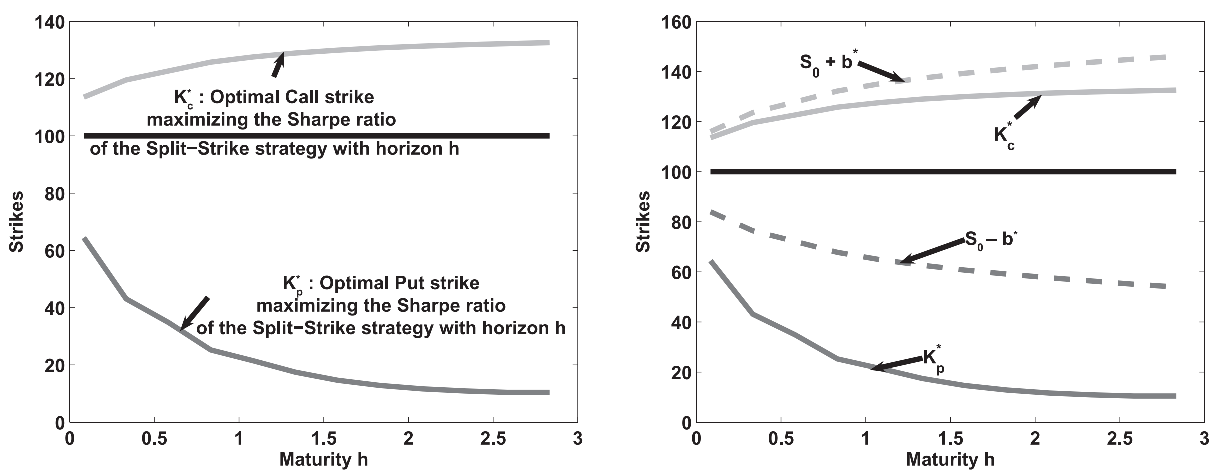
Sharpe Ratio of the Strategy versus Sharpe Ratio of Investing in the Index



Notes: Here $\sigma = 20\%$, $\mu = 0.1$, $r = 0.04$ and $h = T = 1$. The strikes of the call and the put are respectively equal to K_c and K_p.

Exhibit 11

Choice of the Strikes K_c and K_p to Maximize the Sharpe Ratio



Notes: Here $\sigma = 20\%$, $\mu = 10\%$, and $r = 4\%$. The strikes of the call and the put are respectively equal to K_c and K_p. On the left panel, the optimal strikes K_c^* and K_p^* are determined in the general case. On the right panel, the optima are compared with the ones obtained by maximizing the Sharpe ratio when strikes are symmetric with respect to S_0. In this case, the optima are denoted by $K_c = S_0 + b^*$ and $K_p = S_0 - b^*$.

ENDNOTES

The authors thank the editor for his helpful suggestions. They thank Professor Gurdip Bakshi for providing the implied volatility skew data. We would like to acknowledge support from the Natural Sciences and Engineering Research Council of Canada and from Tata Consultancy Services. We also thank Petr Kobelevskiy and William Gornall for excellent research assistance and M. Suchanccki for careful comments on a previous version.

[1]In a Ponzi Scheme, returns to investors come from their own money or money paid by subsequent investors rather than from any actual profit earned. See a discussion on Ponzi Schemes in Bhattacharya [2003].

[2]This is a marketing name for a collar strategy. In other words, this is commonly called a bearish collar strategy.

[3]It is conventional for hedge fund returns to be quoted net of expenses. These expenses include a fee on the total assets plus an incentive fee based on the performance of the fund. See Gregoriou and Lhabitant [2009] as well as Markopolos [2009] and SEC [2005] for more discussion on the unusual fee structure of Madoff's fund.

[4]We compare Madoff's strategy with a split-strike strategy on the S&P 500. The FS prospectus refers to the S&P 100. However results of the empirical study would have been similar or even worse. Indeed Gregoriou and Lhabitant [2009] note that "it would have been prohibitively expensive using S&P 100 Index options" since they *are much less widely used than S&P 500 Index options*.

[5]Other examples of split-strike strategies with alternative strikes have been investigated by Clauss, Roncalli, and Weisang [2009] and their conclusions are similar. The impact of the choice of the strikes in a split-strike conversion strategy will be analyzed from a theoretical perspective later in the article.

[6]The definition of the VIX was changed in 2003. See Carr and Wu [2006]. We used data from the CBOE website so that the current version of the VIX index was used for the entire period. The original VIX was based on S&P 100 Index option prices whereas the new VIX uses options on the S&P 500 Index. In addition the VIX index is only an approximation of the implied volatility of an at-the-money call option. Later we incorporate a volatility skew into the price calculations.

[7]The authors thank Gurdip Bakshi for providing this updated data set. The options consist of 5% out-of-the-money calls and 5% out-of-the-money puts. Each option has 28 days left to maturity. The implied volatilities were obtained from market prices by inverting the Black Scholes formula. A 2% dividend yield is assumed on the index.

[8]See Cox and Rubinstein [1985], Chapter 6, footnote 34 for the first moments and footnote 43 for the covariance between two calls.

[9]Proof available from the authors upon request.

[10]Proof available from the authors upon request.

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