Motivation :

Why should we learn about circuits ?

- integrated circuits :
    - digital ICs
    - analog
- Software CAD tools :
    - process simulators
    - device diodes transistors MOS BJT
    - circuits : SPICE Pspice Hspice
    - systems and interconnects
    - synthesis tools
- Control
- instrumentation
- power systems
- bioengineering

The fundamental laws :

- energy
- information

their

- generation
- conversion
- transmission
- utilization

Variables: charge
current
voltage
power

Model

Theory

Scientific method: observations
postulate a model
establish relationship
among variables
<Laws>

Basic circuit laws: Ohm's
Kirchhoff's } Laws
Coulomb's

Most fundamental: conservation of energy
atomic theory of matter

Systems of units: Systeme international
SI

Basic quantites: length
mass
time
electric current
temperature
luminance

Basic units: meter
kilogram
second
ampere

Electric current:

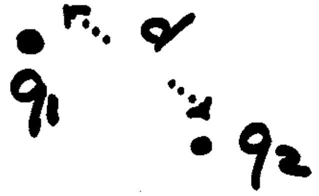
→ Charge: • electron × 6.25×10^{18} = coulomb

↳ 1.6×10^{-19} coulomb

Charles Augustin de Coulomb : 1736-1806

two "Coulomb's laws"

• Law of $F = k \cdot \frac{q_1 \cdot q_2}{d^2}$



→ Average current = $\frac{\Delta q}{\Delta t}$

ampere (A) = $\frac{1C}{1sec}$

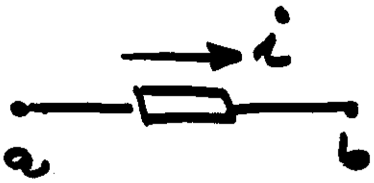
André Marie Ampere : 1775-1836

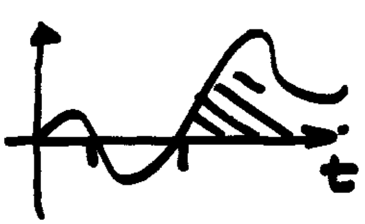
Ampere's law (electrodynamics)

$$i = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t}$$

$$i = \frac{dq}{dt} \Rightarrow q(t) = \int_0^t i(s)\,ds$$

Current reference direction:

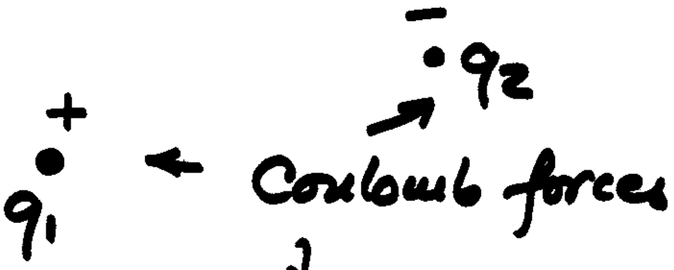




orientation vs. direction

Choose a reference orientation, and only then calculate the value!

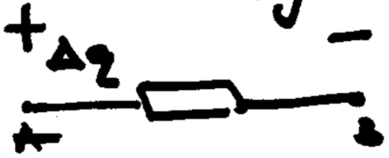
Voltage:



↓

Stored energy
= electric potential energy

w_{AB} potential energy



$$v_{AB} = \frac{dw_{AB}}{dq}$$

potential drop
or voltage

Volts: Alesandro Volta : 1745-1827

Voltage reference:

higher potential + A
lower − B

A B
+ −

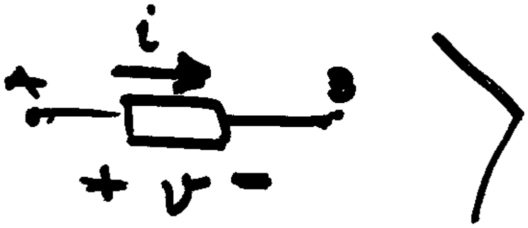
Polarity : = orientation

⟨ voltage across two points
current through an element ⟩

Energy and Power:

↓

Ability to do work: kinetic (motion)
potential (position)



$$\Delta W_{AB} = v \cdot \Delta q$$

$$\frac{\Delta W_{AB}}{\Delta t} = v \cdot \frac{\Delta q}{\Delta t}$$

$$\frac{\Delta W_{AB}}{\Delta t} = v \cdot i$$

↖ average current

↗

the rate at which energy has been expended by other parts of the circuit

Unit: joule (=) newton·meter

mechanical energy delivered = force · distance

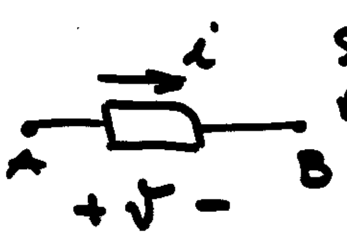
Power:

delivered to a device (element
part of a circuit)

$\rightarrow p(t) = \frac{dw(t)}{dt}$

$w(t) = \int p(x)\,dx$

$\rightarrow p = v \cdot i$

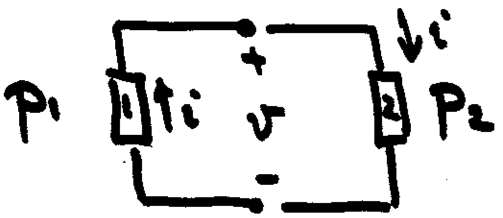


Standard references

James Watt: 1736 – 1819

$1 \text{ Watt} = \frac{1 \text{ Joule}}{1 \text{ sec}}$

Reference.



$p_1 = -vi$
$p_2 = vi$

power delivered to a branch

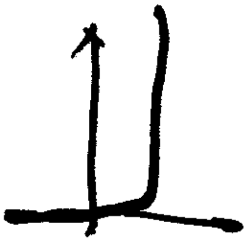
Delivered power = absorbed power

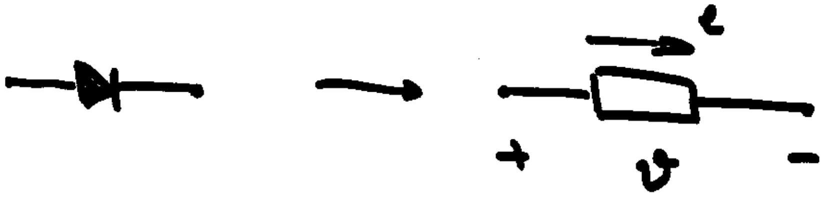
Electric Circuits and Models :

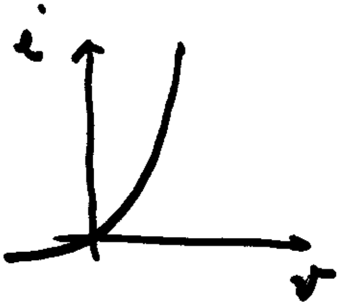
Physical electrical devices:

wires resistors
diodes capacitors
transistors inductors
batteries , generators
transformers , relays, motors
⋮

are connected into physical circuits
↓ ideal models
↳ connected into circuits.



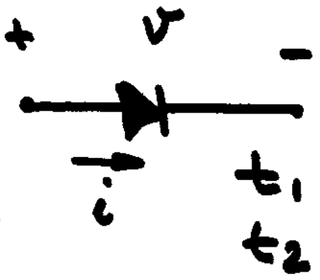


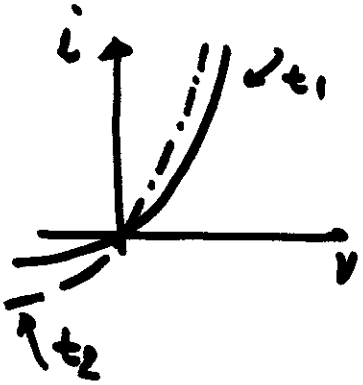


Various conditions require various models!

Ex:

- temperature
- aging
- charge conservation models for transistors





If:

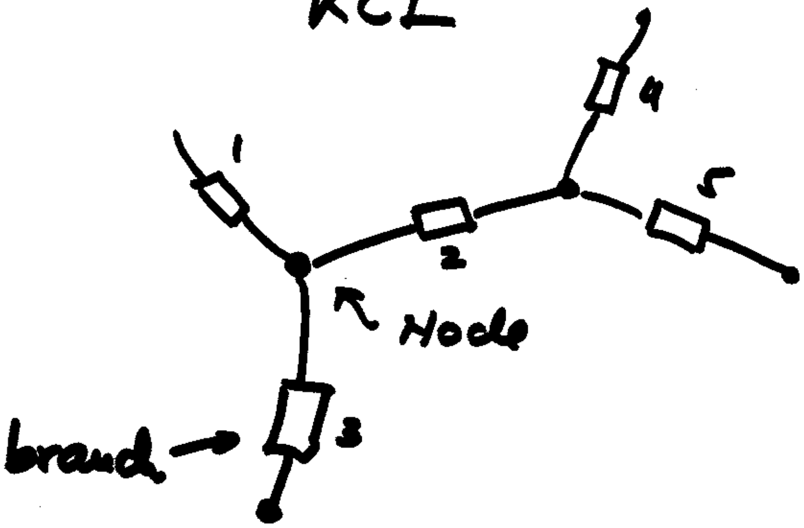
Physical properties are localized:
Geometry of the device does not matter

↳ lumped elements
(circuits)

Otherwise: distributed (important when:
high frequencies
small geometries)
(size)

Kirchhoff's Current Law

KCL

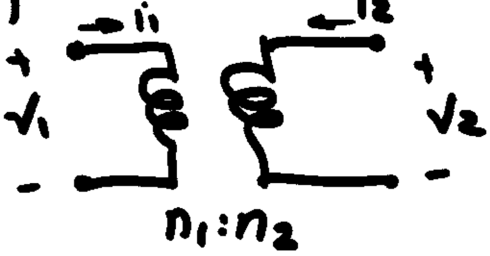


Elements:

- Two-terminal
- three-terminal

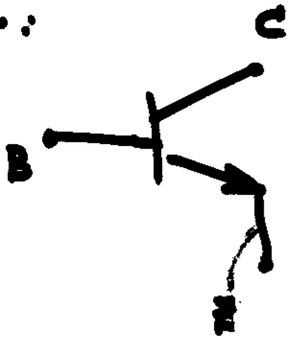
4-terminal
or
two-port

Transformers:



Transistr:

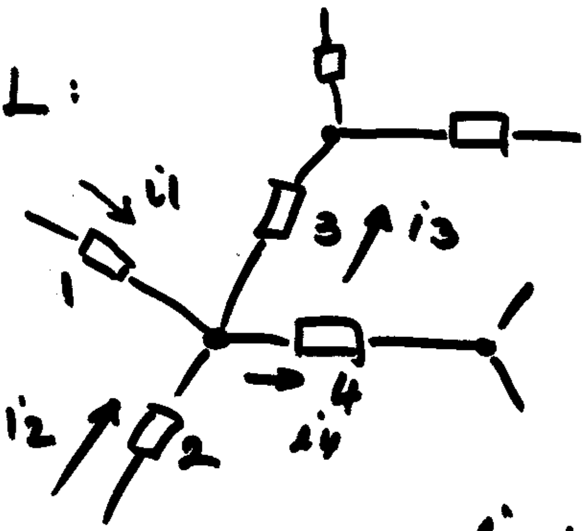
BJT



Three-terminal

(Two-port)

KCL:



- Circuit

vs

- Network

$$i_1 + i_2 - i_3 - i_4 = 0$$

At each node in any electrical network and at each instant of time, the algebraic sum of all currents leaving a node is zero

$\uparrow$

(entering)

Conservation of Charge:

$$\int i_1 ds + \int i_2 ds - \int i_3 ds - \int i_4 ds = 0$$

$$q_1 + q_2 - q_3 - q_4 = 0$$

Principle of conservation!

$\Downarrow$

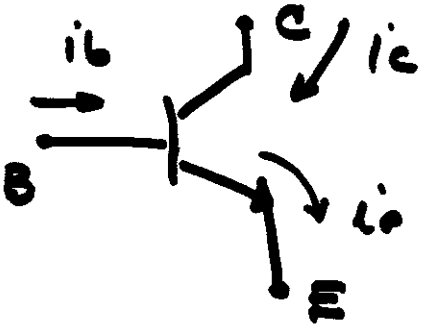
KCL

KCL (2nd form):

At any instant of time, the net current leaving (or entering) an electrical element having any number of terminals must be zero

BJT

Bipolar Junction Transistor



$$i_b + i_c - i_e = 0$$

$$-i_b - i_c + i_e = 0$$

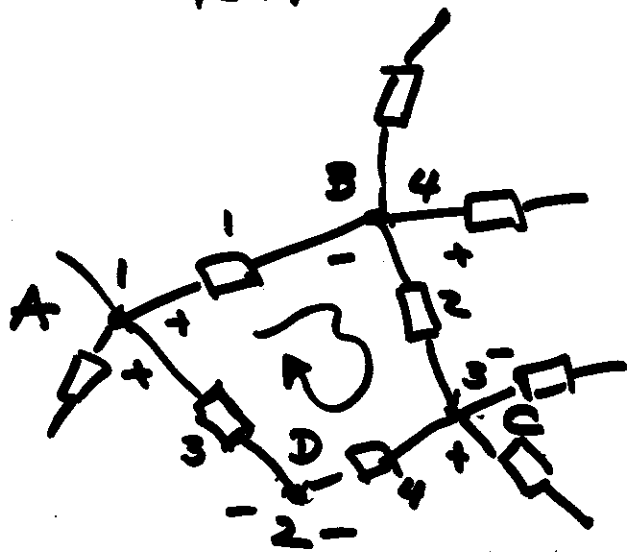
$$\Downarrow$$

$$i_b + i_c = i_e$$

KCL (3rd form):

At each node and at all times, Σ all currents with references oriented away from a node $= \Sigma$ all currents with references oriented toward the node.

Kirchhoff's Voltage law
KVL



$$V_{AB} + V_{BC} + V_{CD} + V_{DA} = 0$$

$$V_{AB} = V_1$$

$$V_{BC} = V_2$$

$$V_{CD} = V_4$$

$$V_{DA} = -V_3$$

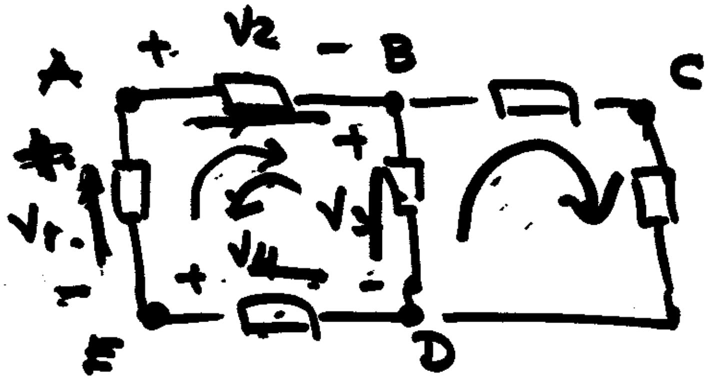
$$V_1 + V_2 + V_4 - V_3 = 0 \quad A —$$

KVL:

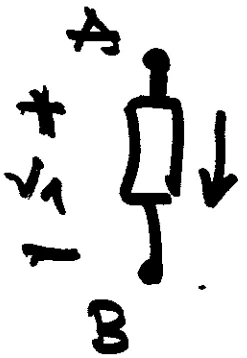
The algebraic sum of all voltages encountered in traversing any closed path in a lumped connected circuit is zero at any instant of time.

KVL (2nd form)

In a lumped connected circuit, the sum of all voltages along a closed path with references that agree with a clockwise traversal of the path equals the sum of all voltages whose references agree with a counterclockwise traversal of the path.



$$v_1 - v_2 - v_3 + v_4 = 0$$ ✓

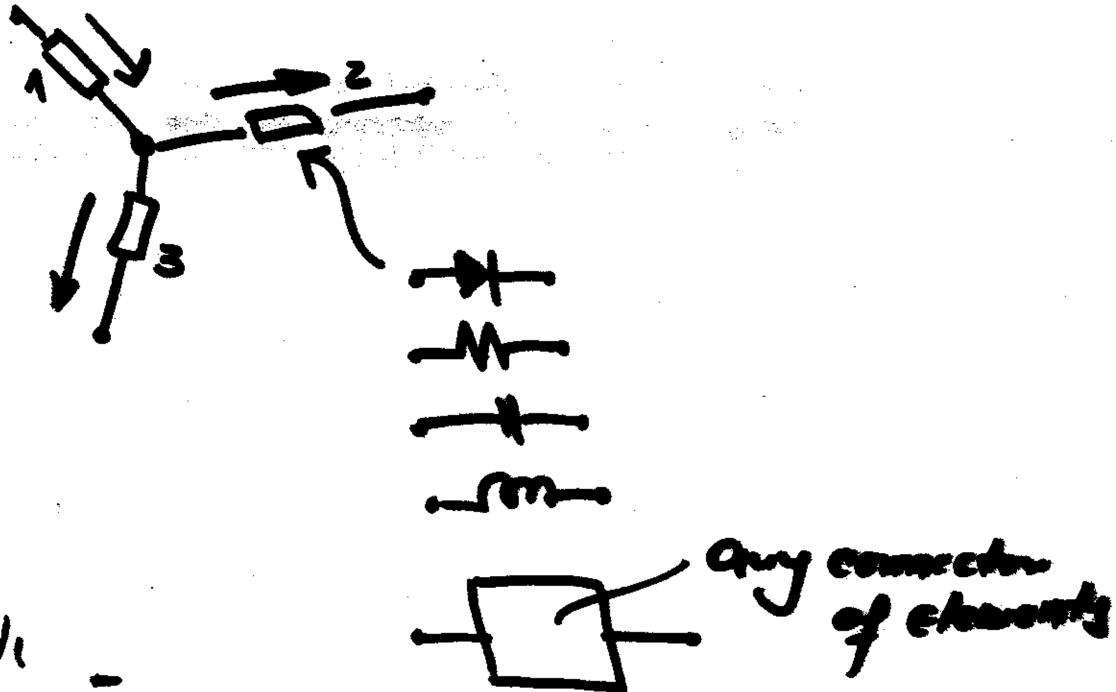


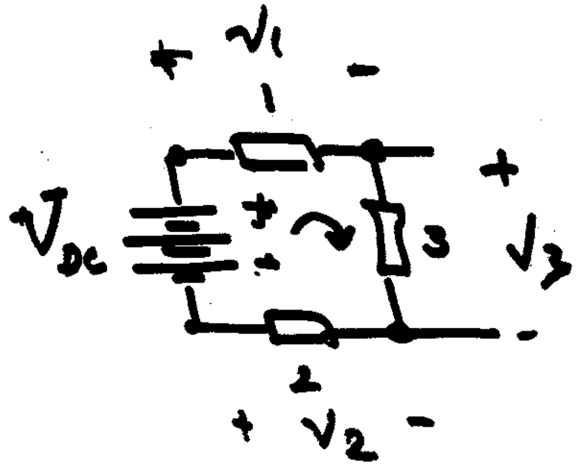
$$v_{AB} = v_1$$

$$v_3 + v_2 - v_1 - v_4 = 0 \qquad (-1)$$

More on KCL and KVL:

— Topological Laws





Used to calculate voltages and currents in a circuit

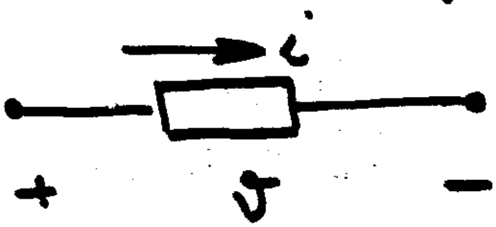
$$V_{DC} - v_1 - v_3 + v_2 = 0$$

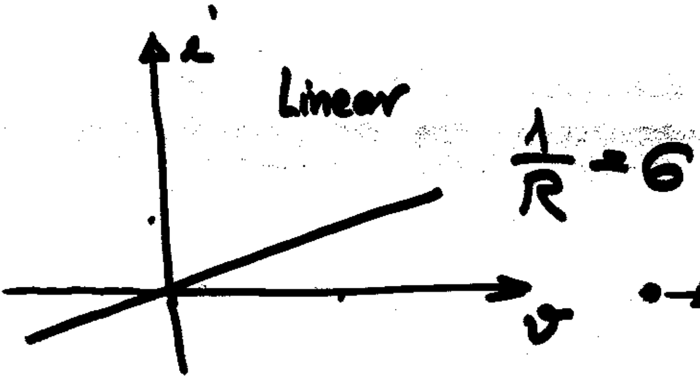
Equation: Sets of linear equations

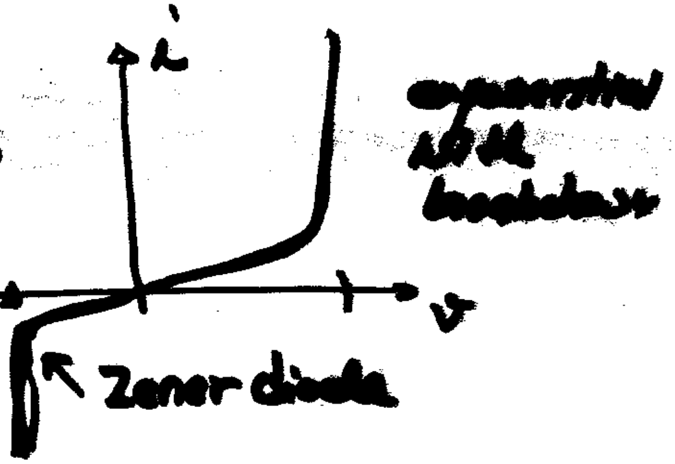
All coefficients +1 or −1 !

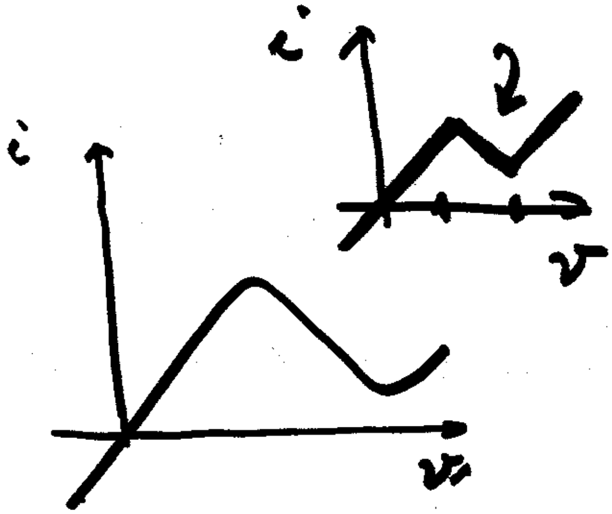
Are they independent ?

Two-terminal components

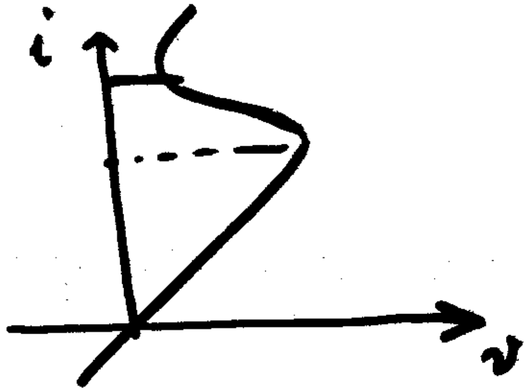






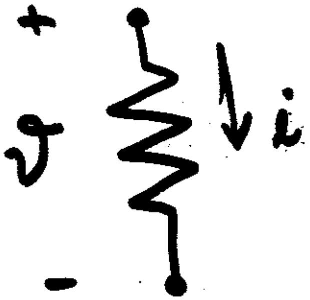


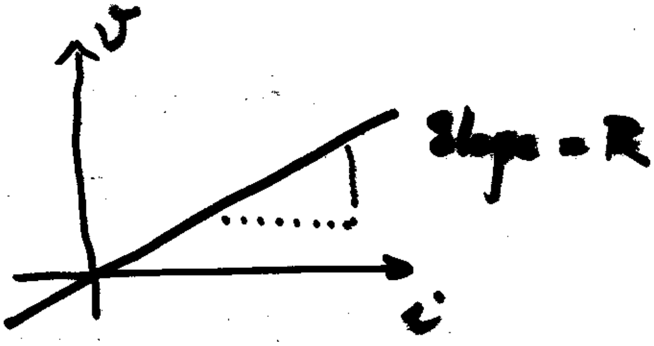
Nonlinear
N-type
(Made of Transistors
and resistors)



Nonlinear
S-type

Linear Resistor:





$$v = R \cdot i$$

↑ constant

≠ $f(v, i)$

Ohm's Law: $v = R \cdot i \quad \langle \forall t \rangle$

Georg Simon Ohm (1789-1854)

R = resistance

$$1\Omega = \frac{1V}{1A}$$

Range: 1Ω $1k\Omega$ $1M\Omega$

$R = 0$: Short circuited 

$R = \infty$: open circuited 

$$i = \frac{1}{R} \cdot v$$

conductance
(Siemens ; mho

reverse of ohm)

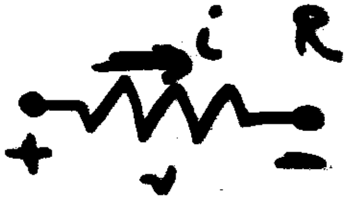
Power and energy:

$$p = vi$$

$$\rightarrow p = Ri^2$$

$$\rightarrow = G \cdot v^2$$

instantaneous!



$$W_R = \int R i^2(s)\,ds$$

$$\int G \cdot v^2(s)\,ds$$

Always absorbed by R
Never delivered to the rest of the circuit

R: power ratings (max power they
tolerance can dissipate)

Passive vs. Active components:

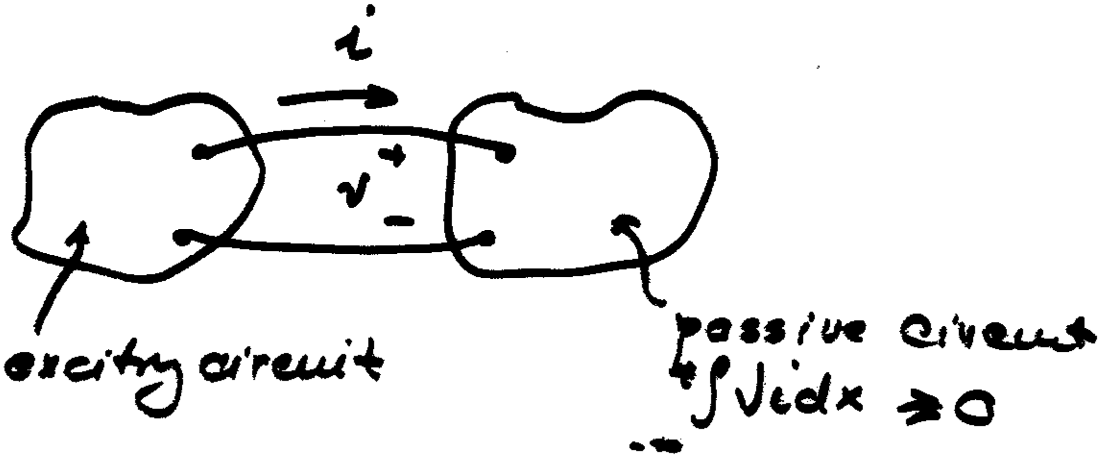
$$w(t) = \int_{-\infty}^{t} v(x)\cdot i(x)\,dx \qquad \forall t$$

$$w(t) = w(0) + \int_{0}^{t} v(x)\cdot i(x)\,dx$$

Passive components: $w(t) \geq 0 \quad \forall t$

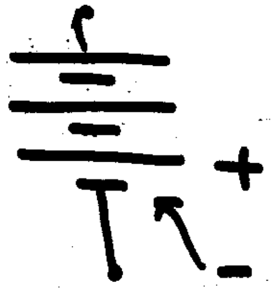
If $w(t) < 0$ for some t

$\downarrow$ Active component



Sources:

batteries



Voltage sources

V_{DC}

phase

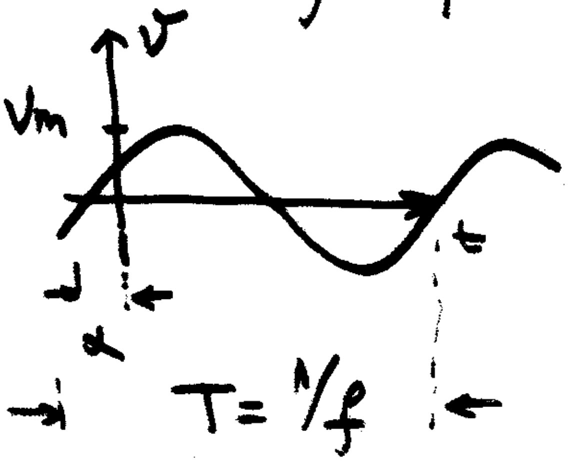
$$v = V_m \sin(\omega t + \alpha)$$

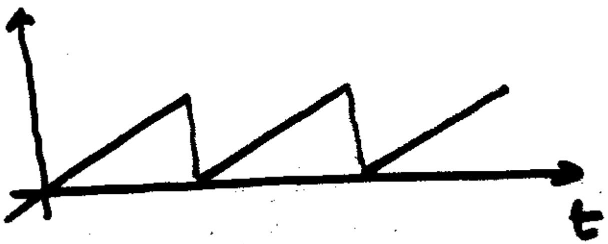
max

frequency

$$\omega = 2\pi f$$

$$f = \frac{1}{T}$$





Time dependent

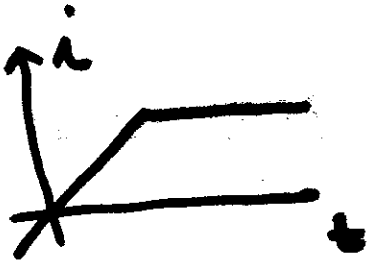
Current sources:

I_{DC}

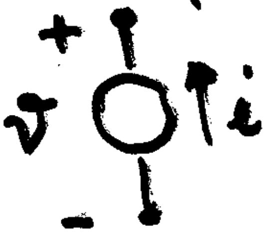
$I_m \sin(\omega t + \alpha)$

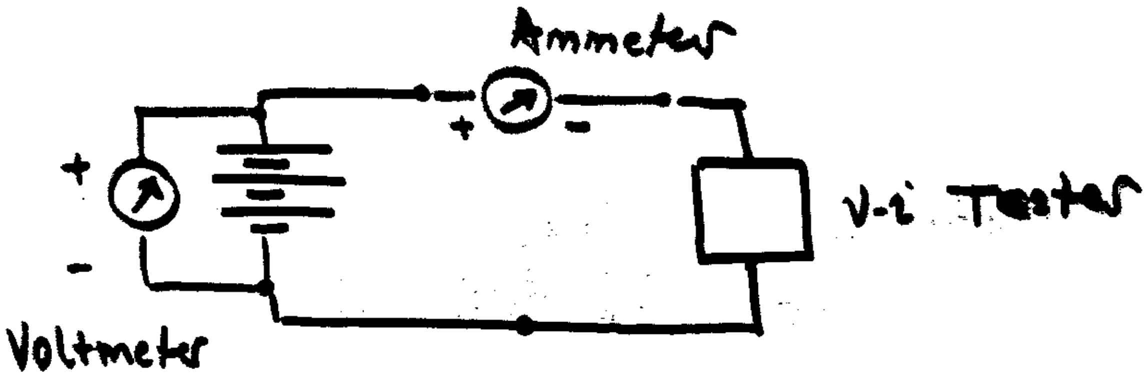
saw-tooth

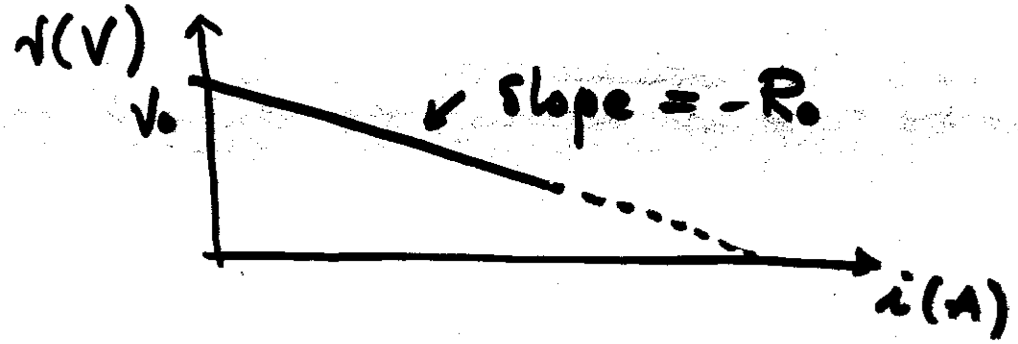
ramp



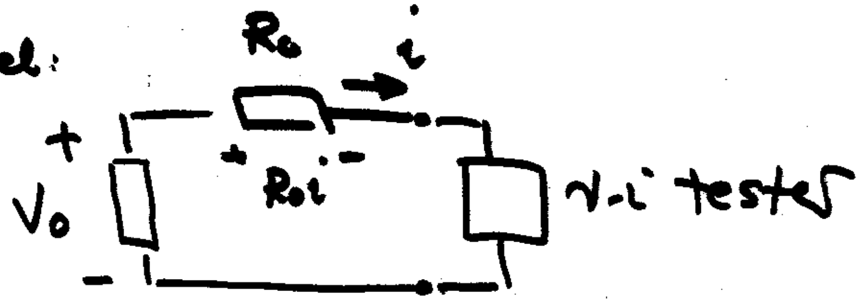
Active elements: $$\underset{\text{generated}}{\underset{\uparrow}{w(t)}} = \int_0^t p(s)\,ds \;\underset{\uparrow}{>}\; 0$$

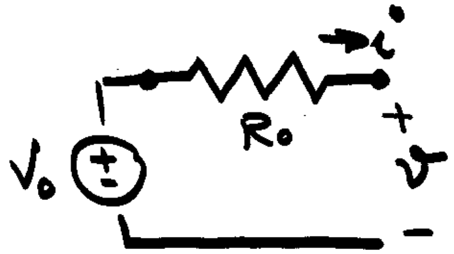


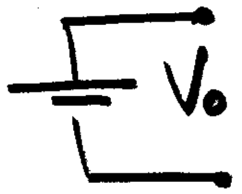




Model:





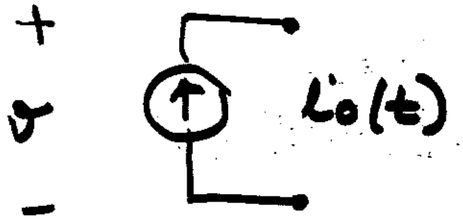


$$v = V_0 - R_0 i$$

↖ if small ⇒ $v = V_0$

Not dependent on i

The current source:

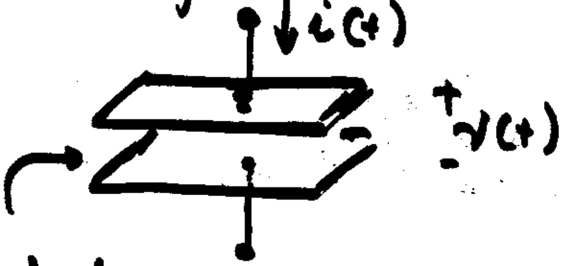


Does not depend on voltage v

Ideal!

<Current sources are often built from voltage sources and nonlinear resistive elements (transistors) to provide current that does not depend on voltage across the sources' terminals>

the Capacitor :



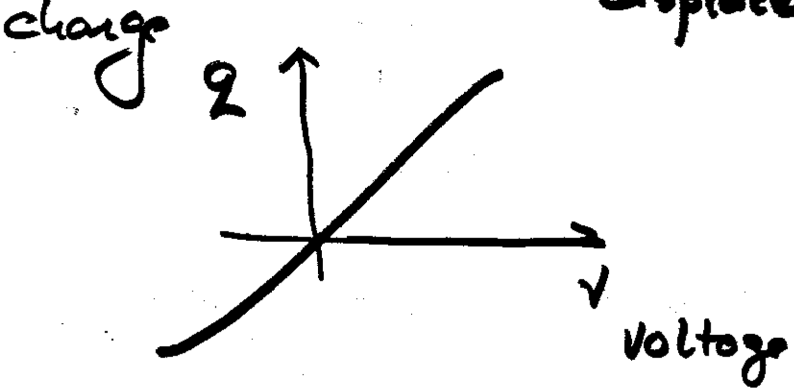
dielectric
- air
- paper
- cloth
- ⋮

electromagnetic field:

James Maxwell (1831-1879)

Maxwell equations:

-displacement current-



$$q = \text{function}(v)$$

$$\underset{\uparrow\text{time}}{q}(t) = \text{functn}\,(\underset{\uparrow\text{time}}{v(t)})$$

Current: $i = \frac{dq}{dt}$; $i = \frac{dq}{dv} \cdot \frac{dv}{dt}$

$$C = \frac{dq}{dv}$$

capacitance

(farad : F)

$\frac{\text{coulomb}}{\text{volt}}$

Aftr: Michael Faraday (1791-1867).

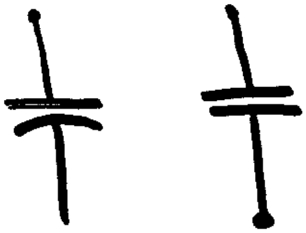
in general: $c = \text{function}(v)$

nonlinear capacitor

If $C = \text{constant}$

then $q = C \cdot v$

$$i = C \frac{dv}{dt}$$



$$i(t) = C \cdot \frac{dv(t)}{dt}$$

$$\Rightarrow \quad v(t) = \frac{1}{C}\int_{t_0}^{t} i(x)\,dx$$

or

$$v(t) = \frac{1}{C}\int i(x)\,dx + \underbrace{v(t_0)}_{\text{constant!}}$$

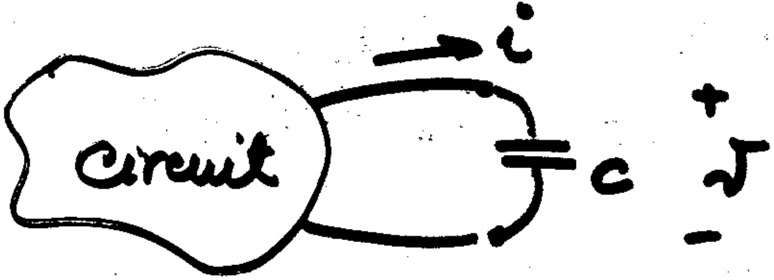
≡ initial value of v at time t_0

$$\text{Manufactured:}\quad \left.\begin{array}{l}\text{shapes}\\ \text{sizes}\\ \text{dielectric}\\ \text{power limits}\end{array}\right\rangle C$$

$$\text{Dielectric strength} = \frac{\text{Voltage}}{\text{Thickness}}$$

$$\text{Voltage rating} = \text{Dielec strength} \times \text{thickness}$$

Capacitr Energy:



Power: $p_c = v i$

$$p_c = v \cdot \underbrace{C \frac{dv}{dt}}$$

Energy:

$$w_c = \int_{-\infty}^{t} C \cdot v \cdot \frac{dv(x)}{dx} \cdot dx$$

$$W_c = \int_{-\infty}^{t} C \cdot v \cdot dv$$

$$= \frac{1}{2} \cdot C \cdot \left[v^2(t) - v^2(0) \right]$$

$$\boxed{W_c(t) - w_c(0)} = \underbrace{\frac{1}{2} C v^2(t)}$$

or. proportn $\sim q^2(t)$

$C \geq 0$

$v^2(t) \geq 0$

Capacitor is a passive element

$$W_c(t) \geq 0 \quad \forall t$$

↑ For any t!

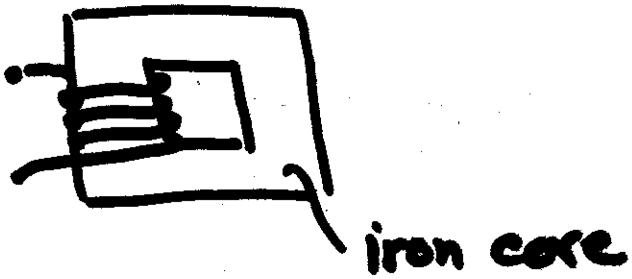
Energy is stored! (Unlike Resistors where it is dissipated into heat!)

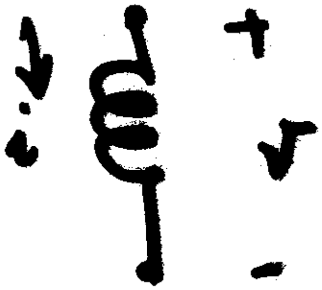
In a non-ideal case:

Some energy will be lost due to "leakage". i.e., a resistance present in the dielectric material

The inductor

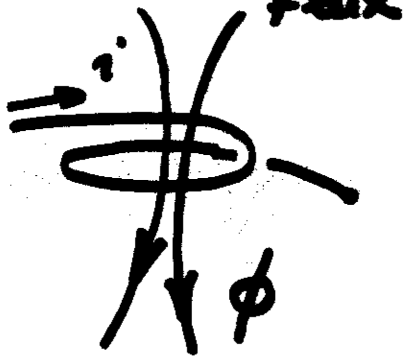
Also a two-terminal element





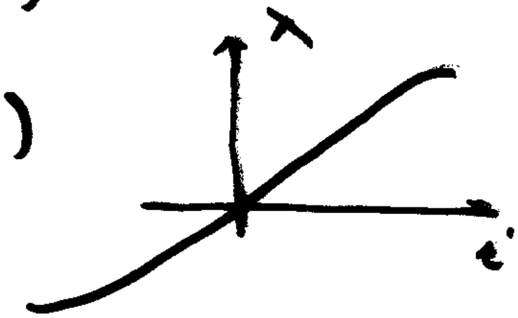
Magnetic flux

Faraday



theory of electromagnetic fields:

Self-flux: $\lambda = f(i)$



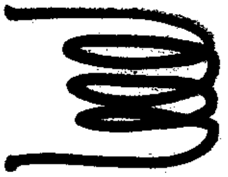
If the dependence is linear:

$$\lambda = \underset{\substack{\uparrow \\ \text{inductance}}}{L} \cdot i$$

$$\text{Henry} = \frac{\text{Weber}}{\text{Ampere}}$$

L depends on: medium (permeability)
geometry

$$L = k \cdot \underset{\nwarrow \text{ Number of turns}}{N^2}$$



Faraday: Discovered induction:

$$v = \frac{d\lambda}{dt}$$

$$v = \frac{d}{dt}(Li)$$

L: may be a function of i, t

Linear:

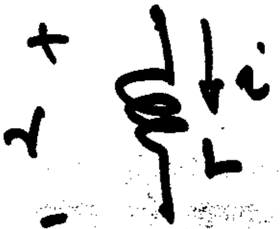
$$v = L \cdot \frac{di}{dt} \leftarrow$$

Hence.

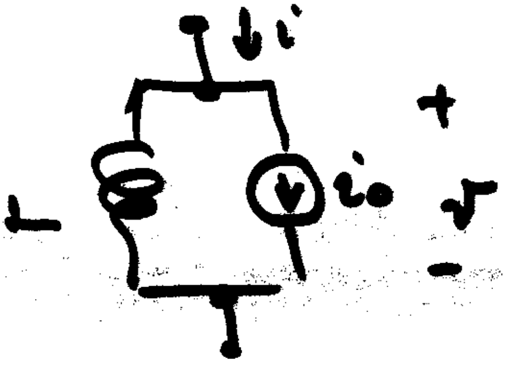
$$i(t) = \int_{t_0}^{t} v(x)dx + i(t_0) \leftarrow \text{initial value}$$

Joseph Henry (1797 – 1878)

Contemporary to Faraday.



No initial energy



With initial energy

Energy:

$$p = vi \quad \leftarrow \text{Power}$$

$$p = Li \cdot \frac{di}{dt}$$

$$W_L(t) = \int_0^t Li \cdot \frac{di}{dt} \cdot dx$$

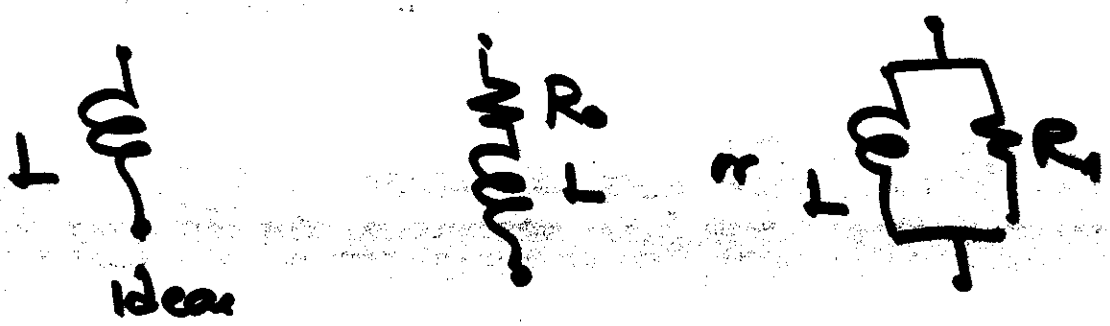
$$W_L(t) = \int_0^t L \cdot i \cdot di$$

$$= \frac{1}{2} L \left\{ i^2(t) - i^2(t_0) \right\}$$

$$W_L(t) - W_L(t_0) = \frac{1}{2} L \cdot i^2(t)$$

Energy is stored !

in case of leakage – some will be transformed into heat.



Conservation of : charge
flux

Hence: Capacitor : $i = C \cdot \frac{dv}{dt}$

continuous

inductor : $v = L \frac{di}{dt}$

continuous.

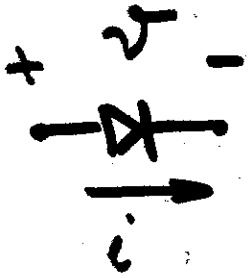
Linear: $\left.\begin{matrix} C>0 \\ L>0 \end{matrix}\right\rangle$ Passive !

The Diode

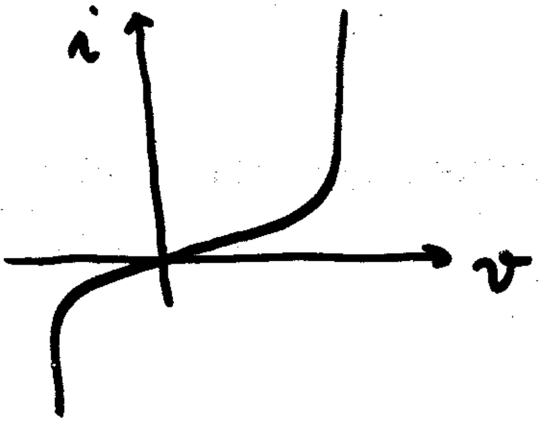
pn-junction : semiconductor device

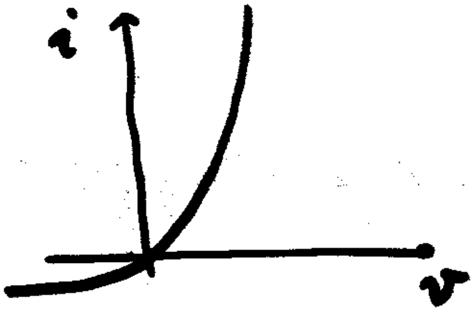
p: materials with "dopings" that add + charges (holes)

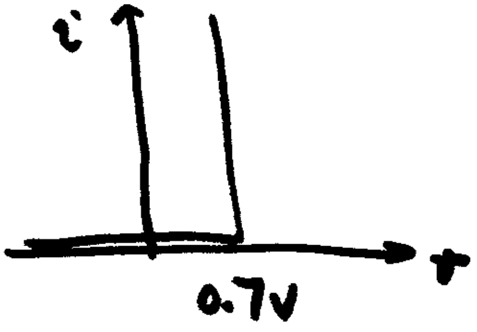
n: − charges (electrons)

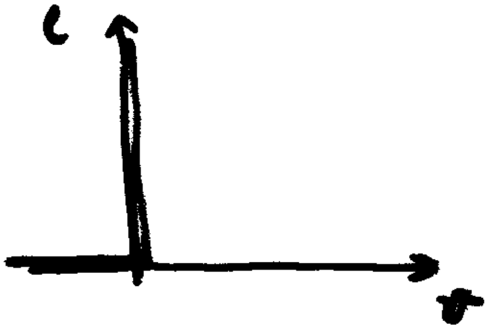


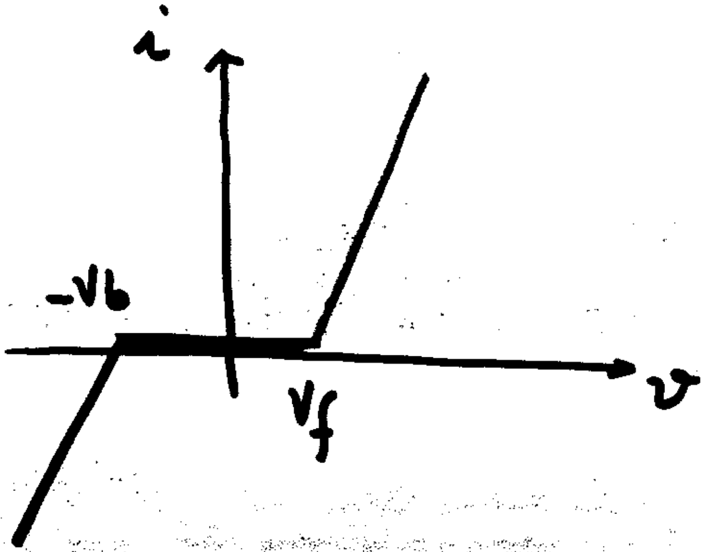
diode
nonlinear element
two-terminal





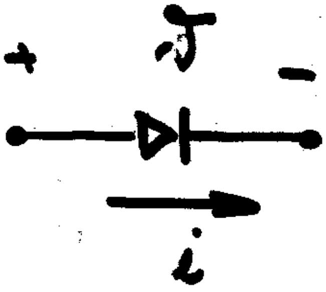






V_f: forward offset
0.7 V

V_b: reverse breakdown

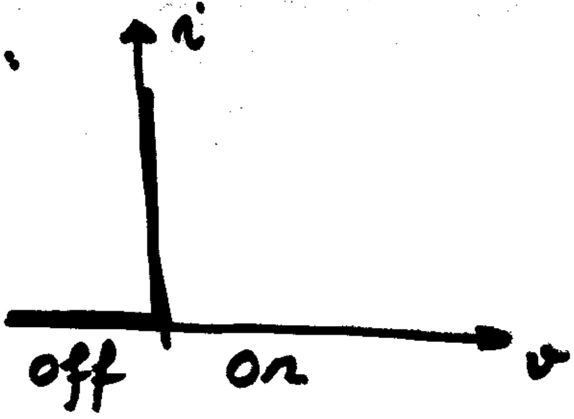


$v < -V_b$: resistor

$-V_b < v < V_f$: $i = 0$

$v > V_f$: resistor

Ideal diode:



$v < 0$ $i = 0$

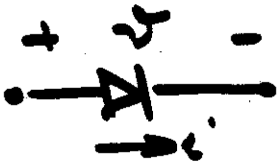
$v > 0$ short

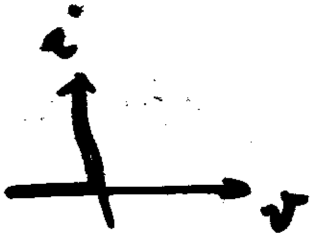
<Switch>

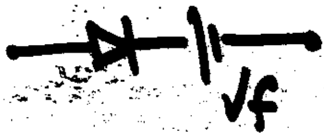
$p = v \cdot i$

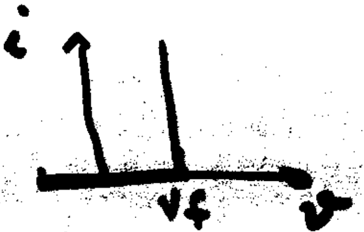
No power dissipation in a diode (ideal) → $p = 0$: Passive Lossless device

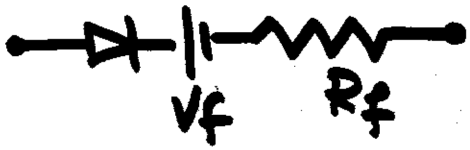
Models:

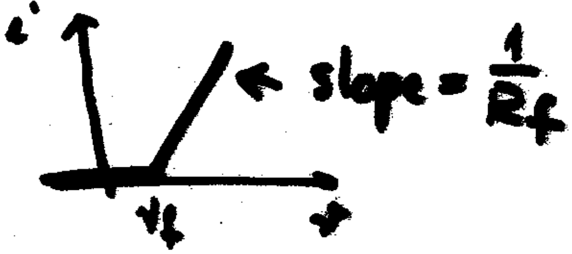


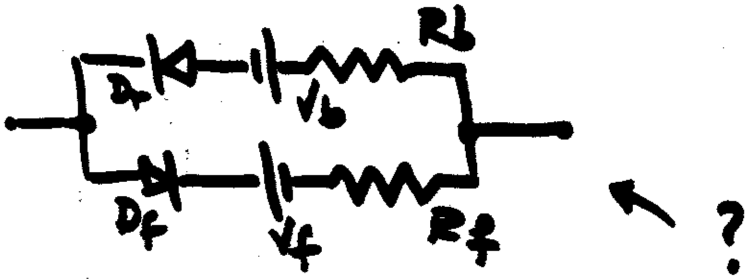












physical

model

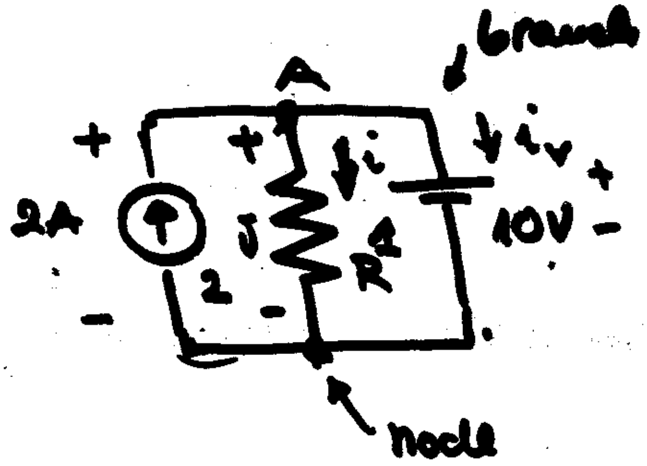
KCL

KVL

Elements: sources
R, C, L, diodes (many more)

Q: What are the values of voltages and currents established in a specific circuit?

Q: What is the power dissipated in an element?



$R = 1\,k\Omega$

$v = ?$

$i = ?$

KCL: $2 = i + i_V$

KVL: $v - 10 = 0$

Ohm(R): $v = 10^3 \cdot i$

No of equat. : 3
unknown : 3

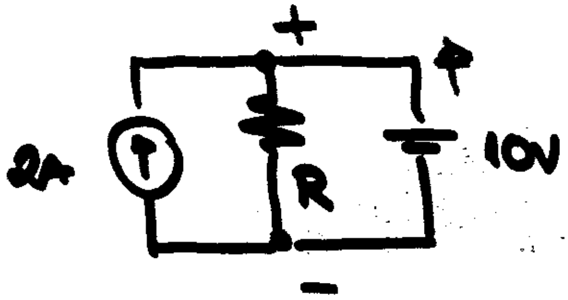
$v = 10\ V$

$i = 10\ mA$

$i_V = 1.99\ A$

#N-1

Problem 10: p. 58 (text)



$P_V = \frac{1}{2} P_1$: both sources deliver power

$V_R = 10V$

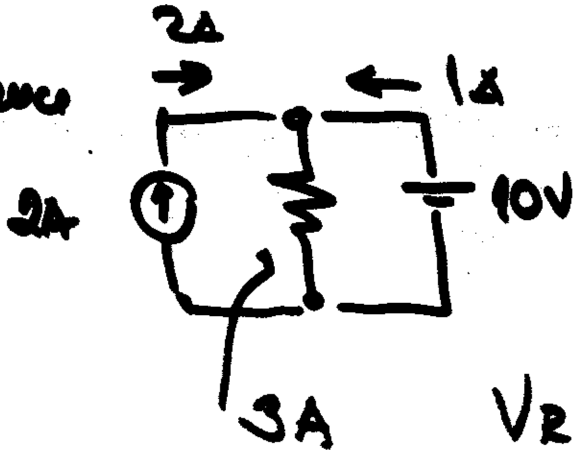
$P_1 = 2 \times 10$, $P_1 = 20W$

$P_V = \frac{1}{2} P_1 \Rightarrow P_V = 10W$

$V_V = 10V \Rightarrow P_V = V_V . I_V$

$I_V = 1A$

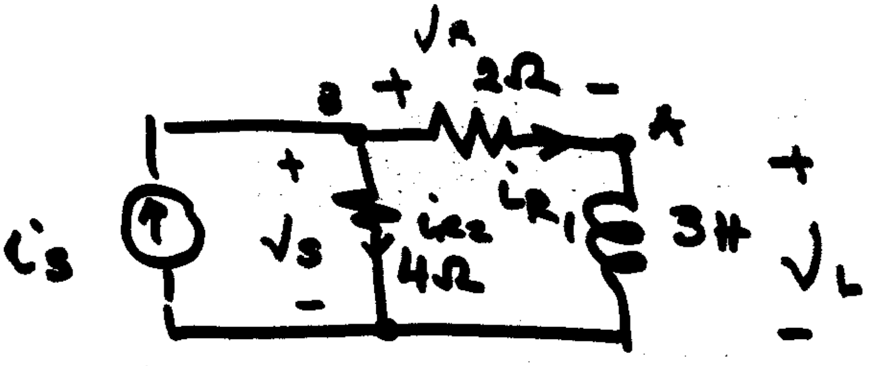
Hence



$V_R = R \cdot I_R$

$10 = R \cdot 3 \Rightarrow R = \frac{10}{3} \Omega$

We can also solve some simple circuits with L's and C's:



i : funct of time
I : DC (constant)

$$v_R(t) = -20e^{-2t} \text{ V}$$

Ohw's law: $V_R = 2 \cdot i_R$

$$i_{R_1} = \frac{1}{2} \cdot V_R \Rightarrow i_{R_1} = -10e^{-2t} \text{ A}$$

$$V_L = L\frac{di_L}{dt}$$

$i_L = i_{R_1}$ (KCL for node A)

$$V_L = 3 \cdot \frac{di_{R_1}}{dt} \; ; \; V_L = 60e^{-2t} \text{ V}$$

KVL: $V_S = V_R + V_L$; $V_S = (-20e^{-2t} + 60e^{-2t})$ V

$$V_S = 40e^{-2t}$$

$$i_{R_2} = \frac{V_S}{4} \text{ (A)} \quad i_{R_2} = 10e^{-2t} \text{ A}$$

KCL: $i_s = i_{R1} + i_{R2}$

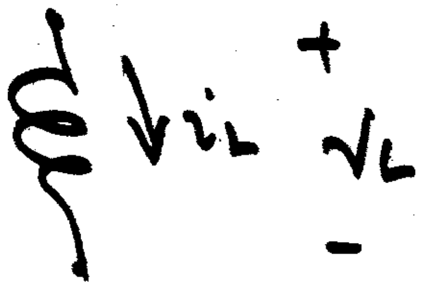
(B)

$i_s = 0$

!

How is this possible?

There is current in a circuit with supply current $i_s = 0$!

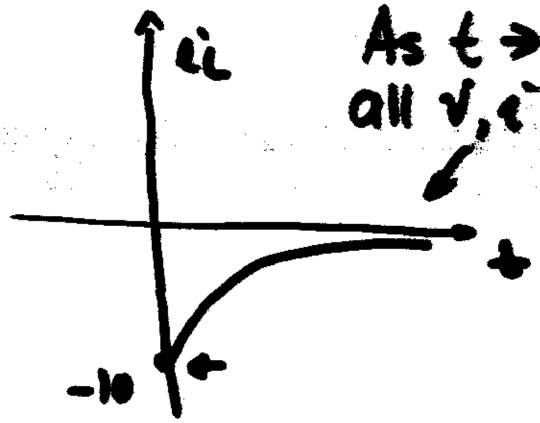


$i_L = -10e^{-2t}$ A

$i_L(0+) = -10$ A

$i_L(0-) = -10$ A

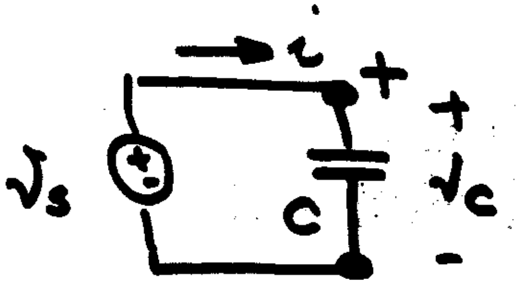
As expected:
As $t \to \infty$
all $v, i \to 0$

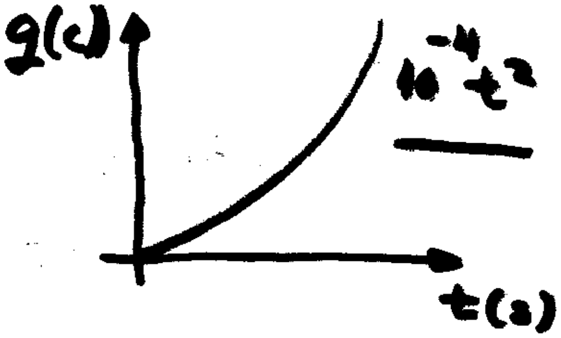


(Law of conservation of electromagnetic flux)

There was electromagnetic energy stored is the inductor that produces currents and voltages in the circuit.

Problem 18: p.59 (text)





C = 0.5 μF

a. Power delivered to the capacitor:

$\rightarrow i = \frac{dq}{dt}$; $\underline{i = 2\cdot 10^{-4} t}$ (A)

$\underline{q = C\cdot v_c}$ $\Rightarrow$ $v_c = \frac{1}{C}\cdot q$

$v_c = \frac{1}{0.5\times 10^{-6}} \times 10^{-4}\cdot t^2$ (V)

$\rightarrow v_c = 200t^2$ (V)

$V_s = v_c \leftarrow$

$p = v_s\cdot i$

$p = 200t^2\cdot 2\cdot 10^{-4}\cdot t$ (W)

$p = 4\times 10^{-2} t^3$ (W)

b. Energy stored in the capacitr at $t = 5s$

$$W(t) = \int_{-\infty}^{t} p(s)\,ds$$

$$= \int_{-\infty} V_c \cdot i_c \, ds$$

$$W(t) = \int_{-\infty}^{t} 4 \times 10^{-2} s^3 \, ds$$

$$W(5) = 4 \times 10^{-2} \cdot \frac{s^4}{4} \Big|_0^5$$

↖

$V_c = 0 \quad t \leq 0$

$i = 0 \quad t \leq 0$

$p = 0 \quad t \leq 0$

$w = 0 \quad t \leq 0$

$$W(5) = 4 \times 10^{-2} \cdot \frac{625}{4}$$

$$= 6.25\,W$$

c. $V_c(t)$ at $t=2s$

$$V_c(t) = 200t^2 \ (V)$$

$$V_c(2) = 200 \times 2^2 \ (V)$$

$$V_c(2) = 800 \ V$$

Resistive circuits: Voltage sources
current sources
R linear
R non-linear
diodes
transistrs.

L,C circuits: voltage sources
current sources
R (all kinds and types)
→ L, C

Active elements: sources

Passive elements: R linear
L
C
(R,L,C ≥ 0)

↓

Dissipative : $vi \geq 0$

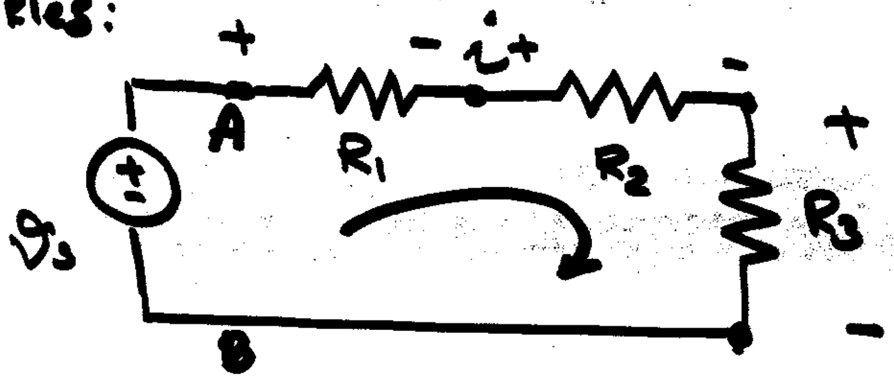
non-dissipative : $v \cdot i = 0$

↑ ideal transformers
shorts, open-circuits

3 Simple Dissipative Structures

Parallel and Series Circuits:

Series:



$$v_{R_1} + V_{R_2} + V_{R_3} = V_S \quad \text{KVL}$$

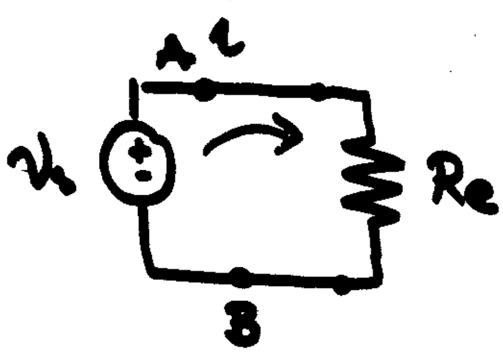
Ohm's law:

$$i \cdot R_1 + i \cdot R_2 + i \cdot R_3 = V_S$$

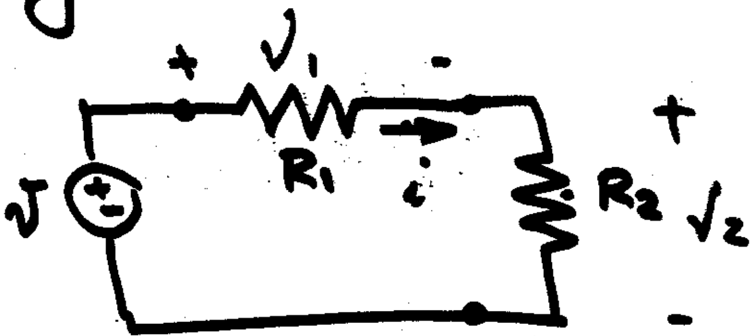
KCL :

$$i_{R_1} = i_{R_2} = i_{R_3}$$

$$i \cdot \underbrace{(R_1 + R_2 + R_3)}_{R_e = R_1 + R_2 + R_3} = V_S$$



Voltage divider:



$v_1 = R_1 \cdot i$

$v_2 = R_2 \cdot i$

$v = v_1 + v_2 \Rightarrow v = R_1 i + R_2 i$

$v = (R_1 + R_2) \cdot i$

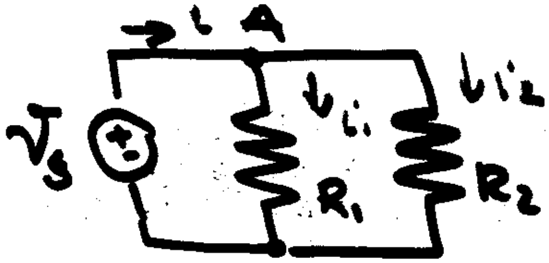
$i = \dfrac{v}{R_1 + R_2} \leftarrow$

$v_1 = R_1 i \Rightarrow v_1 = \dfrac{R_1}{R_1 + R_2} \cdot v \leftarrow$

$v_2 = \dfrac{R_2}{R_1 + R_2} \cdot v \leftarrow$

$$\frac{v_1}{v_2} = \frac{R_1}{R_2}$$

Parallel



$$v_{R_1} = v_{R_2} = v_S$$

$$i_1 R_1 = i_2 R_2$$

$$i = i_1 + i_2$$

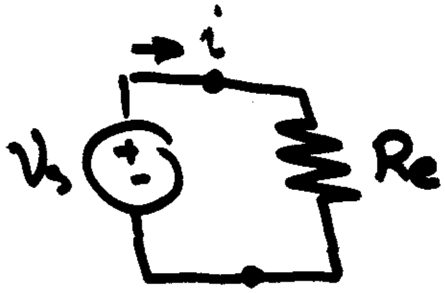
$$i_1 = \frac{v_{R1}}{R_1} \quad \text{or} \quad i_1 = G_1 \cdot v$$

$$i_2 = G_2 \cdot v$$

$$i = (G_1 + G_2) \cdot v$$

$$G_e = G_1 + G_2 \Rightarrow i = G \cdot v \leftarrow$$

$$\frac{1}{R_e} = \frac{1}{R_1} + \frac{1}{R_2} \; ; \; R_e = \frac{R_1 \cdot R_2}{R_1 + R_2}$$



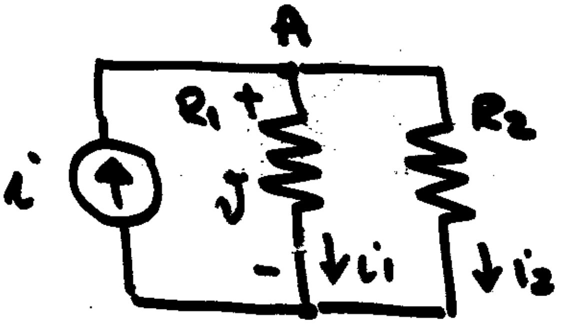
$G = \frac{1}{R}$ — Ω

conductance

resistance

Siemens, Mhos ℧

Current divider:



$i = i_1 + i_2$

$i_1 = G_1 v$

$i_2 = G_2 \cdot v$

$i = (G_1 + G_2) \cdot v \Rightarrow v = \frac{i}{G_1 + G_2}$

$i_1 = \frac{G_1}{G_1 + G_2} \cdot i$

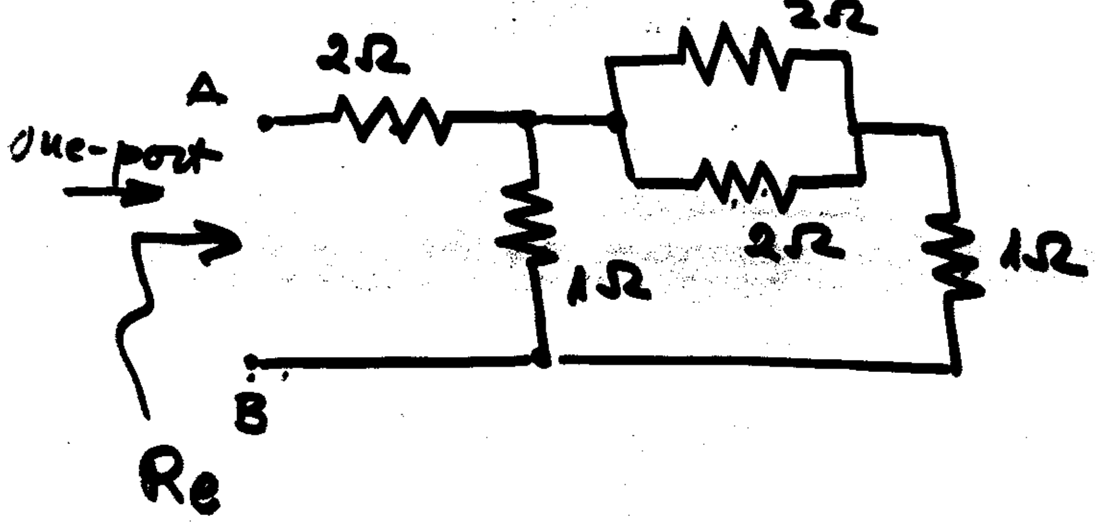
$i_2 = \frac{G_2}{G_1 + G_2} \cdot i$

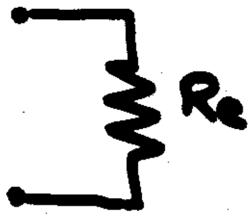
$\frac{i_1}{i_2} = \frac{G_1}{G_2}$ or $\frac{i_1}{i_2} = \frac{R_2}{R_1}$

(Recall: $R = \frac{1}{G}$)

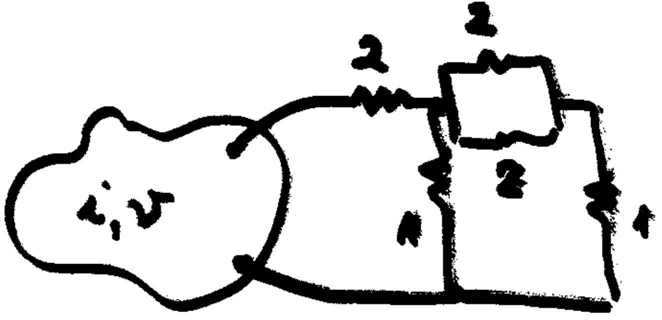
Finding equivalent resistance:

Common procedure for simplyfying circuits:



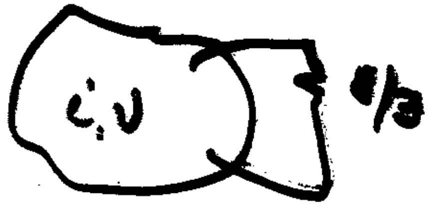


$$Re = 2\Omega + \frac{1\Omega \times \left(\frac{2\Omega \cdot 2\Omega}{2\Omega + 2\Omega} + 1\Omega\right)}{1\Omega + \frac{2\Omega \cdot 2\Omega}{2\Omega + 2\Omega} + 1\Omega}$$

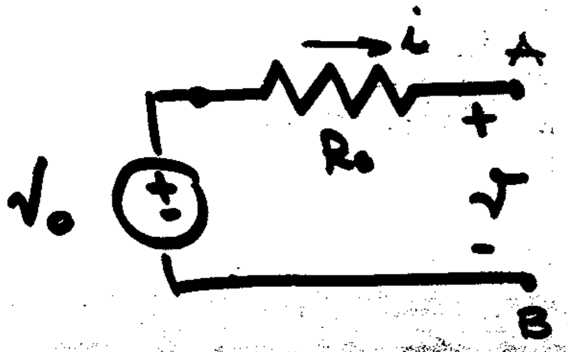


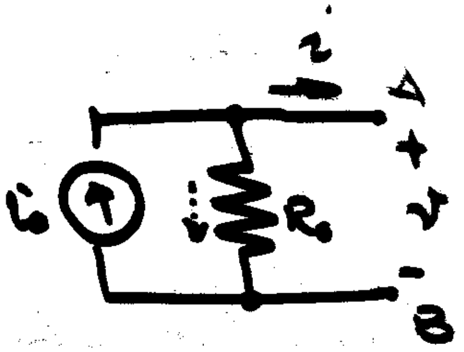
$$Re = 2 + \frac{2}{3}$$

$$Re = \frac{8}{3}\Omega$$



2. Source transformation





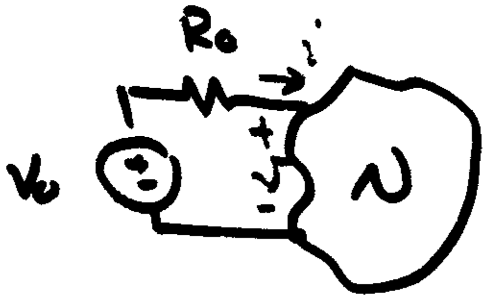
$$v = v_0 - R_0 i \qquad\qquad i = i_0 - \frac{v}{R_0}$$

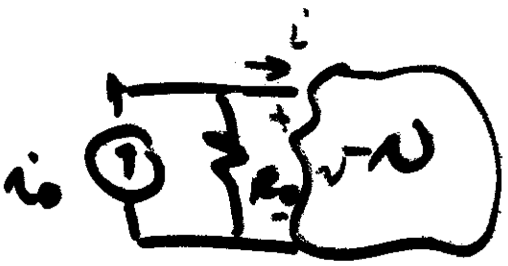
$$\downarrow \qquad\qquad\qquad \downarrow$$

$$i = \frac{v_0}{R_0} - \frac{v}{R_0} \qquad\qquad i = i_0 - \frac{v}{R_0}$$

equivalent if

$$i_0 = \frac{v_0}{R_0} \quad \text{or} \quad v_0 = R_0 \cdot i_0$$



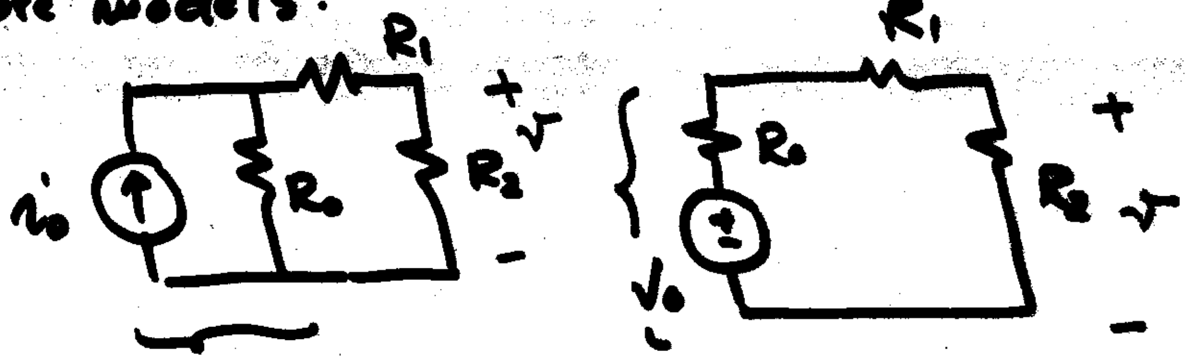


$$i_0 = \frac{v_0}{R_0}$$

$v_0 = R_0 i_0$ is not Ohm's Law

v_0 and i_0 are in different circuits!

Soure models:



$v_0 = R_0 i_0$

accompanid sources

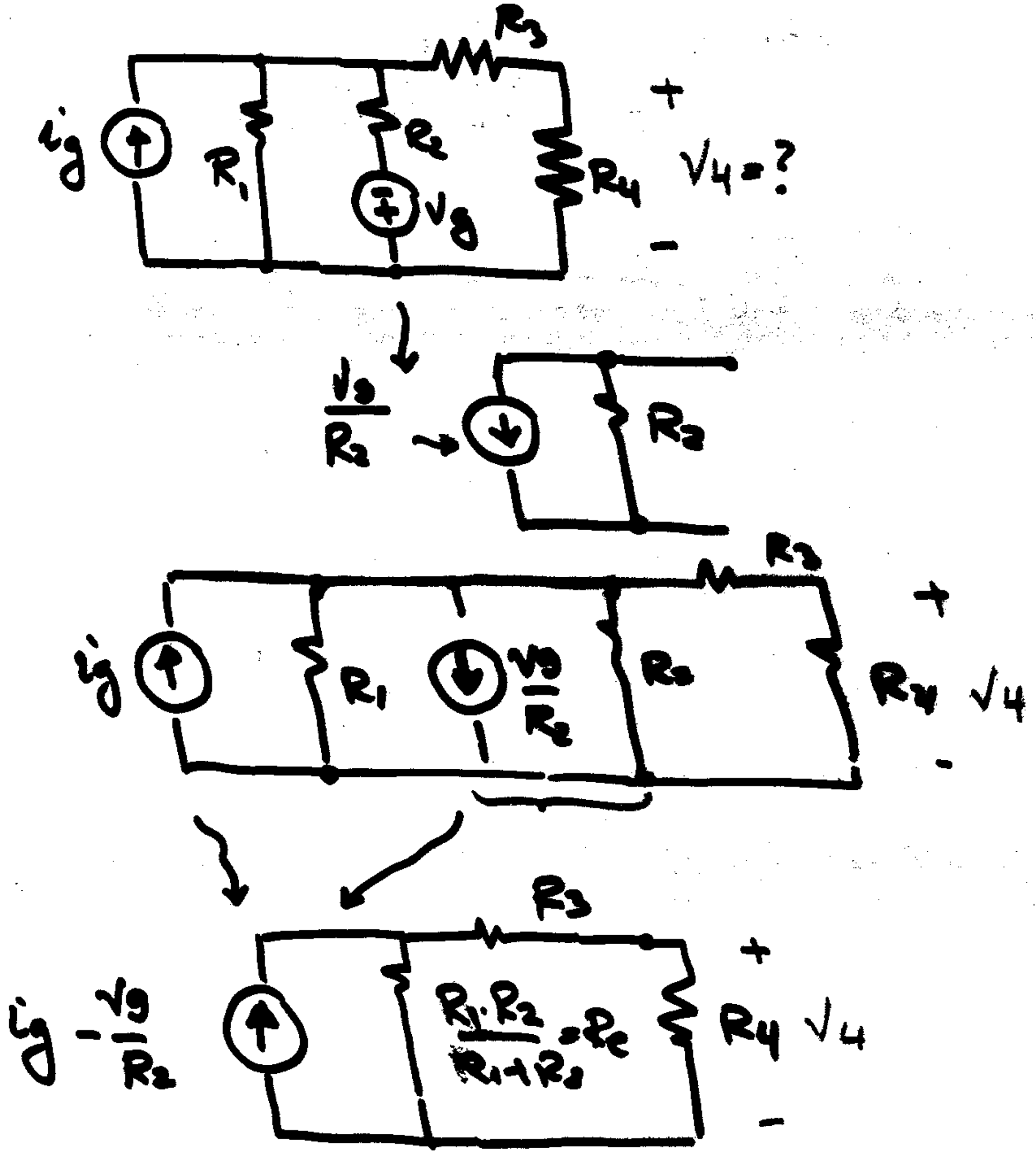
↘ current
voltage

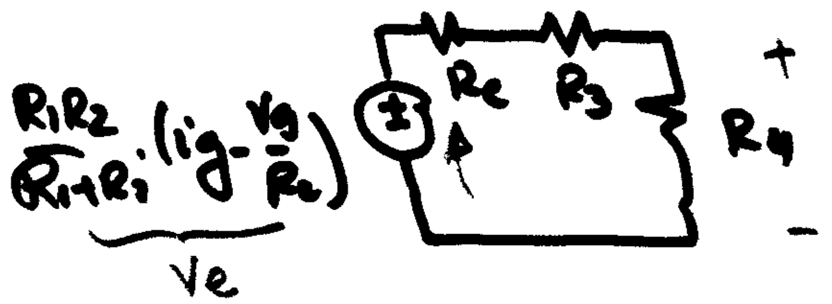
$$v = \frac{R_2}{R_0 + R_1 + R_2} \cdot R_0 i_0$$

voltage divider

Useful when solving and simplifying circuits.

"Solving" circuits: simplification helps





$$i_4 = \frac{V_e}{R_e + R_3 + R_4}$$

$$V_4 = R_4 \cdot i_4$$

3. Basic Principles

Linearity and superposition:

input = x_1 output = y_1

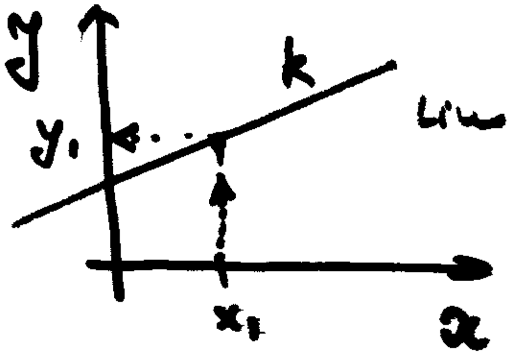
= x_2 = y_2

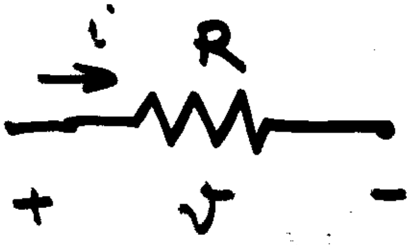
input = $k_1 x_1 + k_2 x_2$

— output = $k_1 y_1 + k_2 y_2$ ← linearity

↖ superposition

Linear system: output is proportional to the input and the principle of superposition holds



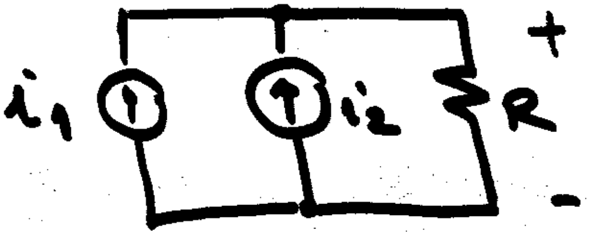


$$\underset{\underset{\text{output }(y)}{\uparrow}}{v} = R\cdot \underset{\underset{\text{input }(x)}{\nwarrow}}{i}$$

linearity holds:

$$y = \underset{\underset{\underset{\text{constant}}{R}}{\downarrow}}{k}\cdot x$$

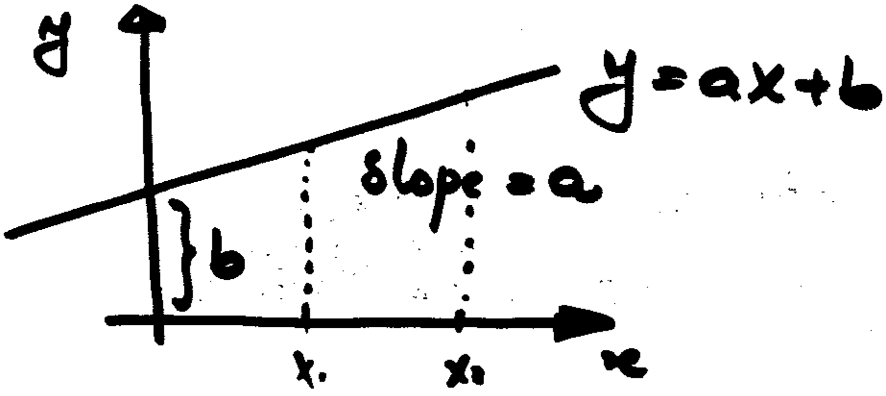
How about superposition?



$$v = R\cdot(i_1 + i_2)$$

$$v = \underbrace{R\cdot i_1}_{v_1} + \underbrace{R\cdot i_2}_{v_2}$$

$$v = v_1 \underset{\underset{\text{superposition}}{\nwarrow}}{+} v_2$$



x_1 : $y_1 = ax_1 + b$

x_2 : $y_2 = ax_2 + b$

Afine

$$x_1 + x_2 : \quad y_3 = a(x_1 + x_2) + b$$
$$= ax_1 + ax_2 + b$$
$$\neq \underset{ax_1+b}{\underset{\downarrow}{y_1}} + \underset{ax_2+b}{\underset{\searrow}{y_2}}$$

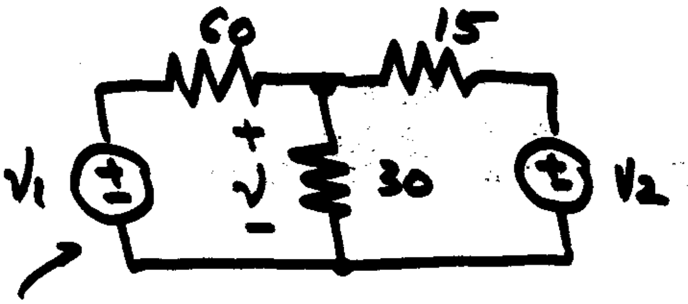
$b = 0$

x_1 : $y_1 = ax_1 + b$

$$kx_1 : \quad y_2 = a\cdot(kx_1) + b$$
$$y_2 = k\cdot\left(ax_1 + \frac{b}{k}\right)$$
$$\neq k\cdot y_1$$

Input : $k.x_1$ output $\neq k.y_1$ Not linear!

How can we use superposition to solve simple circuits?



$v_1 = 21 \sin 400t$

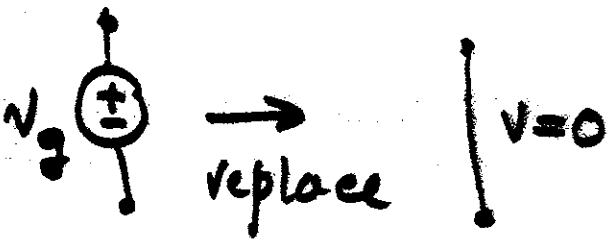
$v_2 = 14 \cos 400t$

Superposition:

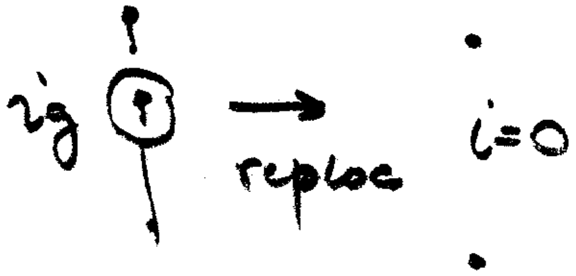
Let v_1 be present, disconnect v_2

< How to disconnect a voltage source! >

$$v_1 \begin{cases} \dfrac{30 \times 15}{45} = 10 \\ v_1 = \dfrac{10}{70} \cdot 21 \sin 400t \end{cases}$$



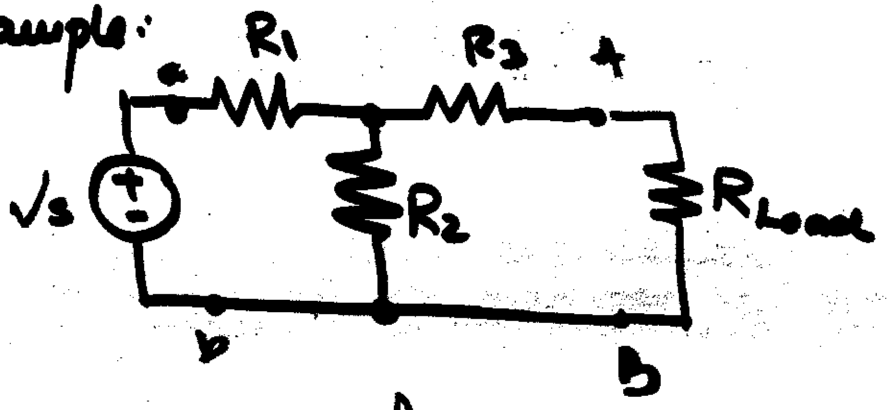
$$v_2 \begin{cases} \dfrac{60 \times 30}{90} = 20 \\ v_2 = \dfrac{20}{35} \cdot 14 \cos 400t \end{cases}$$

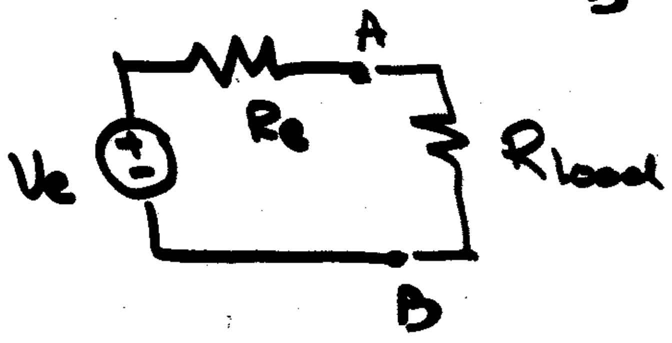


$v = 3 \sin 400t + 8 \cos 400t \leftarrow$

thévenin theorem:

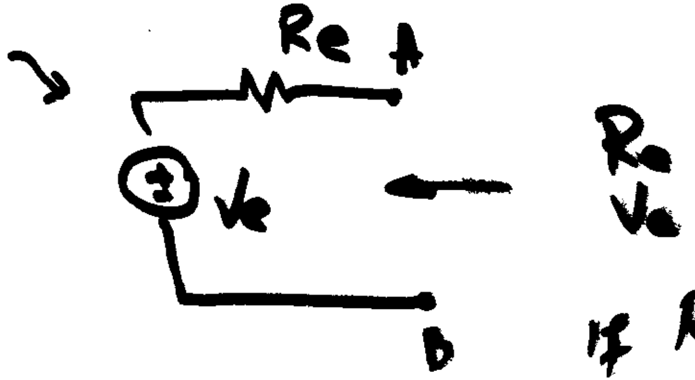
Example:



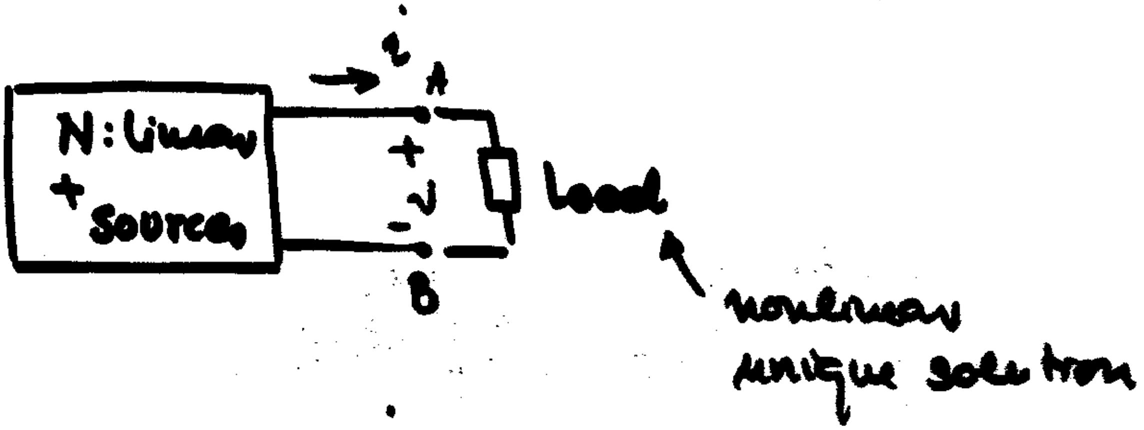


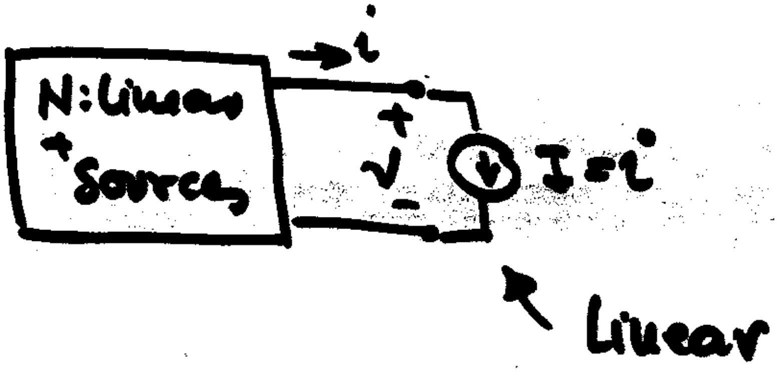
$$V_E = V_S \cdot \frac{R1}{R_1 + R_2}$$

$$R_e = \frac{R_1 \cdot R_2}{R_1 + R_2} + R_3$$

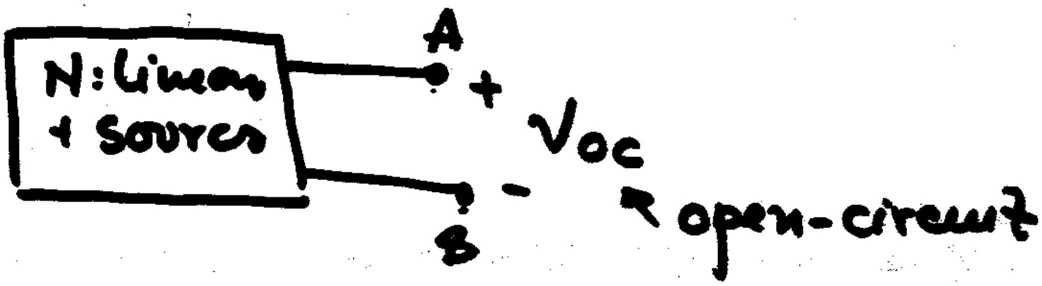


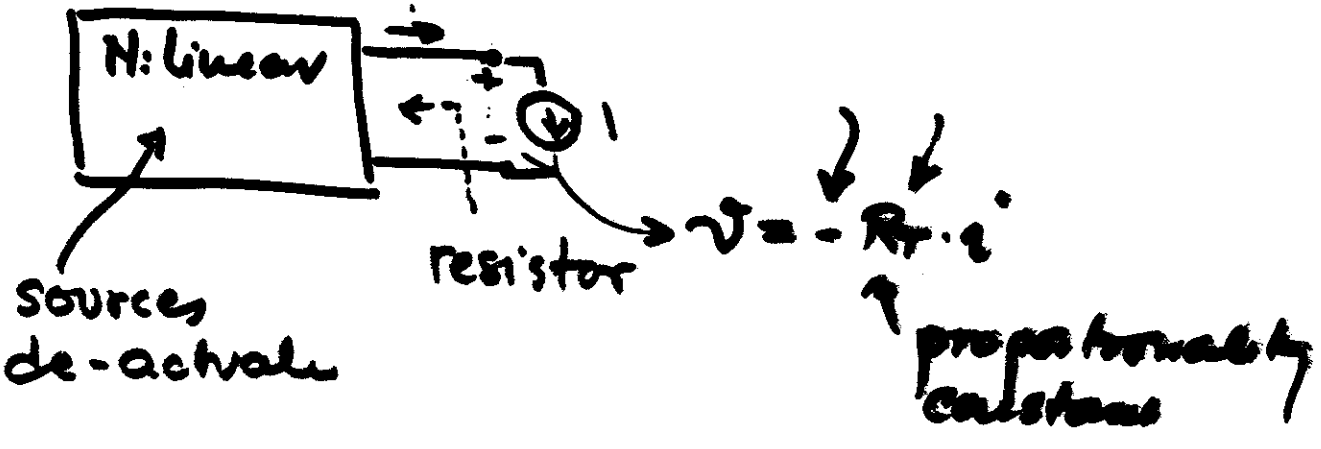
If R_L is removed (open-circuited)





De-activate source I (I = 0)

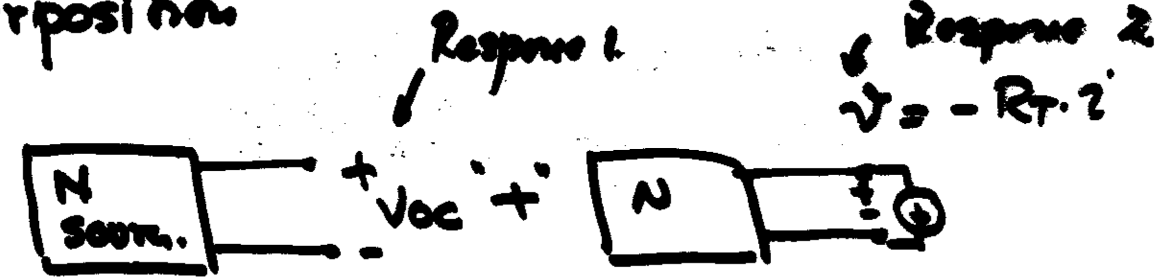




$v = - R_T \cdot i$

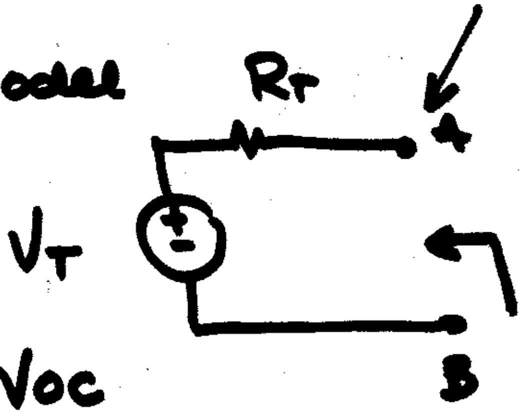
Note:

Superposition



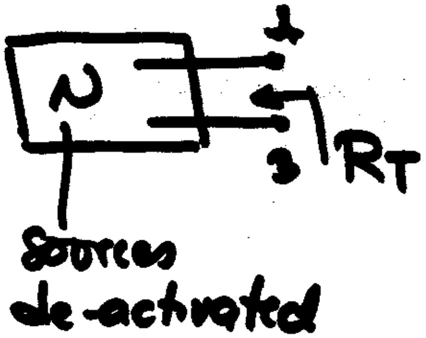
Total response $\underline{V_{oc}} - \underline{R_T} \cdot i$ ←

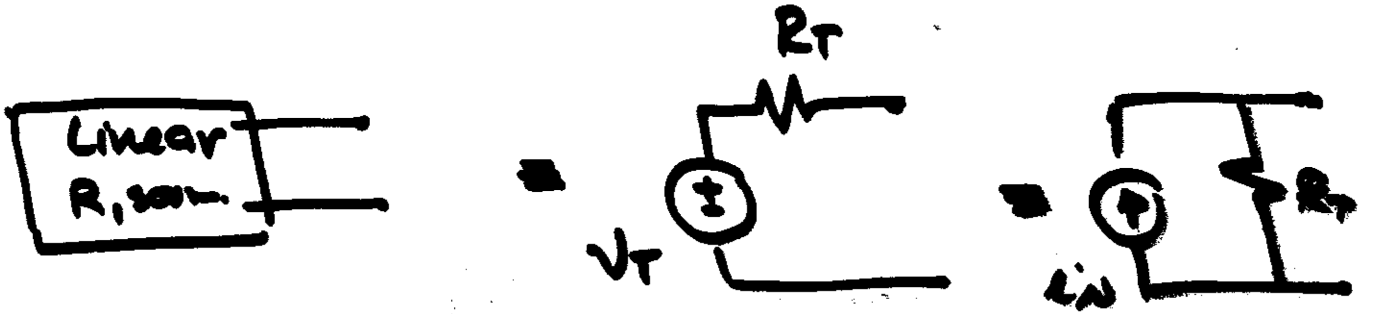
Model



$v_T = V_{oc}$

R_T:





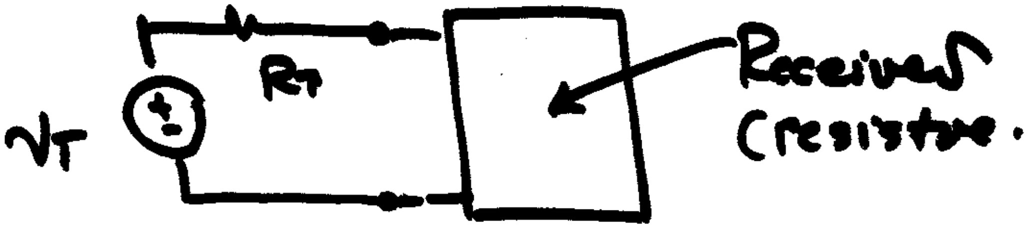
Thévenin (1885) equivalent circuits Norton (1935)

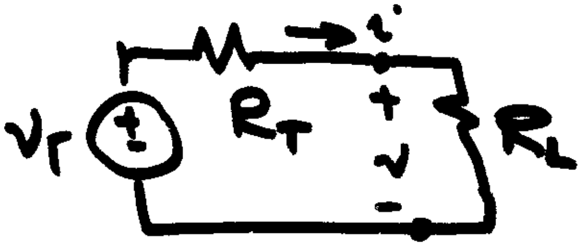
$$V_T = R_T \cdot i_N$$

Not ohm's Law!

V_T, i_N are in different circuits

Maximum power transfer to a load:





$$i = \frac{V_T}{R_T + R_L}$$

$$v = \frac{R_L}{R_T + R_L} \cdot V_T$$

$$P_{\text{delivered to the receiver}} = i \cdot v$$

$$P = \frac{v_T^2 \cdot R_L}{(R_T + R_L)^2}$$

What is the value of R_L so that the power delivered to the receiver is maximal:

$$P(x) = \frac{x}{(R_T + x)^2} \cdot v_T^2 R_L$$

$$\frac{dP}{dx} = \frac{(R_T+x)^2 - 2\cdot(R_T+x)\cdot x}{(R_T+x)^4} v_T^2 R_L$$

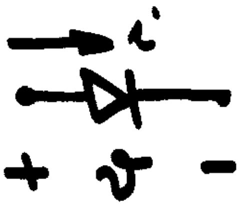
$$\frac{dP}{dx} = 0 \quad \text{if} \quad (R_T+x)^2 - 2(R_T+x)\cdot x = 0$$

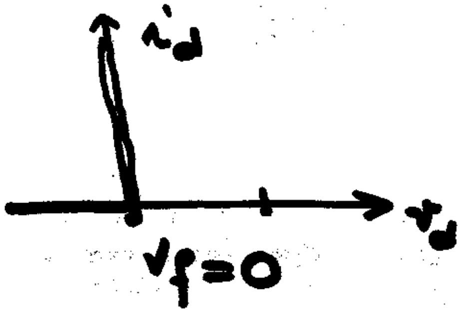
$$\underset{\underset{\neq 0}{\downarrow}}{(R_T+x)} \cdot \underset{\underset{\underline{x = R_T} \text{ or } R_L = R_T \swarrow}{\downarrow}}{(R_T + x - 2x)} = 0$$

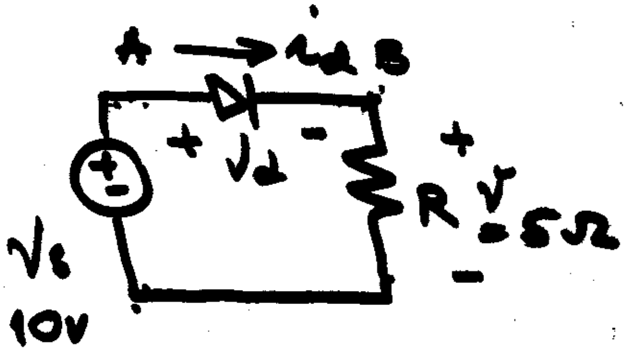
"matching"

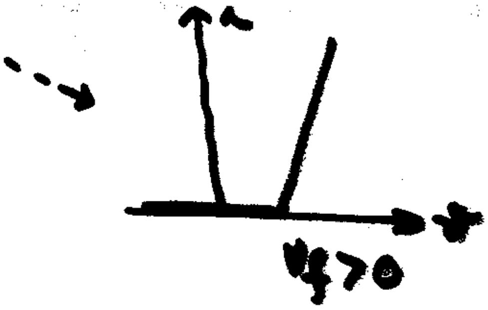
Diode Circuits

- Simplest nonlinear circuits:









$\rightarrow v_s > 0 \Rightarrow v_d = 0 \Rightarrow i = \frac{v_s}{R}$ ON

$v_s < 0 \Rightarrow i_d = 0 \Rightarrow i = 0$ OFF

switch

We can solve simple diode circuits by assuming their state and then checking the circuit for consistency:

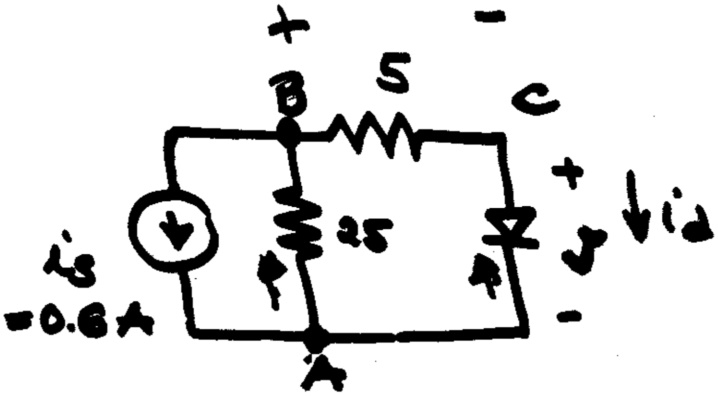
Above: $\underline{v_s > 0}$ OFF $\Rightarrow i_d = 0 \Rightarrow v = 0$

KVL $\Rightarrow v_d = v_s = 10v$

↓

?!?

⟨OFF⟩!

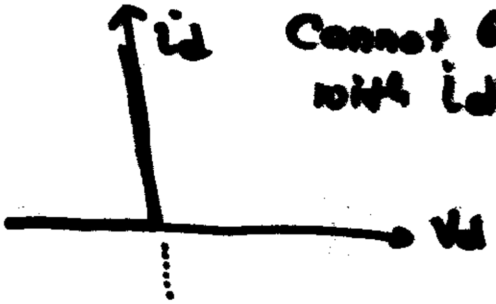


Assume diode ON: $R_e = \frac{5 \cdot 25}{30}\ \Omega$;

$$i_d = \frac{-25}{25+5} \cdot i_s$$

$$i_d = -\frac{25}{30} \cdot 0.6$$

$$i_d = -0.5A \ !!$$

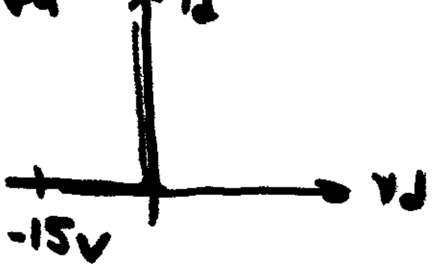


Cannot be ON with $i_d < 0$

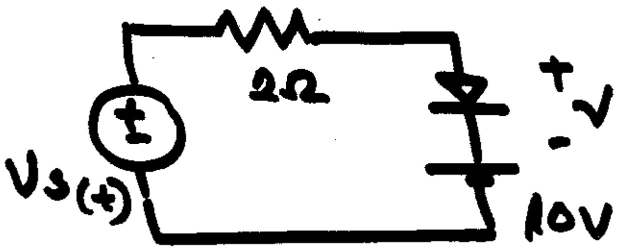
Hence, Assume diode is OFF:

$i_d = 0$

$v_d = -15V$ which is consistent with



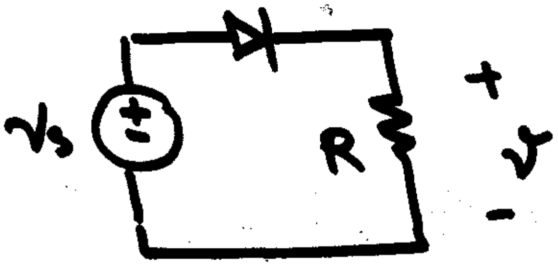
How about time-varying sources:

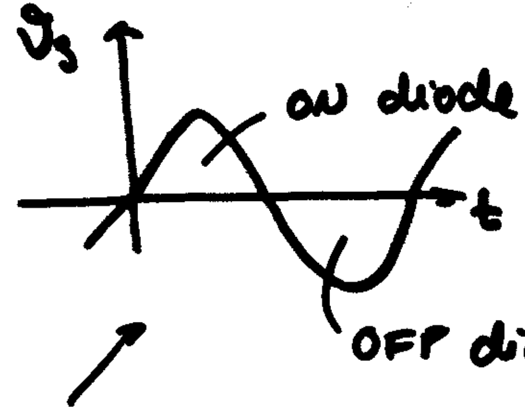


If $V_s > 10V \rightarrow$ Diode is ON $\quad i = \frac{V_s - 10}{2}$

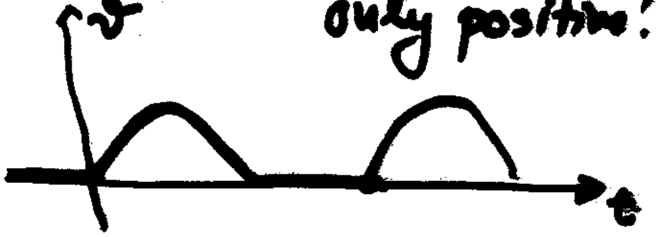
$V_s < 10V$ OFF $\quad i = 0$

Rectifiers:



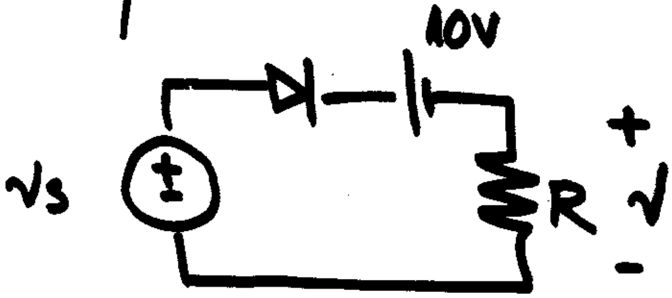


bidirectional



unidirectional!
rectified!

Comparators :



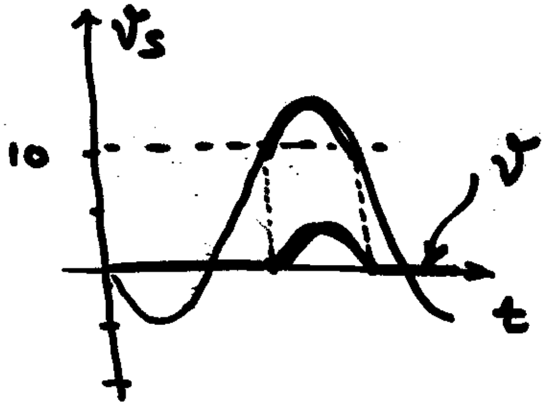
$v = v_s - 10$ if $v_s > 10V$

$v = 0$ if $V_s < 10V$

Hence : 10V is critical :

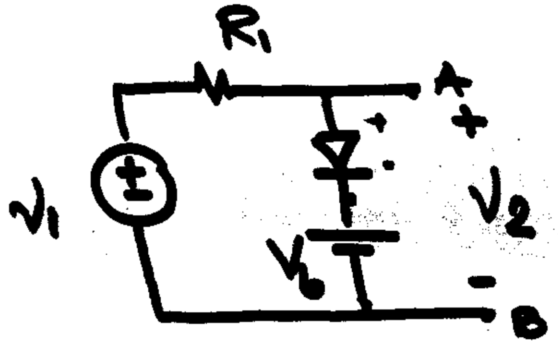
If Input $> 10V$ → there is $v_{acrossR} > 0$

< 0 no voltage



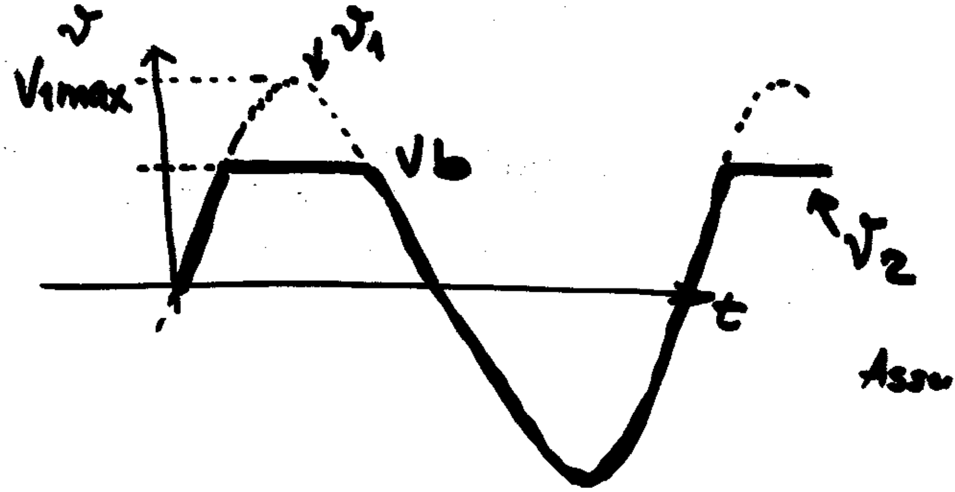
Voltage limiters:

prevents voltage from becoming too large



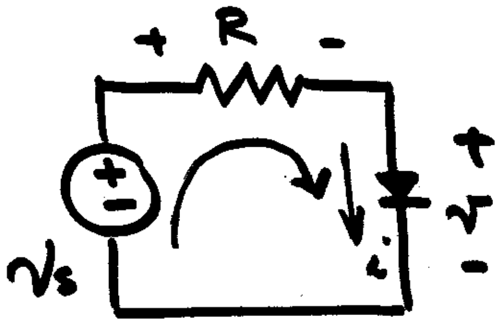
If diode is OFF: $v_2 = v_1$

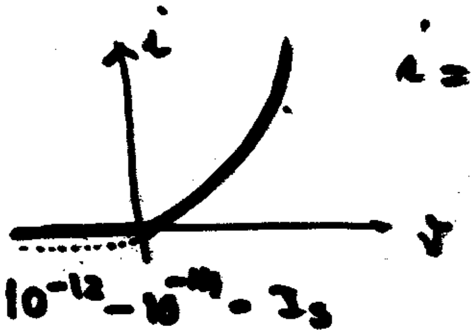
ON: $v_2 = V_b$ (fixed DC)



Assume
$V_b < V_{1max}$

Exponential diodes:





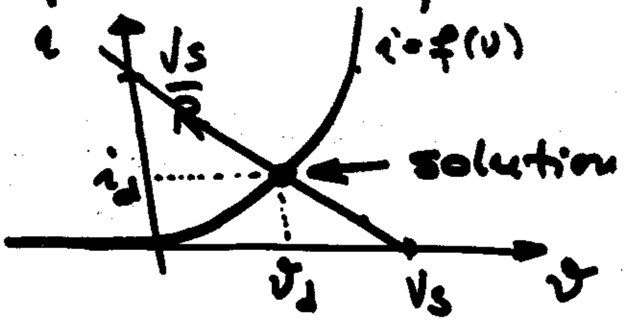
$i = \left(e^{\eta v} - 1\right) I_s$

How can we "solve" circuits with diodes that are not modeled with ideal switches:

KVL: $V_S - Ri - v = 0$ Nonlinear equation.

$V_S - Ri = v$

DC: function of current



$V_S - Ri = v$

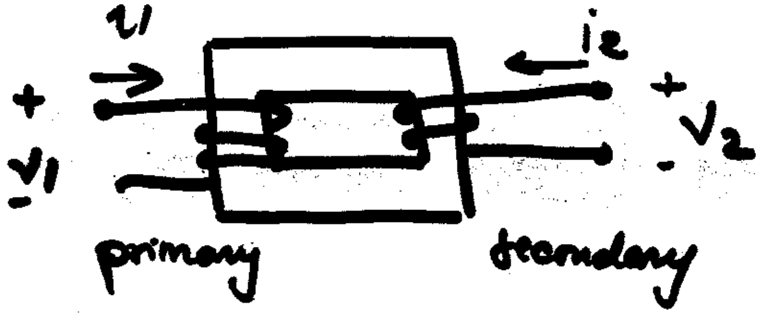
$$i = \frac{V_S - v}{R}$$

Also i = function (v)
diode characteristics

$i = f(v)$

Chapter 4 : Multiterminal Components

1. Ideal transformer



Transformer

4-terminals arranged into 2-ports

$$\frac{v_1}{v_2} = \underbrace{\frac{N_1}{N_2}}_{n} \quad \text{Faraday's Law}$$

n ← turns ratio

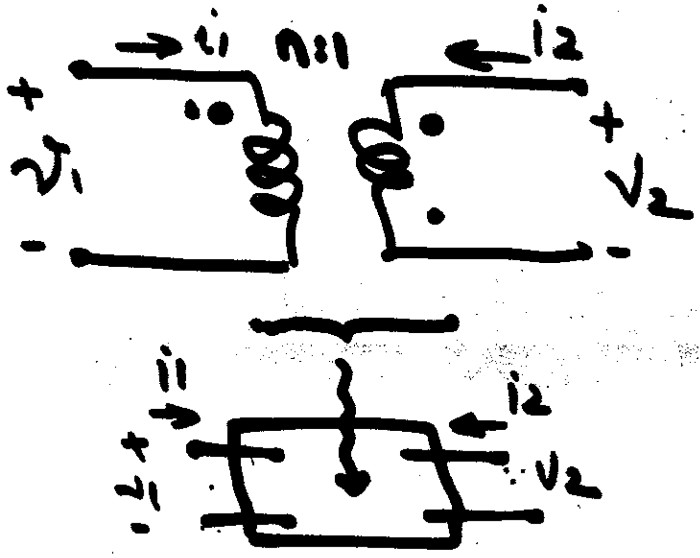
→ $v_1 = n v_2$

$$i_2 = -\frac{N_1}{N_2} i_1 \qquad \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 & +n \\ -n & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ v_2 \end{pmatrix}$$

→ $i_2 = -n \cdot i_1$

Ampère's Law

Circuit symbol:



$$\left\langle \begin{matrix} v_1 = -n \cdot v_2 \\ i_2 = n \cdot i_1 \end{matrix} \right\rangle$$

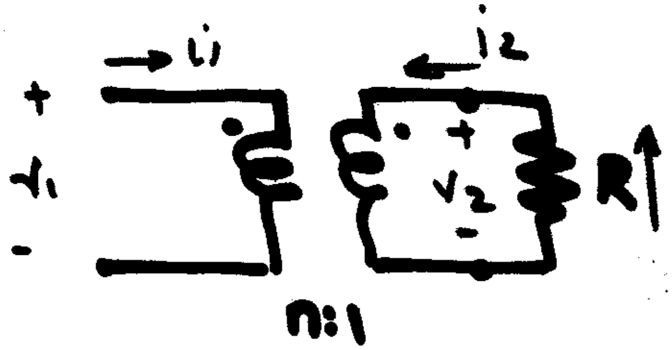
$$\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ v_2 \end{pmatrix}$$

$$p = v_1 i_1 + v_2 i_2$$

$$p = (n v_2) \cdot i_1 + v_2 (-n \cdot i_1)$$

$$\underline{p = 0}$$

passive
→ lossless device



$$\left\langle \begin{matrix} v_1 = n v_2 \\ i_2 = -n \cdot i_1 \end{matrix} \right\rangle \quad \underline{v_2 = -R \cdot i_2} \quad \text{(Ohm's law)}$$

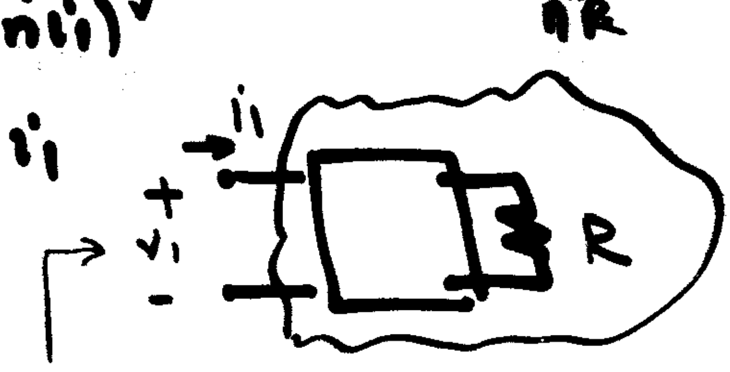
Then:

$v_1 = n \cdot (-R \cdot i_2)$

$v_1 = -n \cdot R \cdot i_2 \checkmark$

$v_1 = -nR(-n i_1) \checkmark$

$v_1 = \underline{n^2 R} \cdot i_1$



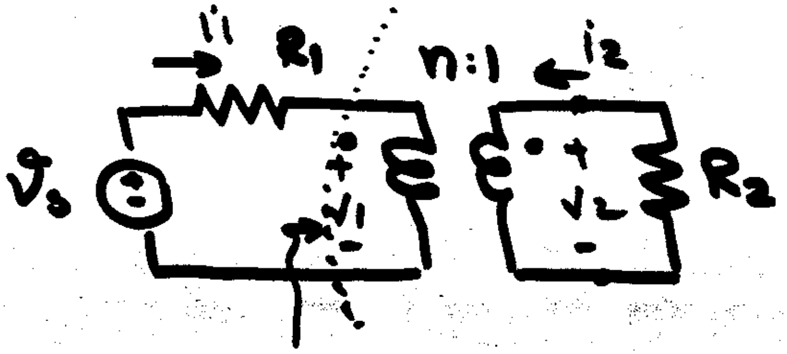
$v_1 = (n^2 R) \cdot i_1$

$v_1 = \text{Resistance} \cdot i_1$

$\text{Resistance} = n^2 R$

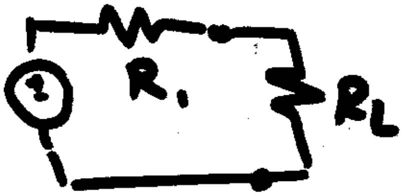
Transformer !

ideal transformer : used for matching!



$R_e = n^2 \cdot R_2$ ✓

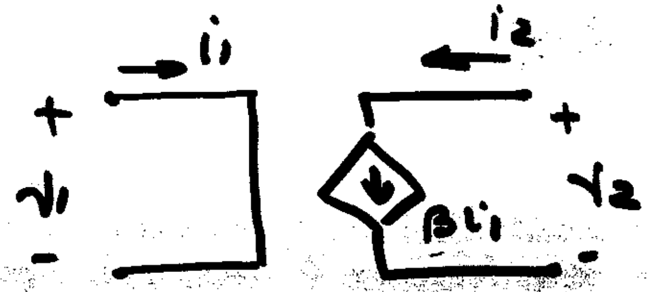
Matching : $\underline{R_1 = n^2 R_2} \Rightarrow \underline{n} = \sqrt{\frac{R_1}{R_2}}$



$R_L = R_1$

Controlled Sources:

<Dependent>

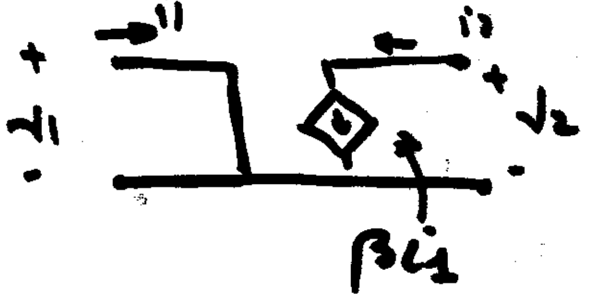


$v_1 = 0$

$i_2 = \beta i_1$

Linear

β = constant



$v_1 = 0$

$\checkmark i_2 = \beta i_1 \checkmark \leftarrow$

$$\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ v_2 \end{pmatrix}$$

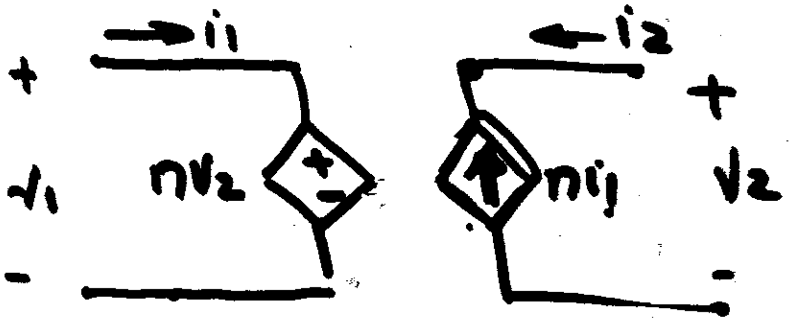
Current controlled current source

<CCCS>

4-types of controlled sources:

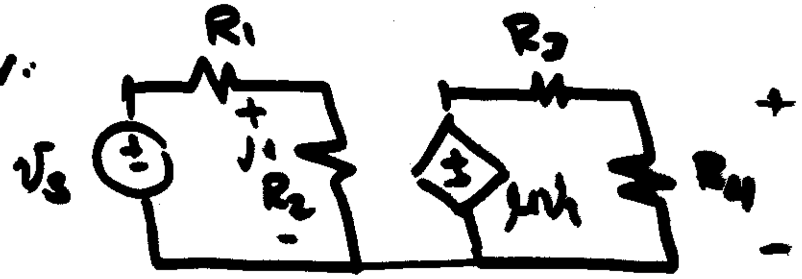
Controlled sources are used to model other circuit elements:

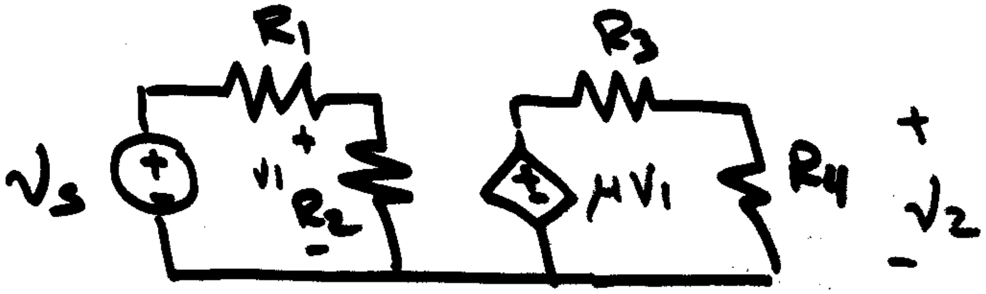
ideal transformers
transistors
op-amps
⋮



$$\left.\begin{array}{l} \longrightarrow v_1 = nv_2 \\ \longrightarrow i_2 = -ni_1 \end{array}\right\rangle \text{ ideal transformer}$$

Amplifier:





KCL, KVL, Ohm's Law

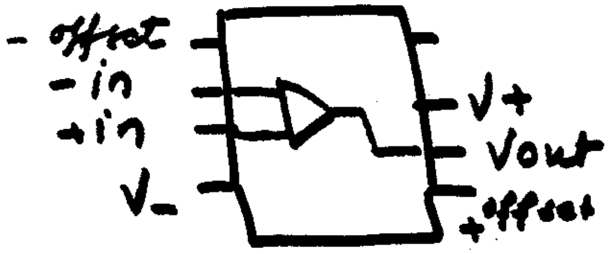
$$v_1 = \frac{R_2}{R_1+R_2}\cdot v_s$$ Voltage divider

$$V_2 = \frac{R_4}{R_3+R_4}\cdot \mu\cdot V_1$$ $-\|-$

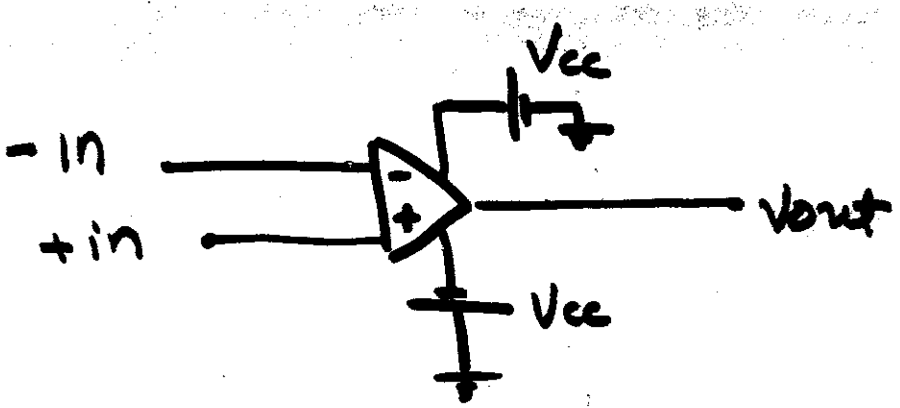
$$v_2 = \frac{R_4}{R_3+R_4}\cdot\frac{R_2}{R_1+R_2}\cdot\mu\cdot v_s$$

Amplifier

Operational Amplifiers



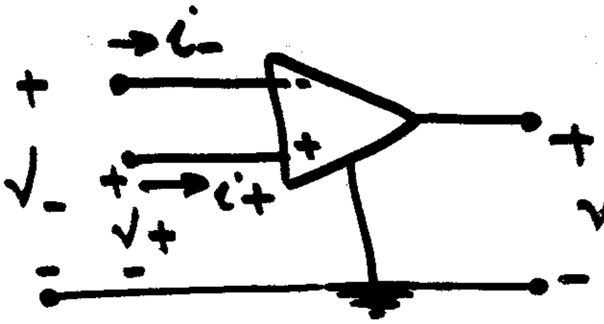
Dual in line package
integrated circuit
chip

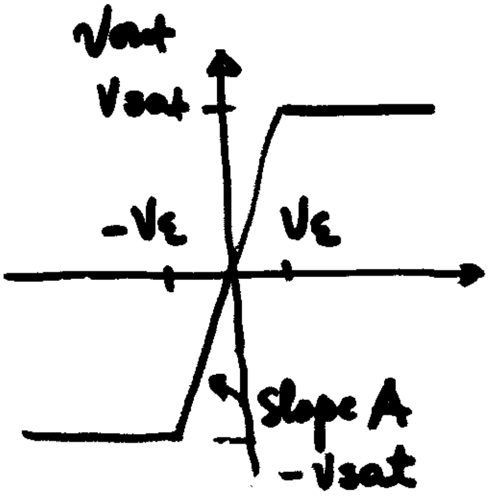


Vcc = 15V
normally

Built from transistors

Simpler model:



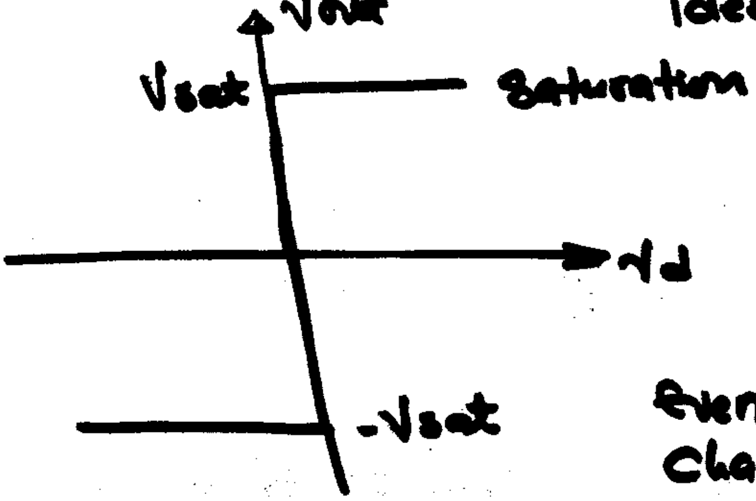


Nonlinear element

$$V_{out} = A \cdot V_d$$

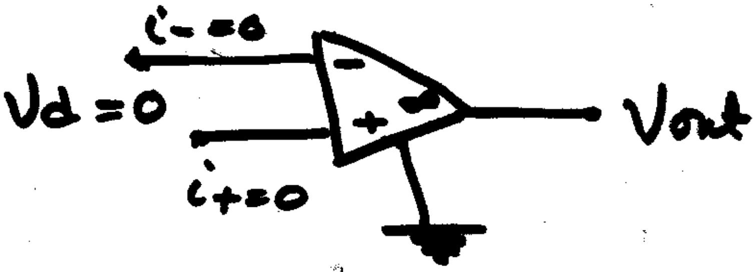
$$-V_\varepsilon < V_d < V_\varepsilon$$

Ideal op-amp:



even more "ideal" characteristics

A : very large "gain".



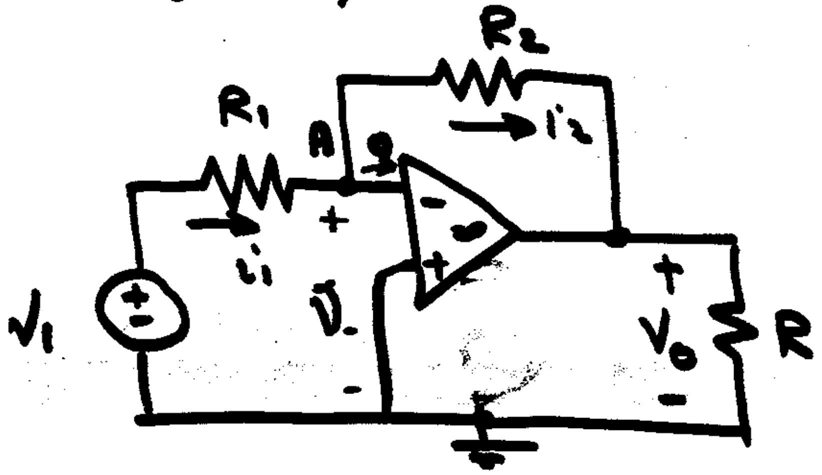
if $A \to \infty$

$$\underbrace{V_{out}}_{\text{finite}} = \underbrace{A}_{\text{infinite}} \cdot \overset{\to 0}{V_d}$$

$$V_+ - V_- = 0$$

⟶ $-V_{sat} < V_o < V_{sat}$

Inverting Amplifier



$i_- = 0 \Rightarrow i_1 = i_2$ KCL

$V_+ - V_- = 0 \Rightarrow V_- = 0$ (because $V_+ = 0$)

KVL: $v_1 - R \cdot i_1 - V_- = 0 \Rightarrow i_1 = \frac{v_1}{R_1}$

$V_- - R_2 \cdot i_2 - V_o = 0 \Rightarrow V_o = -R_2 \cdot i_2$

But, $i_1 = i_2$ (KCL) $\Rightarrow V_o = -R_2 \cdot \frac{V_1}{R_1}$

$$V_o = -\frac{R_2}{R_1} \cdot v_1$$

$$v_o = k \cdot v_1 \quad \leftarrow$$

Voltage gain: $k = -R_2/R_1$

$|V_o| < V_{sat} \Rightarrow |V_i| < \frac{R_1}{R_2} \cdot V_{sat} \quad \leftarrow$

Only true if $|V_o| < V_{sat}$

$$\left| -\frac{R_2}{R_1} V_1 \right| < V_{sat}$$

If: $|V_1| < \frac{R_1}{R_2} \cdot V_{sat}$

then: $V_o = -\frac{R_2}{R_1} \cdot V_i$

otherwise: $|V_o| = V_{sat}$

$-\frac{R_2}{R_1}$ = voltage gain

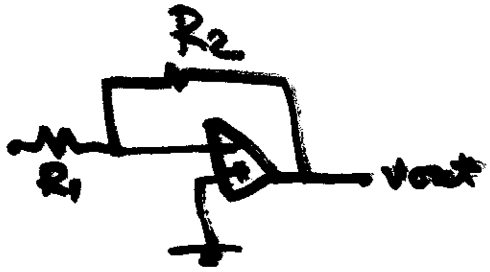
Closed loop gain

Recall: A = open loop gain

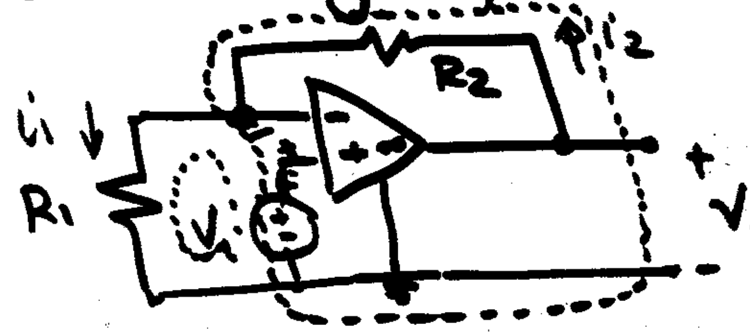
$$V_o = -\frac{R_2}{R_1} \cdot V_1$$

↖ inverting op-amp

R_2 : Feedback resistor



Non-inverting amplifer



$$i_- = 0 \Rightarrow i_1 = i_2$$

$$v_- - v_+ = 0 \Rightarrow v_- = v_i$$

$$v_- - R_1 \cdot i_1 = 0 \Rightarrow i_1 = \frac{v_i}{R_1}$$

$$v_o - R_2 \cdot i_2 - v_- = 0$$

$$v_o = R_2 \cdot i_2 + \underset{\nwarrow = v_i}{v_-}$$

$$v_o = R_2 \cdot \underset{\downarrow}{i_2} + v_i$$

$$v_o = R_2 \cdot i_1 + v_i$$

$$v_o = R_2 \cdot \frac{v_i}{R_1} + v_i$$

$$\rightarrow v_o = \left(1 + \frac{R_2}{R_1}\right) \cdot v_i \checkmark$$

$$\swarrow$$

$$v_o = k \cdot v_i$$

$$k = 1 + \frac{R_2}{R_1}$$

non-inverting

And: $|v_o| < V_{sat} \Rightarrow \underline{|v_i|} < \underline{\frac{R_1}{R_1+R_2} \cdot V_{sat}}$

the voltage follower

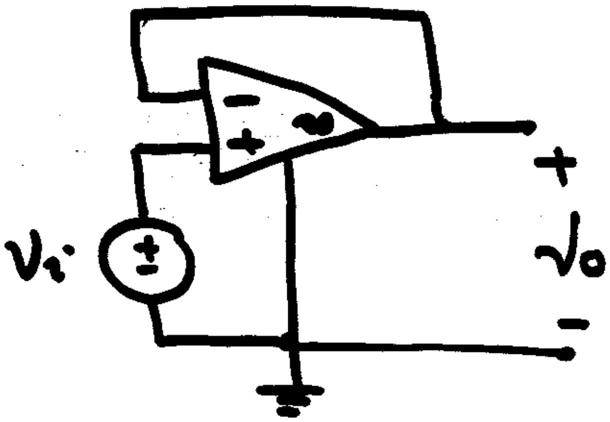
Assume $R_2 = 0$ (feedback resistor)

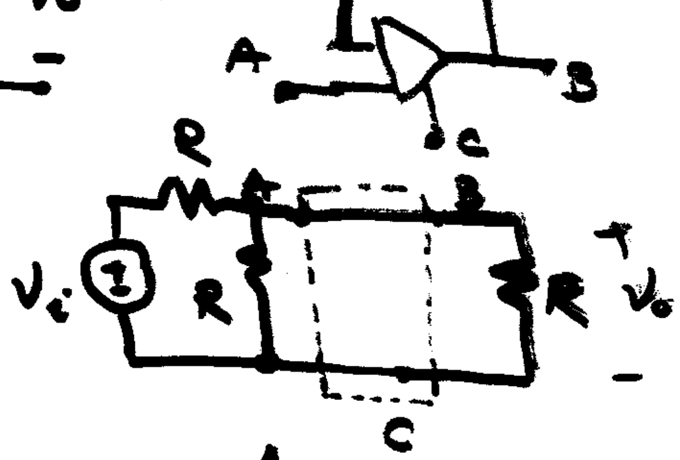
$$v_o = \left(1 + \frac{R_2}{R_1}\right) \cdot v_i$$

$R_2 = 0$

$v_o = v_i$ gain $= 1$

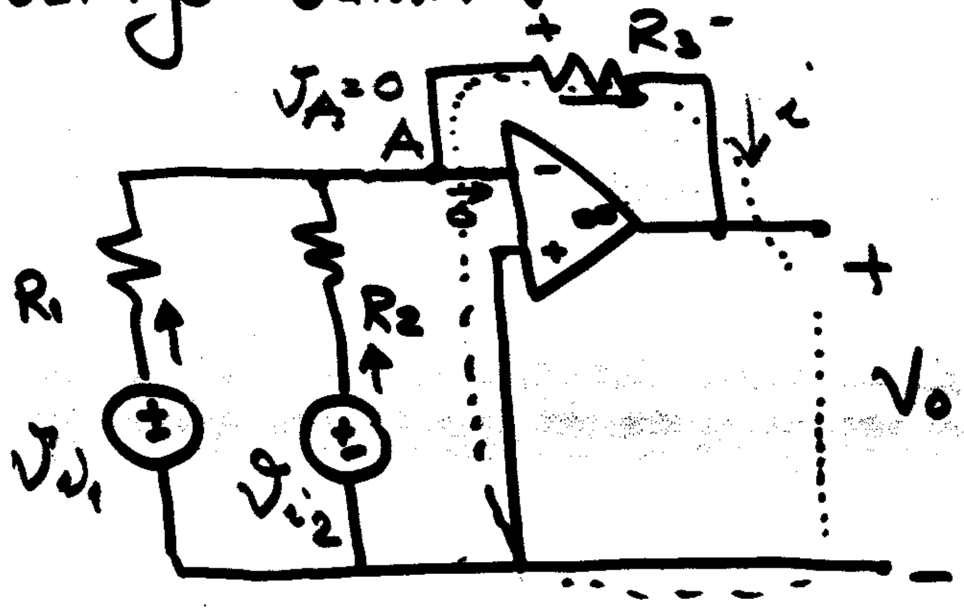
R_1 : does not matter





$v_o = \frac{1}{2} v_i$

Voltage Summer:



$$i_{i1} = \frac{v_{i1}}{R_1}$$

$$i_{i2} = \frac{v_{i2}}{R_2}$$

$$i = i_{i1} + i_{i2} \Rightarrow i = \frac{v_{i1}}{R_1} + \frac{v_{i2}}{R_2}$$

KVL: $\searrow v_o + R_3 \cdot i = 0$

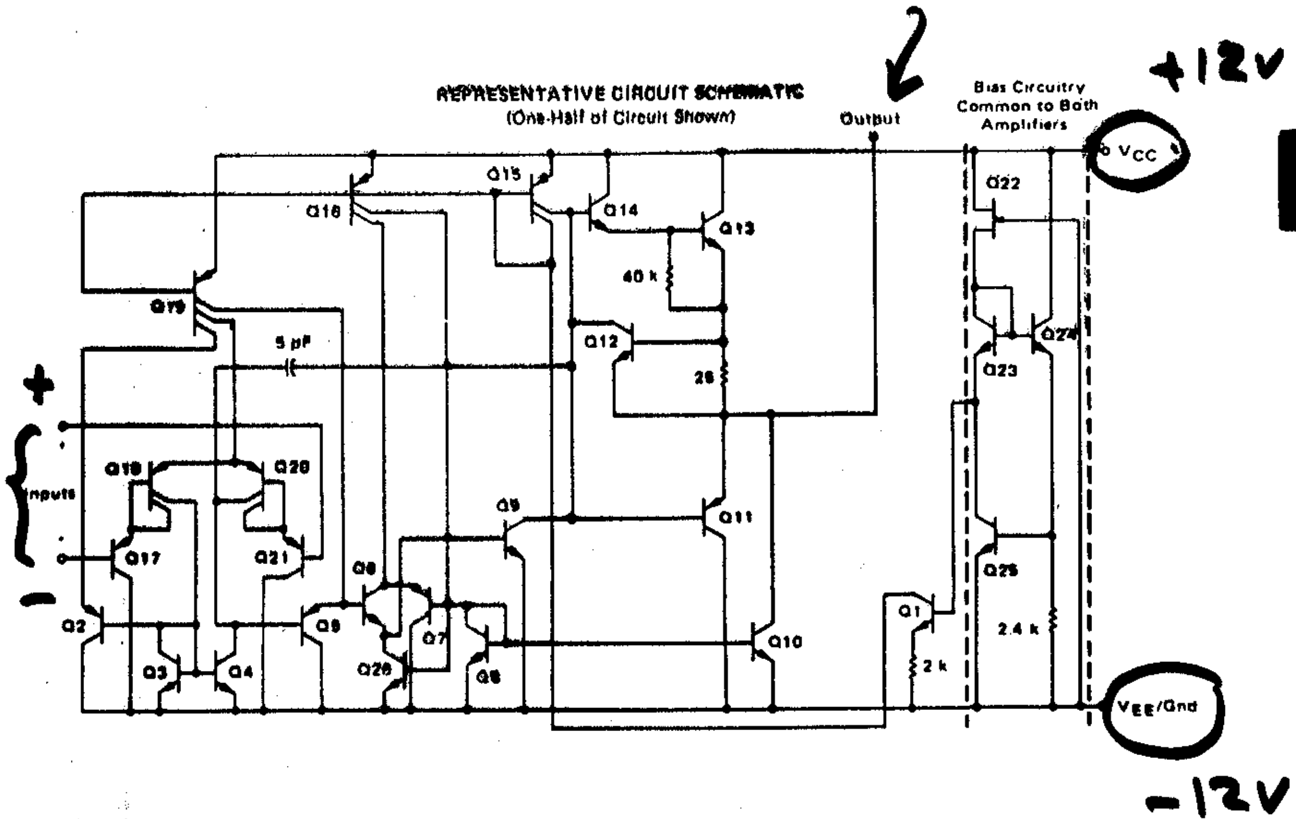
$$v_o = -R_3 \cdot \left(\overset{\text{input 1}}{\frac{v_{i1}}{R_1}} \underset{\text{sum}}{+} \overset{\text{input 2}}{\frac{v_{i2}}{R_2}} \right)$$

	LM2904			
Max	Min	Typ	Max	Unit
				mV
7.0 9.0	– –	2.0 –	7.0 10	
–	–	7.0	–	µV/°C
50 150	– –	5.0 45	50 200	nA
–	–	10	–	pA/°C
-250 -500	– –	-45 -50	-250 -500	nA
				V
28.3 28	0 0	– –	24.3 24	
V_{CC}	–	–	V_{CC}	V
				V/mV
– –	– –	100 –	– –	
–	–	-120	–	dB
–	50	70	–	dB
–	50	100	–	dB
3.3	0	–	3.3	V
				V
– –	22 23	– 24	– –	
20	–	5.0	20	mV
–	20	40	–	mA
– –	10 –	20 –	– –	mA µA
60	–	40	60	mA
				mA
3.0 1.2	– –	1.5 0.7	3.0 1.2	

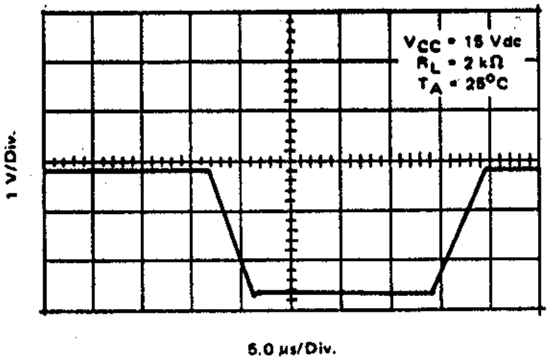
e common mode voltage range
both inputs can go to +32 V
M2904)

out to V_{CC} can cause excessive
Destructive dissipation

REPRESENTATIVE CIRCUIT SCHEMATIC
(One-Half of Circuit Shown)



LARGE SIGNAL VOLTAGE
FOLLOWER RESPONSE



CIRCUIT DESCRIPTION

The LM158 Series is made using two internally compensated, two-stage operational amplifiers. The first stage of each consists of differential input devices Q20 and Q18 with input buffer transistors Q21 and Q17 and the differential to single ended converter Q3 and Q4. The first stage performs not only the first stage gain function but also performs the level shifting and transconductance reduction functions. By reducing the transconductance a smaller compensation capacitor (only 5 pF) can be employed, thus saving chip area. The transconductance reduction is accomplished by splitting the collectors of Q20 and Q18. Another feature of this input stage is that the input common-mode range can include the negative supply or ground, in single supply operation, without saturating either the input devices or the differential to single-ended converter. The second stage consists of a standard current source load amplifier stage.

MOTOROLA

LM158, LM258, LM358, LM2904

Specifications and Applications Information

DUAL DIFFERENTIAL INPUT OPERATIONAL AMPLIFIERS

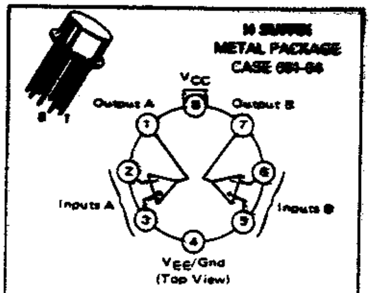
SILICON MONOLITHIC INTEGRATED CIRCUIT

DUAL LOW POWER OPERATIONAL AMPLIFIERS

Utilizing the circuit designs perfected for recently introduced Quad Operational Amplifiers, these dual operational amplifiers feature 1) low power drain, 2) a common mode input voltage range extending to ground/V_{EE}, 3) Single Supply or Split Supply operation and 4) pin outs compatible with the popular MC1558 dual operational amplifier. The LM158 Series is equivalent to one-half of an LM124.

These amplifiers have several distinct advantages over standard operational amplifier types in single supply applications. They can operate at supply voltages as low as 3.0 Volts or as high as 32 Volts with quiescent currents about one-fifth of those associated with the MC1741 (on a per amplifier basis). The common mode input range includes the negative supply, thereby eliminating the necessity for external biasing components in many applications. The output voltage range also includes the negative power supply voltage.

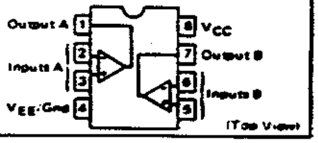
- Short Circuit Protected Outputs
- True Differential Input Stage
- Single Supply Operation: 3.0 to 32 Volts
- Low Input Bias Currents
- Internally Compensated
- Common Mode Range Extends to Negative Supply
- Single and Split Supply Operation
- Similar Performance to the Popular MC1558


H SUFFIX
METAL PACKAGE
CASE 601-04

J SUFFIX
CERAMIC PACKAGE
CASE 693-02

N SUFFIX
PLASTIC PACKAGE
CASE 626-05

D SUFFIX
PLASTIC PACKAGE
CASE 751-02
SO-8



MAXIMUM RATINGS (T_A = +25°C unless otherwise noted)

Rating	Symbol	LM158 LM258 LM358	LM2904	Unit
Power Supply Voltages				Vdc
Single Supply	V_{CC}	32	26	
Split Supplies	V_{CC}, V_{EE}	±16	±13	
Input Differential Voltage Range (1)	V_{IDR}	±32	±26	Vdc
Input Common Mode Voltage Range (2)	V_{ICR}	-0.3 to 32	-0.3 to 26	Vdc
Input Forward Current (3) (V_I < -0.3 V)	I_{IF}	50	–	mA
Output Short Circuit Duration	t_S	Continuous		
Junction Temperature	T_J			°C
Ceramic and Metal Packages		175		
Plastic Package		150		
Storage Temperature Range	T_{stg}			°C
Ceramic and Metal Packages		-65 to +150		
Plastic Package		-55 to +125		
Operating Ambient Temperature Range	T_A			°C
LM158		-55 to +125	-	
LM258		-25 to +85	–	
LM358		0 to +70	–	
LM2904		–	-40 to +85	

(1) Split Power Supplies

(2) For Supply Voltages less than 32 V for the LM158/258/358 and 26 V for the LM2904, the absolute maximum input voltage is equal to the supply voltage

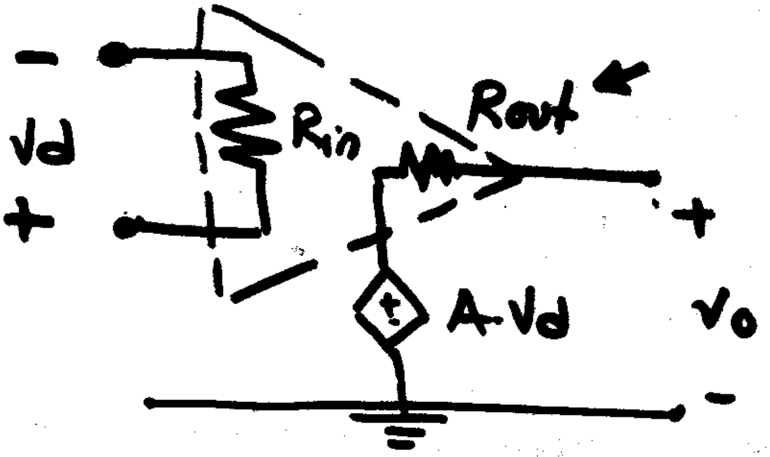
(3) This input current will only exist when the voltage is negative at any of the input leads. Normal output states will reestablish when the input voltage returns to a voltage greater than 0.3 V

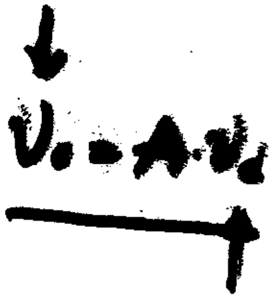
ORDERING INFORMATION

Device	Temperature Range	Package
LM158H	−55 to +125°C	Metal Can
LM158J		Ceramic DIP
LM2904D	−40 to +85°C	SO-8
LM2904H		Metal Can
LM2904J		Ceramic DIP
LM2904N		Plastic DIP
LM258D	−25 to +85°C	SO-8
LM258H		Metal Can
LM258J		Ceramic DIP
LM258N		Plastic DIP
LM358D LM358H	0 to +70°C	SO-8 Metal Can
LM358J		Ceramic DIP
LM358N		Plastic DIP

Nonideal op-amp models

Assumptions:
1. R_{in} large, but $< \infty$
2. A large, but $< \infty$
3. R_{out} small, but > 0





Circuit analyzed by applying
KCL
KVL

Transistors and their models

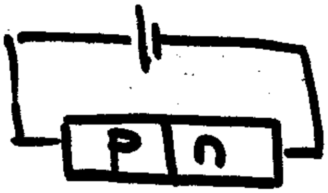
invented by W. Shockley
J. Bardeen
J. Brattain
} Bell Laboratories 1953

BJT: Bipolar junction transistor

MOS: Metal Oxide Silicon transistor

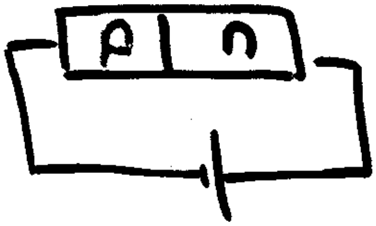
FET: Field Effect transistor

JFET: junction FET



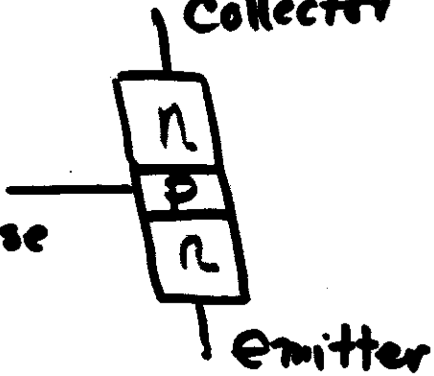
Semiconductor junction
<diode>

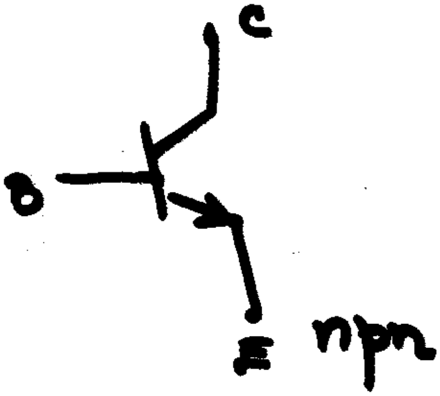
Forward biased

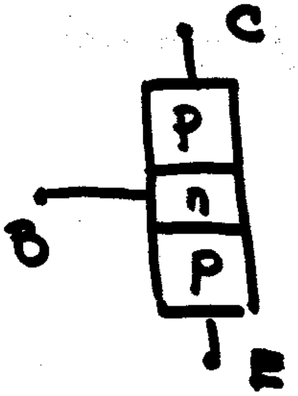


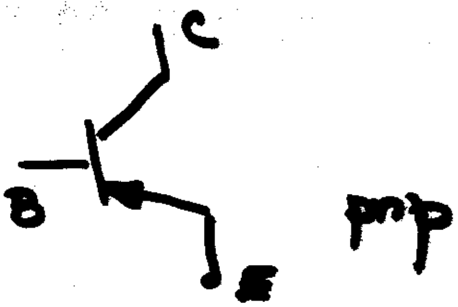
Reverse biased

Transistr

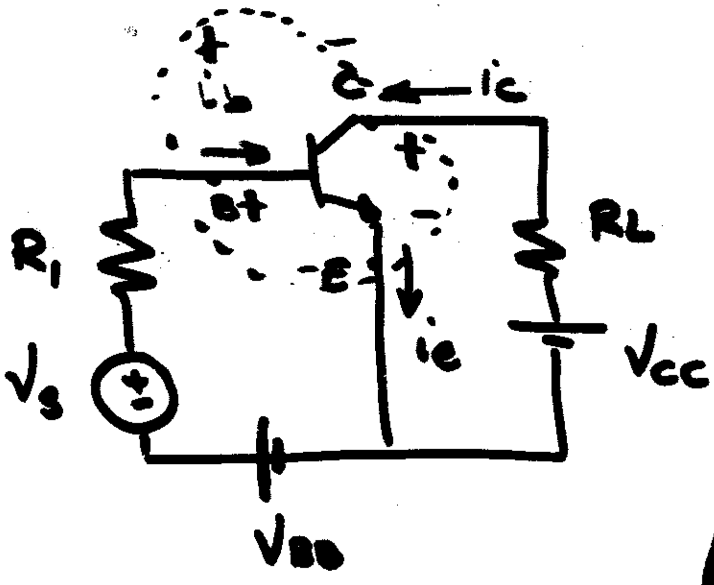






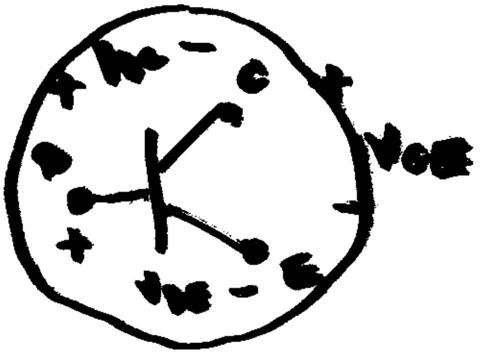


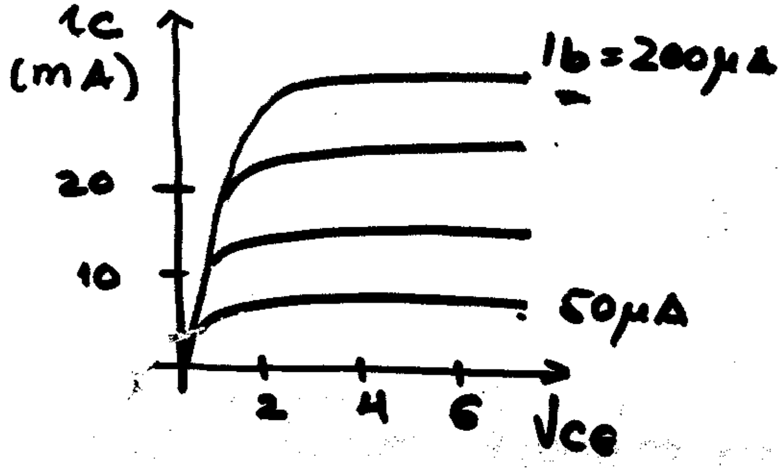
BJT

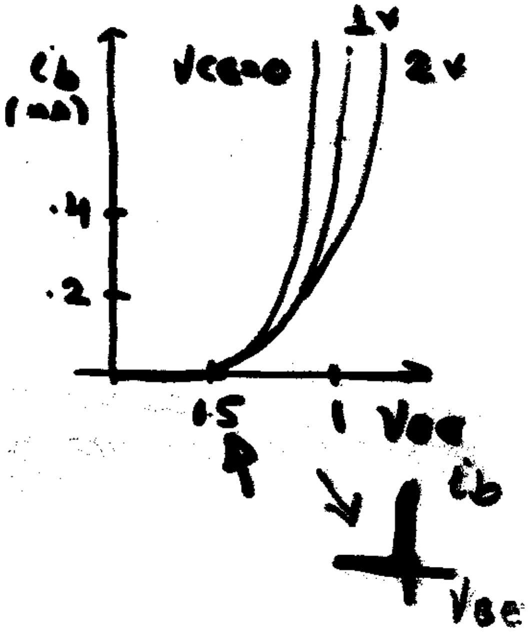


$i_e = i_b + i_c$

$$V_{BC} = \underbrace{V_{BE} - V_{CE}}_{BJT}$$





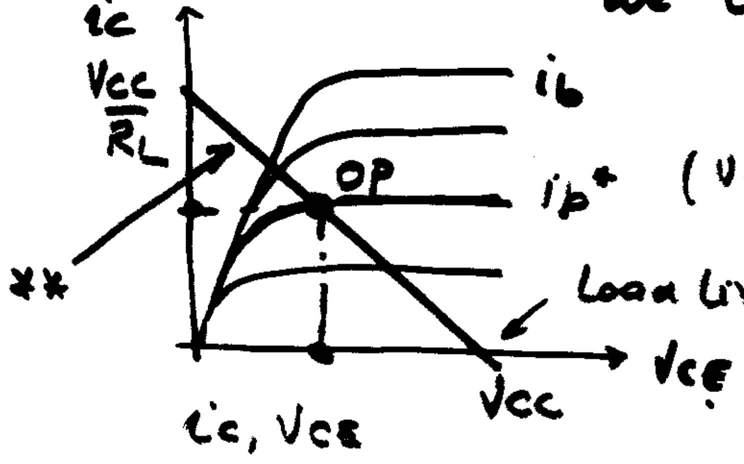


For the circuit called transistor amplifier:

$$* \quad \underline{i_B} = -\frac{1}{R_1}\cdot \overset{0.2}{V_{BE}} + \frac{\overset{\checkmark}{V_{BB}}}{R_1 \checkmark} \leftarrow \text{KVL}$$

$$** \quad \underset{\checkmark}{i_c} = -\frac{1}{R_L}\cdot \underset{\checkmark}{V_{CE}} + \frac{V_{CC}}{R_L} \nwarrow \text{KVL} \quad \underline{V_S = 0}$$

Load line analysis (similar to the analysis we did for diode circuits)



Fixed!

(Vbe = 0.2V)

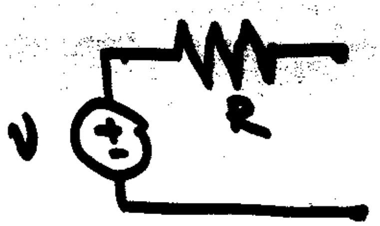
Load Line: $i_c = -\frac{1}{R_L}\cdot V_{CE} + \frac{V_{CC}}{R_L}$

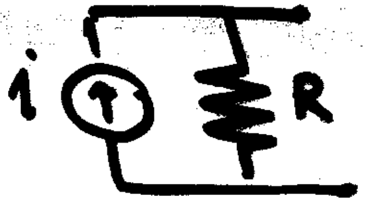
OP: Operating point (DC)

General Analysis of dissipative circuits

Systematic approach:

Sources:





Accompanied!

<Essential!>

Branch variables:

Voltages : v
Currents : i

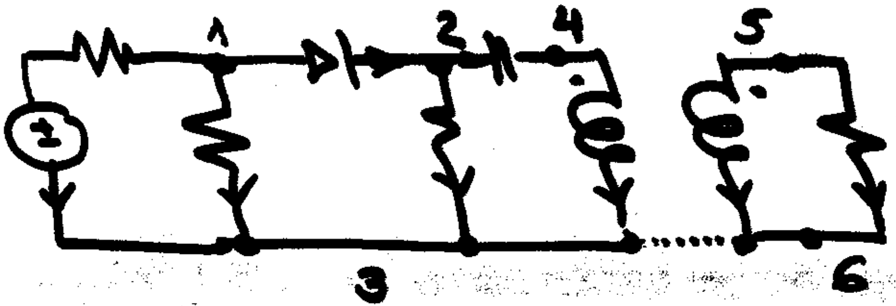
For every branch: - Two variables

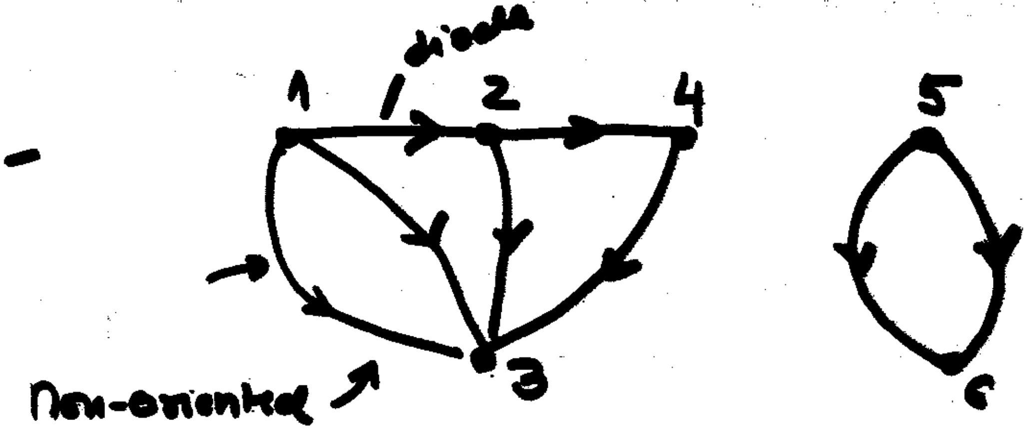
- One equation relating v and i

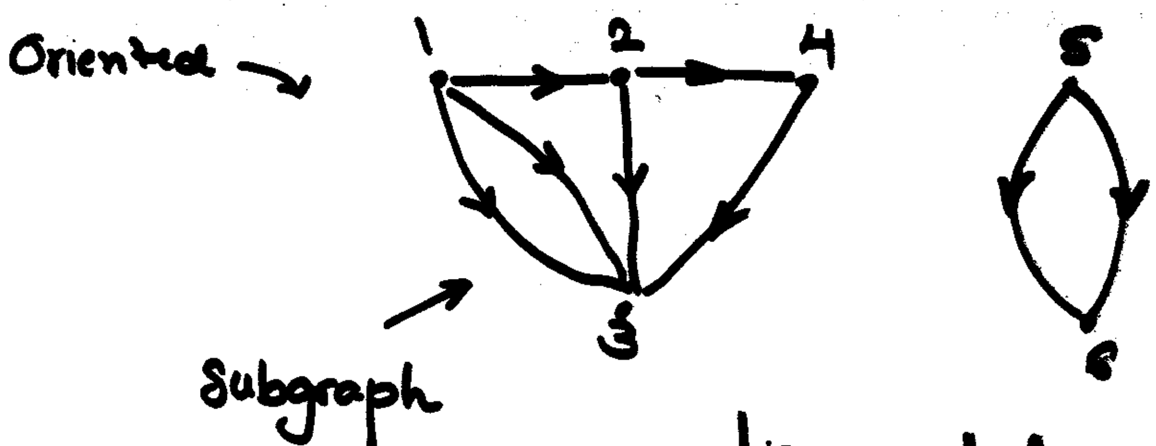
ex: Ohm's law

or graph if nonlinear
function

Graph of a circuit:



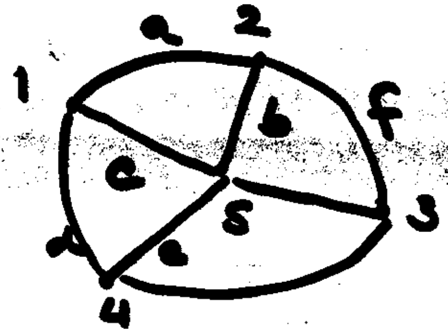


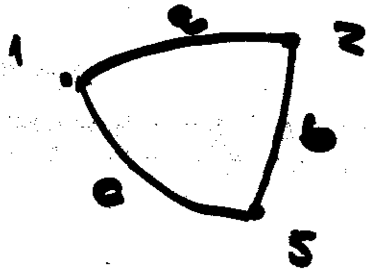


disconnected graph

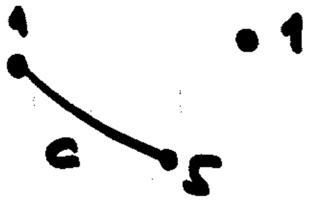
Loops:

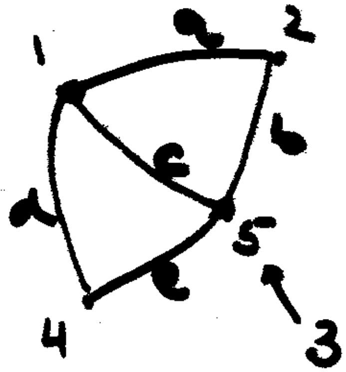
A connected subgraph that has exactly two branches of the subgraph incident at each of its nodes





Loop

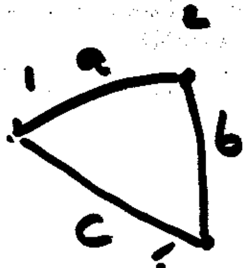




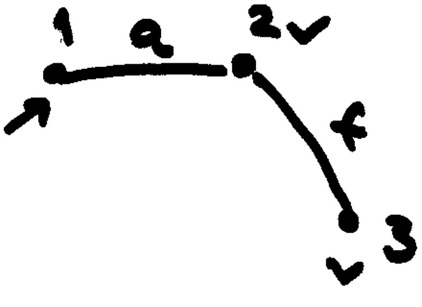
× loop

3 branches incident to node 5.

"Incident"

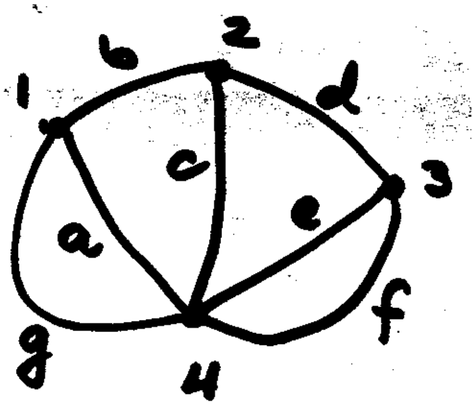


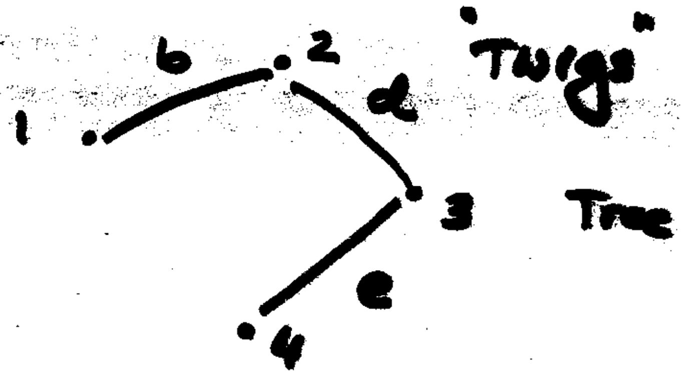
a is incident to nodes 1 and 2.



Tree :

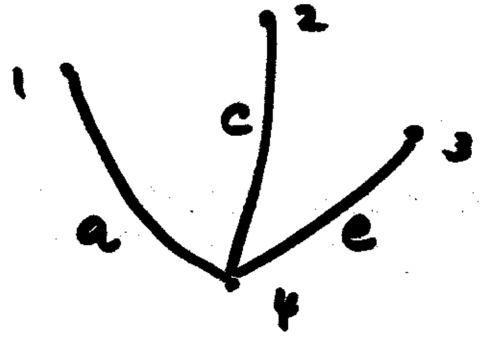
A connected subgraph consisting of a set of branches which connect all the nodes of the graph without forming any loops



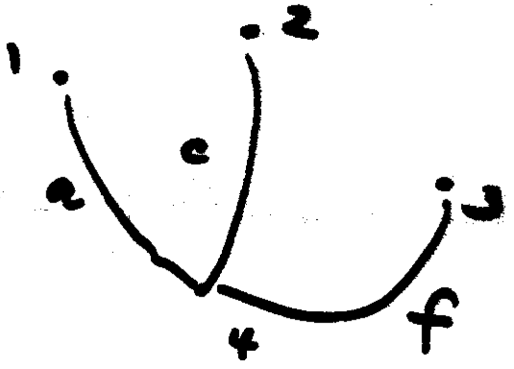


"Twigs"

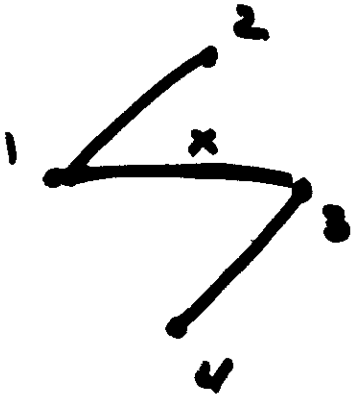
Tree

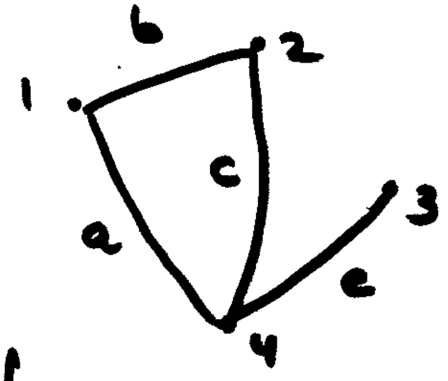


Tree



Tree





X Tree

Loop: a, b, c

↓

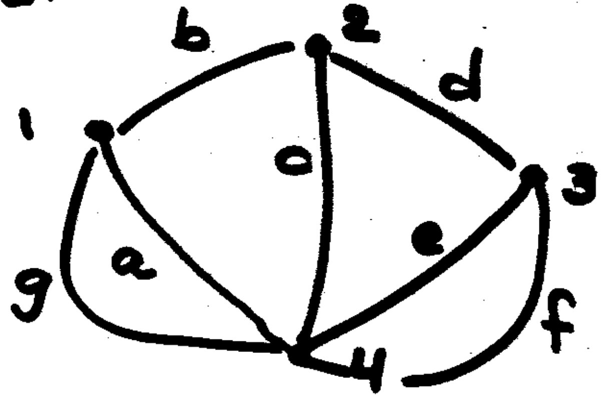
⟨co-trees⟩ Links

There are many trees in a given graph.
(the number is finite.)

In a graph with: b : branches
n : nodes

We can choose a tree:

Example.



n = 4
b = 7

Example

Tree: a, c, e (twigs) $\# = \widetilde{n-1}$ (3)

Co-tree: b, d, f, g (links) $\# = \underbrace{b-(n-1)}$ (4)

example

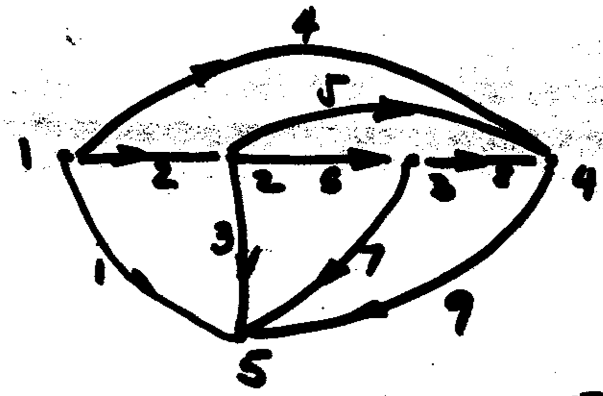
In any connected graph:

No. of twigs = n-1

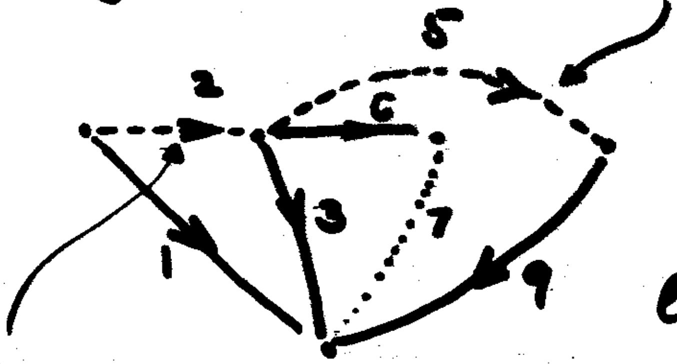
No. of links = b-(n-1) ⟨The rest of branches⟩

Independent Voltage Equations:

Number = $\underbrace{n-1}$:= number of twigs



Why?



KVL:

$v_5 + v_9 - v_3 = 0$

$v_5 = \underbrace{v_3 - v_9}_{\text{Twigs}}$

link

KVL:

$v_2 + v_3 - v_1 = 0$

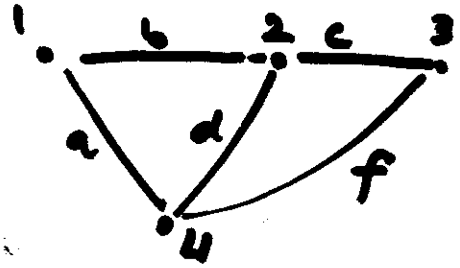
$v_2 = \underbrace{v_1 - v_3}_{\text{Twigs}}$

Link

Fundamental loops: A loop formed by adding any one link to a tree.

→ 1, 2, 3

→ 3, 5, 9 Also: 3, 6, 7 But not: 7, 8, 9 the number of independent voltages = the number of twigs.



$n = 4$
$n-1 = 3$
$b = 5$

# of independent voltages $= 3$
V_a, V_b, and V_c

In a graph having b branches and n nodes, there are $b-(n-1)$ fundamental loops

Number of independent KVL equations is exactly

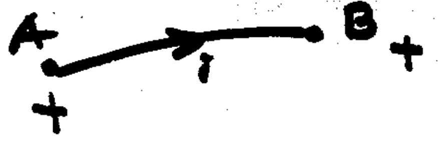
$b-(n-1)$

$5 - 3 = 2$ ← fundamental loops
independent KVL equations

Node voltages as a basis set:

n-1 : node voltages

- Each twig voltage = difference in voltage of the nodes at its two ends

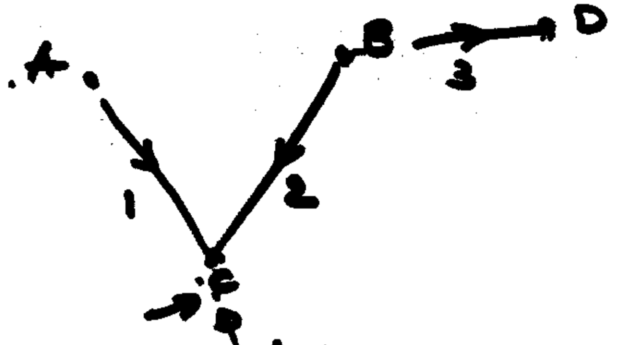


$v_1 = V_A - V_B$ ↖ V_{B0}

V_{AB} ↖ V_{A0}

→ 0 (datum)

- All nodes lie on a tree. Tree is connected; there must be a path in the tree from each node to every other node. Node voltages can be expressed in terms of <u>twig</u> voltages



datum node
<Reference>

useful to choose a node to which most branches are connected

$V_{A\,\text{wrt}\,C} = v_1 \leftarrow$

$V_{B\,\text{wrt}\,C} = v_2$

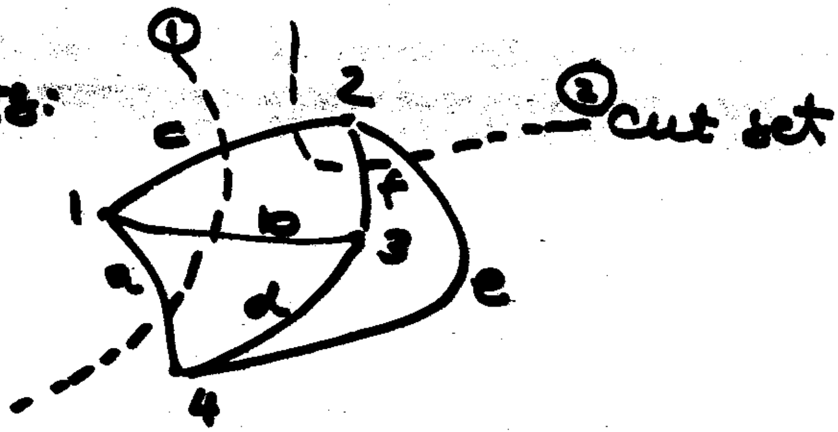
$V_{D\,\text{wrt}\,C} = V_2 - V_3$

The set of node voltages is a basis set.

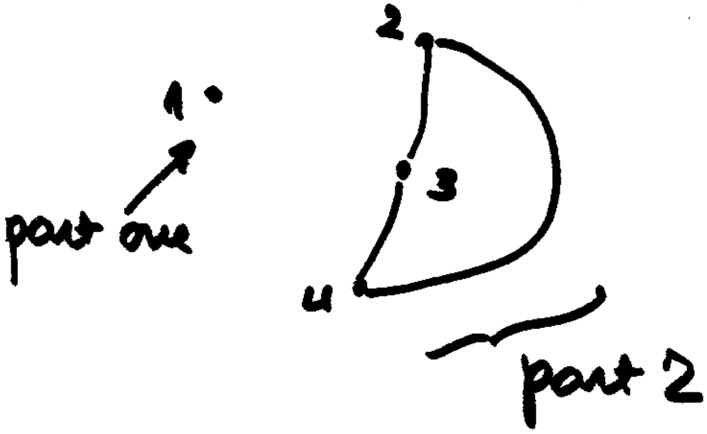
All branch voltages can be obtained from node voltages.

How about currents?

Cut Sets:



It is a set of the smallest number of branches of a graph which, when cut, divide it into two parts:



Cut-set: a, b, c ①

c, f, e ②

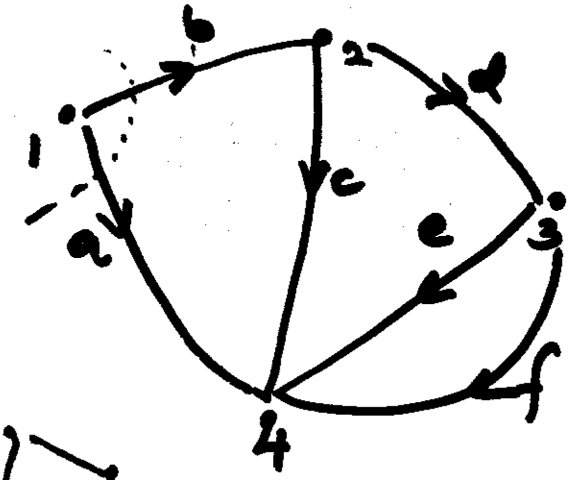
a, b, c, f ×

Fundamental cut set : contains only one twig

there are n-1 fundamental cut sets
<For each tree
there are n-1 twigs>

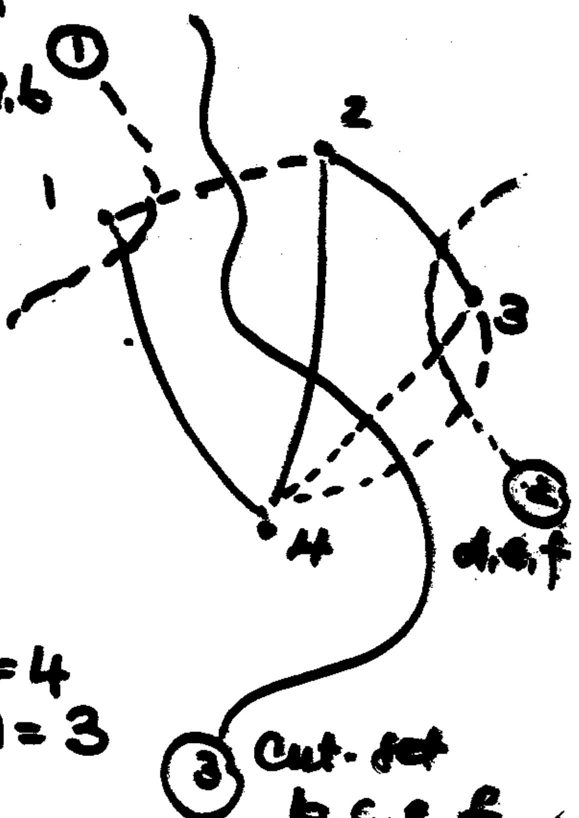
independent Current equations :

In a graph having n nodes, the number of independent KCL equations is exactly n-1.



Node
:

graph



n=4
n-1= 3

Cut set
b,c,e,f ✓

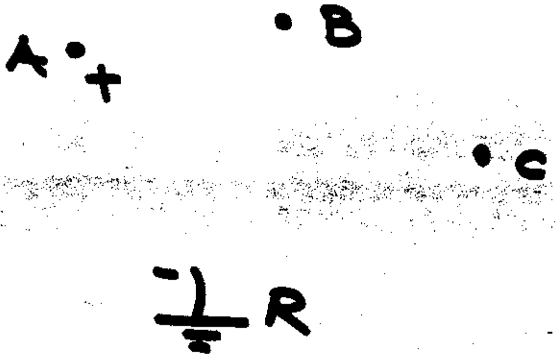
$i_a + i_b = 0$ ✓ ①

$-i_d + i_e + i_f = 0$ ②

$i_b - i_c - i_e - i_f = 0$ ③

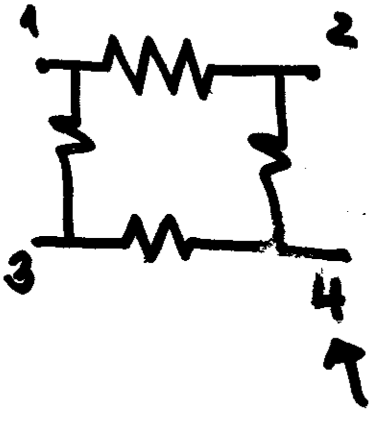
Node Equations:

1. Choose a datum (reference) node



$$v_{AR} = v_A$$
$$v_{BR} = v_B$$
$$v_{CR} = v_C$$

$$\underset{\text{branch voltage}}{v_{AB}} = \underset{\text{node voltage (wrt datum node)}}{v_A} - v_B$$



$$v_{12} = v_1 - v_2$$
$$v_1 \rightarrow v_{14} \qquad v_2 \rightarrow v_{24}$$

datum node

Node-voltage analysis
<or: Node Analysis>

Procedure:

1. Write KCL equations at all nondatum nodes

   n-1 equation with $\underline{b}$ unknowns

   ↑ currents

2. Eliminate currents by using Ohm's Law

   $i = G \cdot v$

   n-1 equation with b unknowns

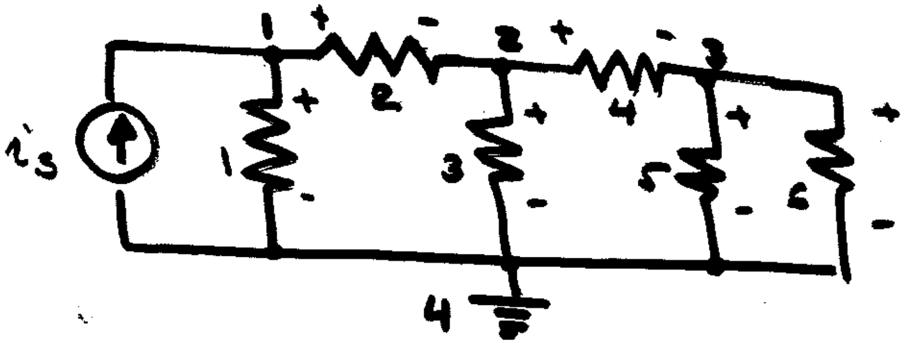
   ↑ voltage

3. Eliminate b branch voltages by expressing them in terms of n-1 node voltages

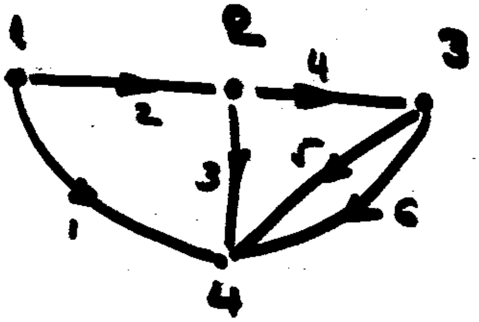
   n-1 equations with n-1 unknowns

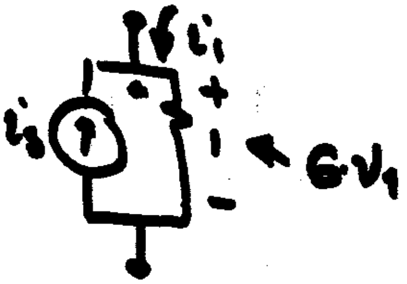
   ↑ node voltages

Example:



Graph:





$\cancel{i_1} + i_s = G \cdot V_{b1}$

$i_1 = G \cdot V_{b_1} - i_s$

$i_1 + i_2 = 0$

Step1: $-i_2 + i_3 + i_4 = 0$

$-i_4 + i_5 + i_6 = 0$

Eq: 3 unknow: 6

Step 2: $G_1 V_{b1} - i_s + G_2 V_{b2} = 0$

$-G_2 V_{b2} + G_3 V_{b3} + G_4 V_{b4} = 0$

$-G_4 \cdot V_{b4} + G_5 \cdot V_{b5} + G_6 \cdot V_{b6} = 0$

Eq: 3 unknown: 6

Step 3: $v_{b1} = V_1 - V_4$ ↙ datum node

$\Rightarrow v_{b1} = v_1$

$v_{b2} = v_1 - v_2$

$v_{b3} = v_2$

$v_{b4} = v_2 - v_3$

$v_{b5} = v_3$

$v_{b6} = v_3$

Hence:

$$G_1 \cdot v_1 + G_2 \cdot (v_1 - v_2) = i_s$$

$$-G_2 \cdot (v_1 - v_2) + G_3 \cdot v_2 + G_4 \cdot (v_2 - v_3) = 0$$

$$-G_4 (v_2 - v_3) + G_5 \cdot v_3 + G_6 \cdot v_3 = 0$$

or: Eq: $\underline{\underline{3}}$ unknown: $\underline{\underline{3}}$

$$(G_1 + G_2) \cdot v_1 - G_2 \cdot v_2 = i_s$$

$$-G_2 \cdot v_1 + (G_2 + G_3 + G_4) v_2 - G_4 \cdot v_3 = 0$$

$$-G_4 \cdot v_2 + (G_4 + G_5 + G_6) v_3 = 0$$

Matrix form:

$$\begin{bmatrix} G_1+G_2 & -G_2 & 0 \\ -G_2 & G_2+G_3+G_4 & -G_4 \\ 0 & -G_4 & G_4+G_5+G_6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} i_s \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\substack{\uparrow \\ 3\times3 \\ (n-1)\times(n-1) \\ n=4}}{G} \cdot \underset{\substack{\nwarrow \\ \text{node voltages}}}{v} = i_s \leftarrow \text{sources}$$

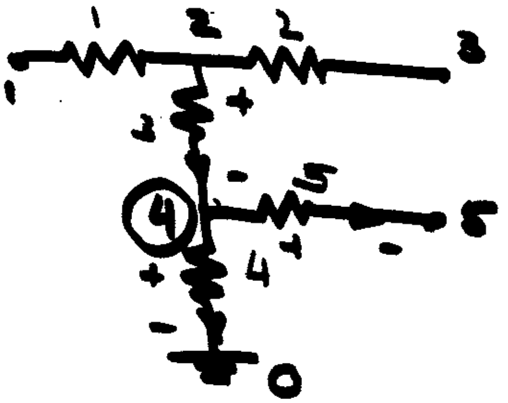
This is a general algorithm!

In a circuit with resistors only:

G = symmetric
diagonal elements: all positive
off diagonal element: all negative.

useful!

⟨Above is not true if controlled sources are present⟩



$$-i_3 + i_4 + i_5 = 0$$

$$-G_3 \cdot \underset{\substack{\downarrow\\ V_2-V_4}}{V_{b3}} + G_4 \cdot \underset{\substack{\uparrow\\ V_4}}{V_{b4}} + G_5 \cdot \underset{\substack{\uparrow\\ V_4-V_5}}{V_{b5}} = 0$$

$$-G_3 \cdot (V_2 - V_4) + G_4 V_4 + G_5 (V_4 - V_5) = 0$$

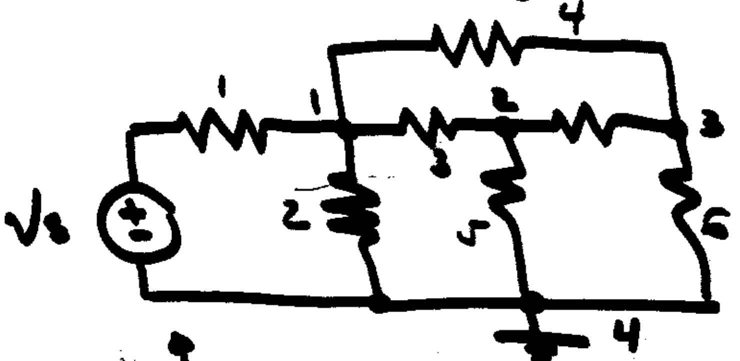
$$\underset{\substack{\uparrow\\ -}}{-} G_3 \cdot V_2 + (\underbrace{G_3 + G_4 + G_5}_{\text{all } +}) V_4 \underset{\substack{\uparrow\\ -}}{-} G_5 V_5 = 0$$

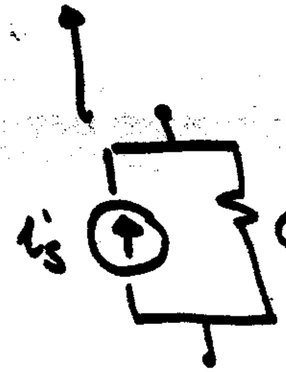
Conductance between nodes $\underline{\underline{2}}$ and $\underline{\underline{4}}$

Sum of all conductances connected to node $\underline{\underline{4}}$

Conductance between nodes $\underline{\underline{5}}$ and $\underline{\underline{4}}$

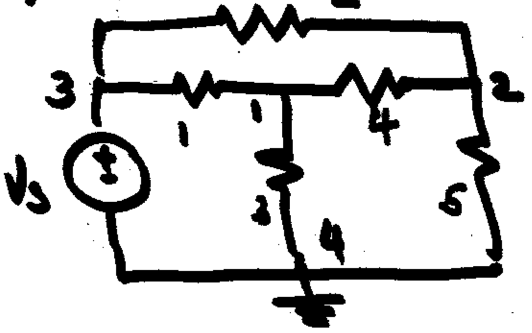
How about voltage sources?





accompanied sources are easily transformed into each other!

But, how about:



$V_3 = V_S$ known!!

No need to write KCL equation for node $\underline{\underline{3}}$.

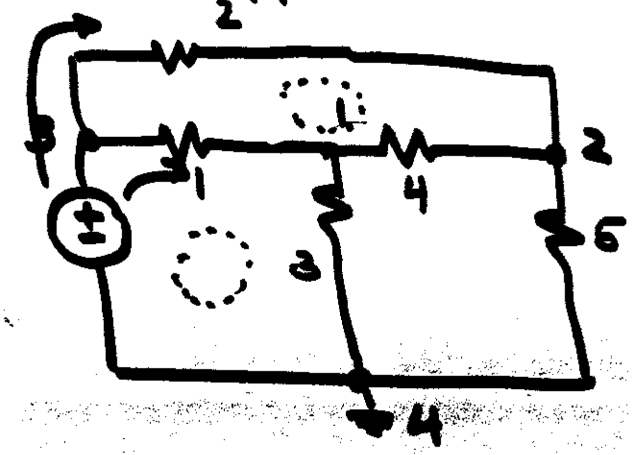
Node equations: $G_1\cdot(V_1-\overset{V_S\downarrow}{V_3}) + G_3\cdot V_1 + G_4(V_1-V_2)=0$

$G_2(V_2-\underset{\uparrow V_S}{V_3}) + G_4(V_2-V_1) + G_5\cdot V_2 = 0$

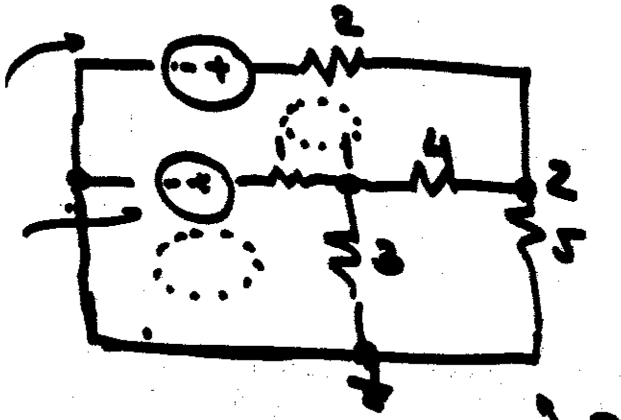
Eq: $\underline{\underline{2}}$ unknown V_1, V_2

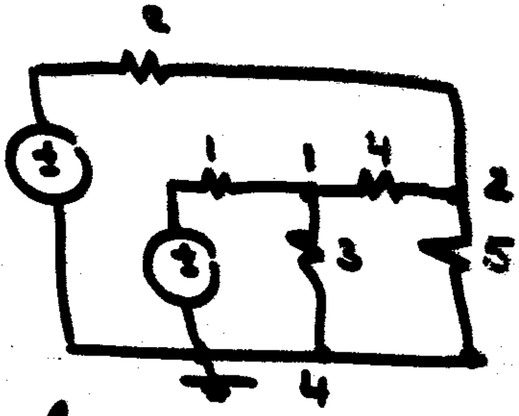
$\underline{\underline{2}}$

Another approach : source shift



Voltage source shift:





'Same' equations describe both circuits.

Loop and mesh equations:

b : branches
n : nodes

n-1 : KCL equations ⟨fundamental cut-sets⟩

b-(n-1): KVL equations ⟨fundamental loops⟩

2b unknowns : b branch voltages
b currents.

b : equations relating branch voltages and currents

Hence: $\left.\begin{array}{l} n-1 \\ b-(n+1) \end{array}\right\rangle$ b equations $\left(\begin{array}{c} KVL \\ KCL \end{array}\right)$

$+\ \underline{b}$ (Ohm's law)

2b equations
with 2b variables (voltages currents) in a branch

We need to solve 2b equations!

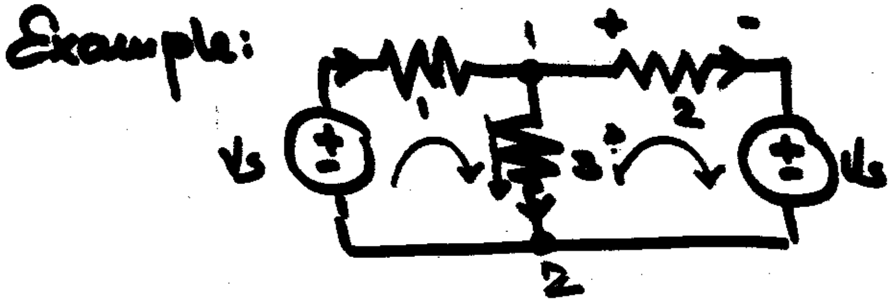
Can we do better?

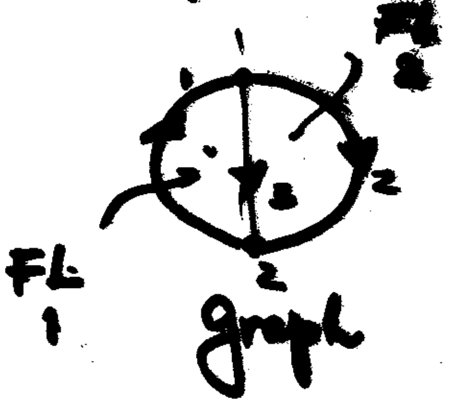
↓ smaller set of equations to solve simultaneously

Yes: Node equations: n-1 simultaneous equations.

Another approach: Loop and mesh equations

Example:





FL: fundamental loop

$v_{b1}+v_{b3}=0$ KVL

$v_{b2}-v_{b3}=0$

$-i_1+i_2+i_3=0$ KCL

$\langle v_{b1} = -\overset{\checkmark}{V_s} + R_1 i_1 \rangle$

$v_{b2} = \underset{\checkmark}{V_s} + R_2 i_2$

(accompanied sources)

Eliminate branch voltages from KVL equation by using Ohm's law:

$$-V_s + R_1 i_1 + R_3 i_3 = 0$$

$$V_s + R_2 i_2 - R_3 i_3 = 0$$

$$\uparrow \quad i_3 = i_1 - i_2$$

$$\left.\begin{array}{ll} 1. & (R_1+R_2) i_1 - R_3 i_2 = V_s \\ 2. & -R_3 i_1 + (R_2+R_3) \cdot i_2 = -V_s \end{array}\right\rangle \begin{array}{l} 2 \text{ equations} \\ 2 \text{ unknowns} \end{array}$$

↑

current
i_1
i_2

Link currents:
- 1
- 2

$$\begin{bmatrix} R_1+R_2 & -R_3 \\ -R_3 & R_2+R_3 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_s \\ -V_s \end{bmatrix}$$

$$\uparrow$$

$$R \cdot i = V_s$$

Loop equations

Algorithm for mesh equations:

1. Identify meshes and assign a mesh current
2. Write KVL in terms of branch voltages
3. Eliminate branch voltages by Ohm's law
4. Replace each branch current by sum or difference of mesh currents.

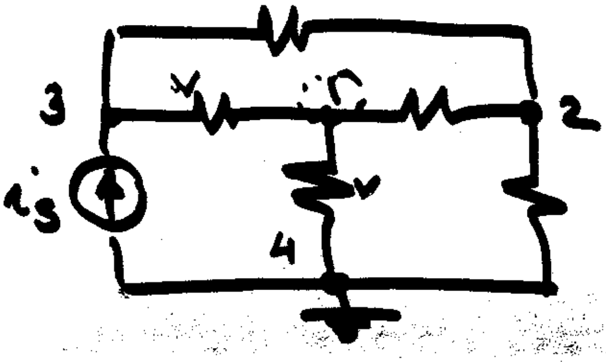
For our example:

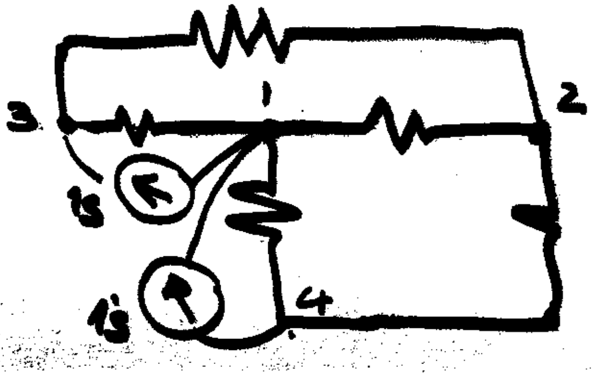
$$-V_s + R_1 \cdot i_{m1} + R_3 \cdot (i_{m1} - i_{m2}) = 0$$

$$R_2 i_{m2} + V_s - R_3 \cdot (i_{m1} - i_{m2}) = 0$$

$$\left. \begin{aligned} (R_1 + R_3) i_{m1} - R_3 i_{m2} &= V_s \\ -R_3 i_{m1} + (R_2 + R_3) i_{m2} &= -V_s \end{aligned} \right\rangle$$

How about unaccompanied current sources



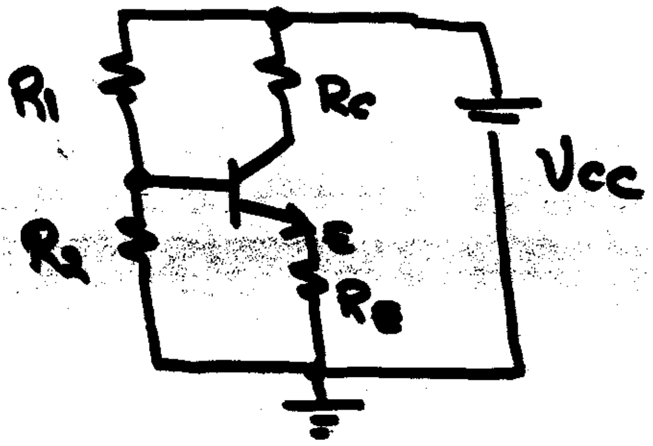


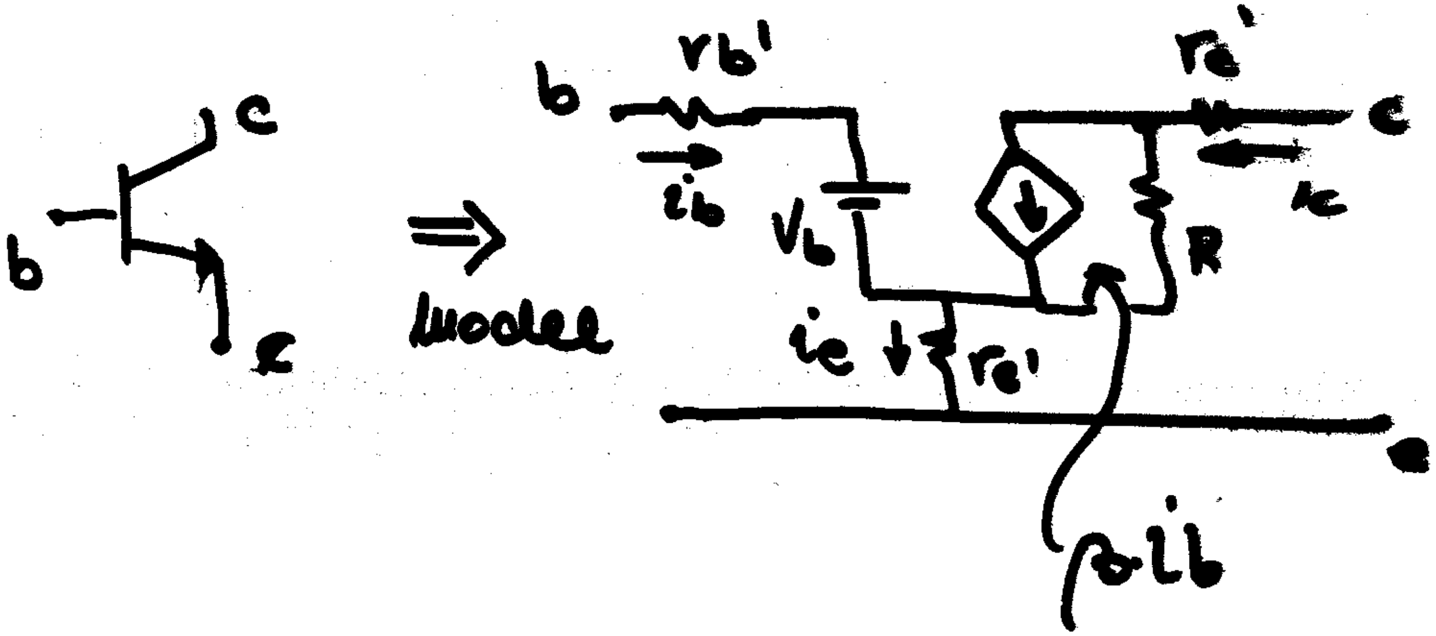
Current source shift!

KCL is the same in (a) and (b)!

Now: each source is accompanied and can be transformed into a voltage source.

Simple amplifier circuit with a bipolar transistor.





V_b: 0.2-0.7 V

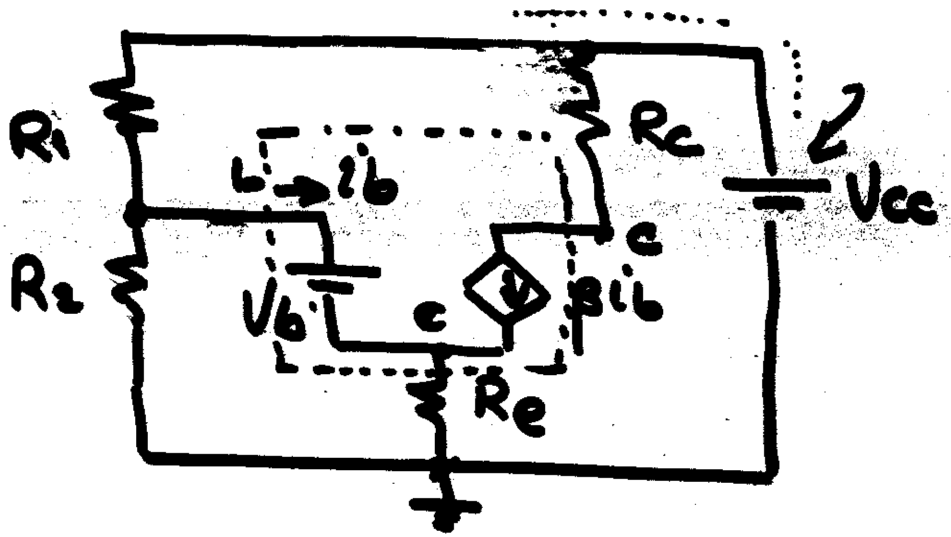
⟨technology dependent⟩

Simplified model: $r_b'=0$ $r_e'=0$
$r_c'=0$

Sometimes even: $V_b=0$, $R=\infty$

Redraw the circuit:

<Replace BJT with its model>



r_b'

$r_c' = 0$

r_e'

$R = \infty$

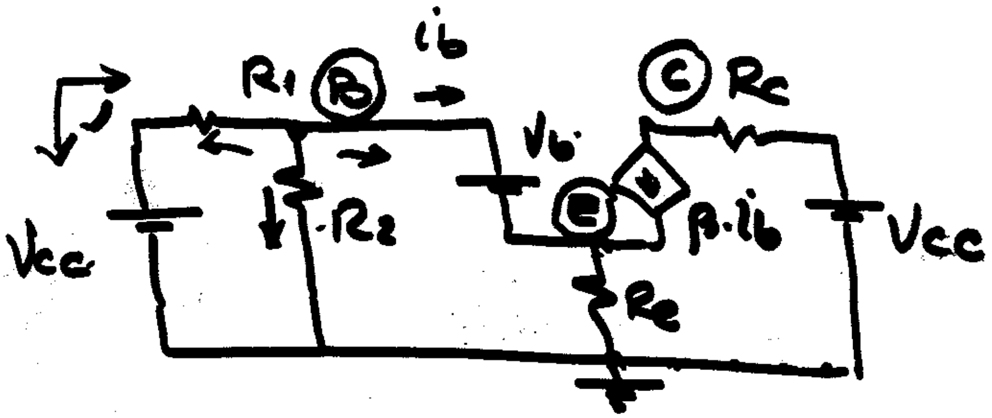
$i_c = \beta \cdot i_b$

<often used in BJT circuits>

Write nodal equations

1. there are unaccompanied independent voltage sources

<Do the "voltage source shift' before attempting to write equations>



Nodal equations:

< Note: Vb is unaccompanied by a resistor!

Special attention requires for nodes B and E! >

Option: Do source shift on Vb

Ⓑ

$$G_1\cdot(\overset{\checkmark}{V_B}-V_{CC}) + G_2\cdot V_B + \underline{\underline{i_b}} = 0$$

Ⓔ $$-\underline{i_b} + G_E\cdot V_E - \beta\cdot\underline{i_b} = 0 \leftarrow$$

Ⓒ $$+\beta\cdot i_b + G_C\cdot(V_C - V_{CC}) = 0$$

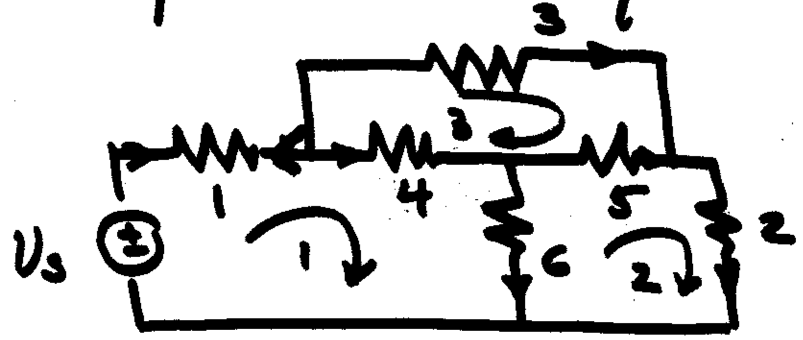
V_B
V_E and $\underline{i_b}$
V_C

Additional equation: $V_B - V_E = V_b$

4 equations
4 unknowns

How would you solve it by "shifting" Vb first?

Loop or Mesh equations:



KVL for each mesh:

Mesh ①

$$+V_S - R_1\cdot i_1 - R_4\cdot(i_1-i_3) - R_6\cdot(i_1-i_2)=0$$

or

$$-V_S + R_1 i_1 + R_4(i_1-i_3) + R_6(i_1-i_2)=0$$

Mesh ②

$$-R_2\cdot i_2 + R_6\cdot(i_2-i_1) + R_5(i_2-i_3)=0$$

Mesh ③

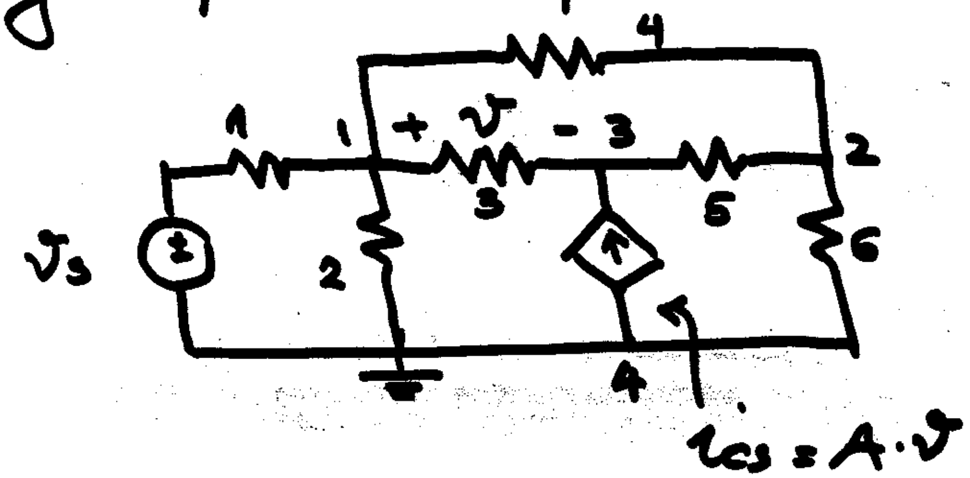
$$R_3\cdot i_3 + R_5(i_3-i_2) + R_4(i_3-i_1)=0$$

$$\left.\begin{matrix} i_1 \\ i_2 \\ i_3 \end{matrix}\right\} \text{\underline{Mesh} currents}$$

$i_{b_1} = i_1$; $i_{b_2} = i_2$, $i_{b_3} = i_3$

$i_{b4} = i_1 - i_3$
$i_{b5} = i_2 - i_3$
$i_{b6} = i_1 - i_2$

Fig 26 <p.210> example:



Resistance

$G = 1/R$

conductance

Datum node : 4

Voltage controlled current source

Node ①

current

$$G_1(V_1 - V_s) + G_2 \cdot V_2 + G_3 \cdot (V_1 - V_3) + G_4(V_1 - V_2) =$$

Node ②

$$G_4 \cdot (V_2 - V_1) + G_5(V_2 - V_3) + G_6 \cdot V_2 = 0$$

Node ③

$$G_3 \cdot (V_3 - V_1) + G_5(V_3 - V_2) - i_{c3} = 0$$

$$i_{c3} = A \cdot v$$

$V_1 - V_3$

3 equations with 3 unknowns! $i_{c3} = A(V_1 - V_3)$

General dissipative circuits

Circuits with controlled sources:

<continued>

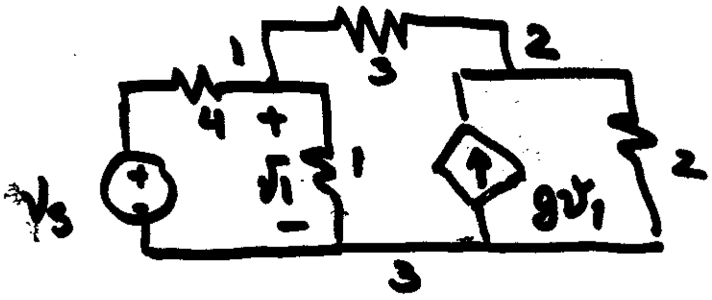


Fig.23 p.212

Nodal equations: <easy!>

① $G_4\cdot(v_1 - v_s) + G_1\cdot v_1 + G_3(v_1 - v_2) = 0$

② $G_3\cdot(v_2 - v_1) + G_2\cdot v_2 - g v_1 = 0$

$$(G_1+G_3+G_4)\cdot v_1 - G_3\cdot v_2 = G_4\cdot v_s$$

$$-(G_3+g)v_1 + (G_2+G_3)\cdot v_2 = 0$$

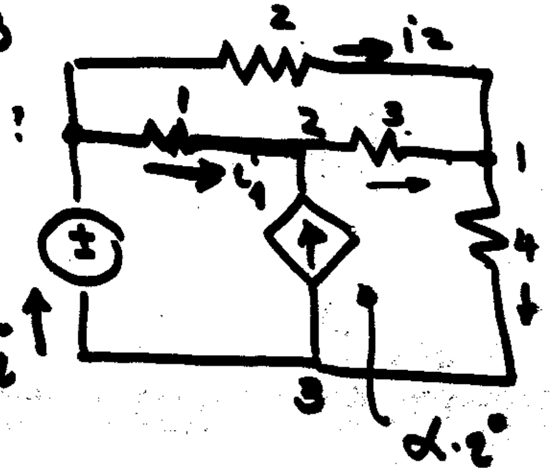
$$\underset{\uparrow}{\begin{bmatrix} G_1+G_3+G_4 & -G_3 \\ -G_3-g & G_2+G_3 \end{bmatrix}} \underset{\uparrow}{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}} = \underset{\uparrow}{\begin{bmatrix} G_4 v_s \\ 0 \end{bmatrix}}$$

conductance. node voltages current

g=0: symmetr. conductance.

g≠0: <u>Not</u> symmetr. any longer

Fig. 29:
P. 213



Nodal equations:

Why? + { 1. current controlled source αi
2. current is the controlling variable i

− {3. unaccompanied voltage source

Options: 1. source shift <easy>

2. keep Vs and worry about it after the first step

3. Add a resistor to Vs and later let
$R \to 0$!

Why is option 1 not good!

① Because $\underline{\underline{i}}$ is the controlling variable of the $\underline{CCC}S$: After the source shift, it disappears!

② Not bad!

③ interesting and novel <at least to us> approach.

Nodal equations:

② $G_1 \cdot (V_2 - V_S) - \alpha i + G_3 \cdot (V_2 - V_1) = 0$

① $G_2 \cdot (V_1 - V_S) + G_3 \cdot (V_1 - V_2) + G_4 \cdot V_1 = 0$

<3 unknowns>

$i = i_1 + i_2$

$$\left.\begin{aligned} i_1 &= G_1 \cdot (V_S - V_2) \\ i_2 &= G_2 \cdot (V_S - V_1) \end{aligned}\right\} !$$

MC

Hence:

① $(G_2+G_3+G_4)V_1 - G_3\cdot V_2 = G_2\cdot V_S \quad \leftarrow$

② $G_1(V_2-V_S) - \alpha\cdot\left[G_1\overbrace{(V_S-V_2)+G_2\cdot(V_3-V_1)}^{i}\right] \leftarrow$

$+G_3\cdot(V_2-V_1)=0$

And

② $(-G_3-\alpha\cdot G_2)V_1 + (G_1+\alpha G_1+G_3)V_2$

$= G_1\ (1+\alpha)V_S\ +\alpha\cdot G_2\cdot V_S$

Finaly

① $(\overset{\checkmark}{G_2}+\overset{\checkmark}{G_3}+\overset{\checkmark}{G_4})\overset{\swarrow}{V_1} - G_3\cdot V_2 = G_2\cdot V_S$

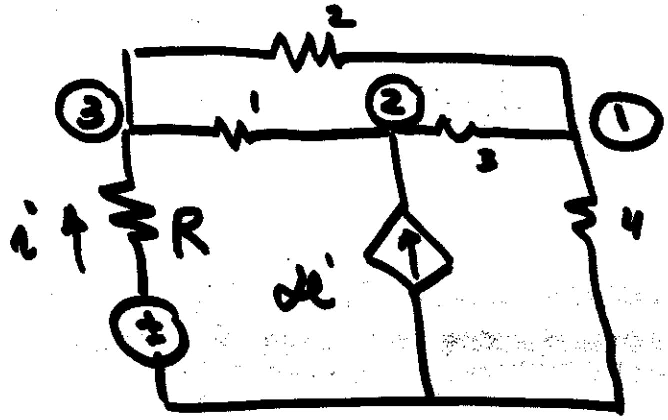
$-(\alpha G_2+G_3)V_1 + (G_1(1+\alpha)+G_3)V_2$

$= \{G_1+\alpha(G_1+G_2)\}V_S$

Check: If $\alpha=0$ is it symmetric!

Yes

What about adding R !



Node equations:

$i = (v_s - v_3) G$

$G = \frac{1}{R}$

① $G_3 \cdot (V_1 - V_2) + G_2 \cdot (V_1 - V_3) + G_4 \cdot V_1 = 0$

② $G_1 \cdot (V_2 - V_3) + G_3 (V_2 - V_1) - \alpha i = 0$

③ $G_1 \cdot (V_3 - V_2) + G_2 \cdot (V_3 - V_1) + \underset{\uparrow}{G} (V_3 - V_s) = 0$

$G = 1/R$

added!

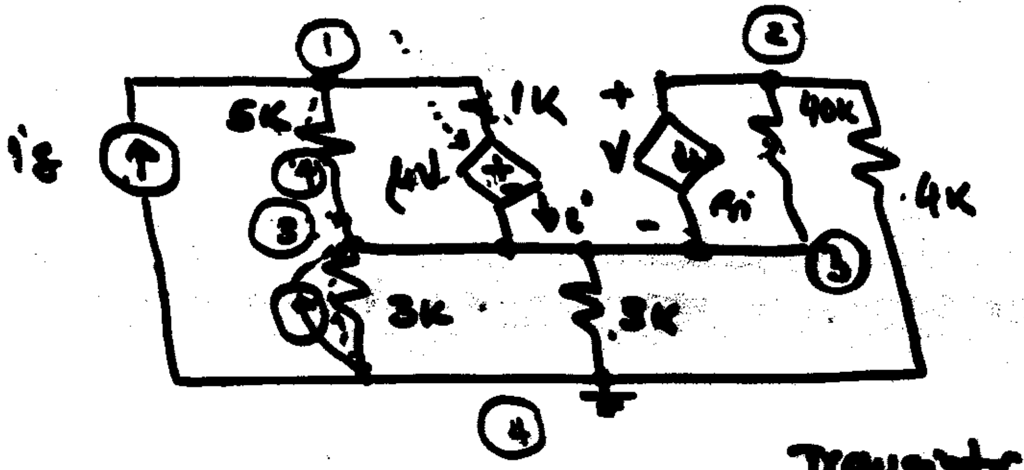
3 eq

3 unknown voltages

Solve these and at the end let $R \to 0$ ||

$G \to \infty$

Design problem #2 <p.237>



$\beta = 100$

$\mu = 2.5 \cdot 10^{-4}$

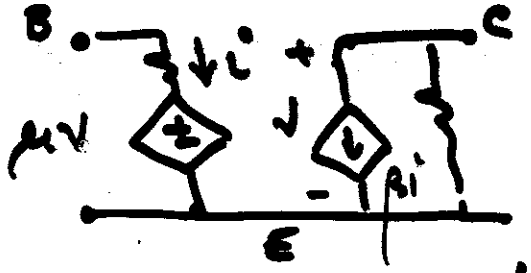
Transistor amplifier

Fig. DP2

a) Node equation

b) mesh equations

Transistor model:



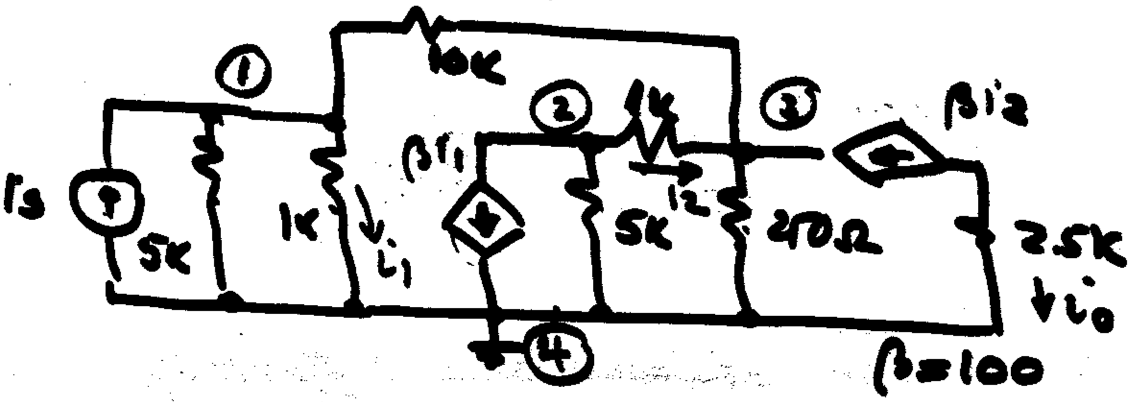
$n = 4$ $n-1 = \underline{3}$ node

$b = 6$ $b-(n-1) = \underline{3}$ mesh

independent equations

More detailed model than the model we use in the past.

Design problem #3 <p. 240>



two-transistor amplifier $i_o/i_s = ?$

a) Nodal equation !

$n=4$ $n-1=3$ ← much easier to write!

$b=6$ $b-(n-1)=3$

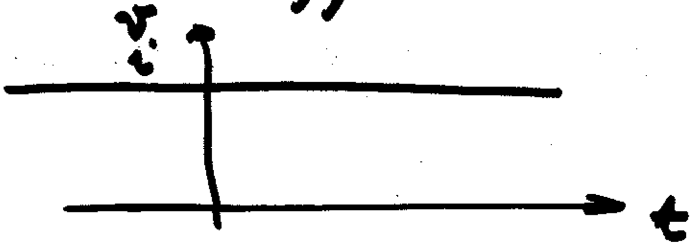
Signals and First-Order Circuits

Circuits with R resistors
C capacitors
L inductors

First order circuits:

RC or RL (but not L and C simultaneously!)

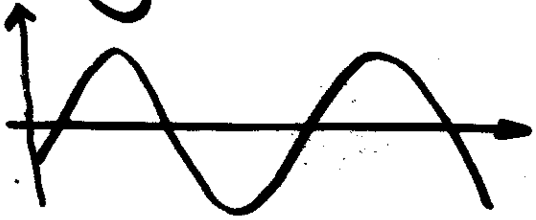
until now supplies were mostly: DC



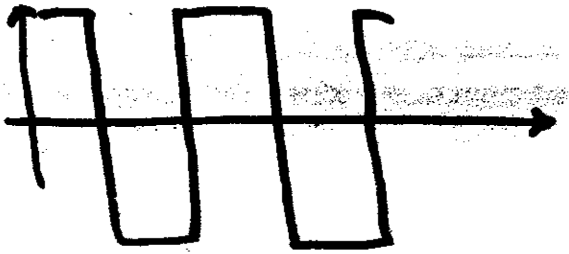
In a linear circuit with R, controlled source,

resulting currents and voltages were also DC.

Function generators can produce:

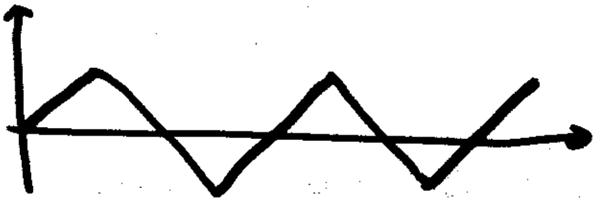


sinusoidal



square

sawtooth

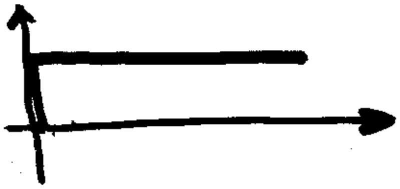


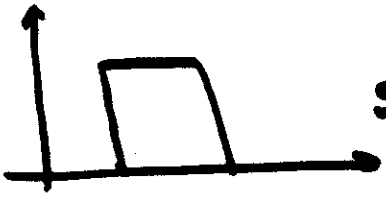
triangular

<All : periodic>

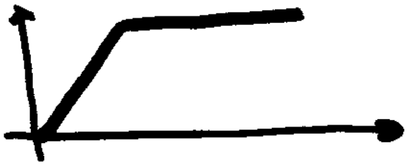
or

step





square-pulse



ramp

exponential

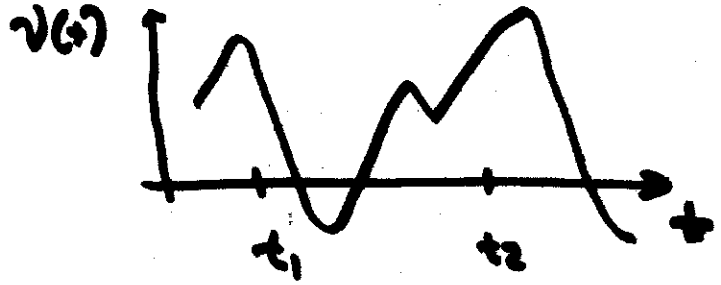
Periodic:

$$f(t+nT) = f(t) \quad -\infty<t<\infty$$

T = period

$f = 1/T$ frequency

Amplitude: largest value



$$\text{Average} = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} v(x)dx$$

If periodic:

$$= \frac{1}{T}\int_{t_1}^{t_1+T} v(x)dx$$

Root-mean-square value:

Resistor : i : current R : resistance periodic excitation

$$W = \int_{t_1}^{t_1+T} R \cdot i^2(x)dx$$

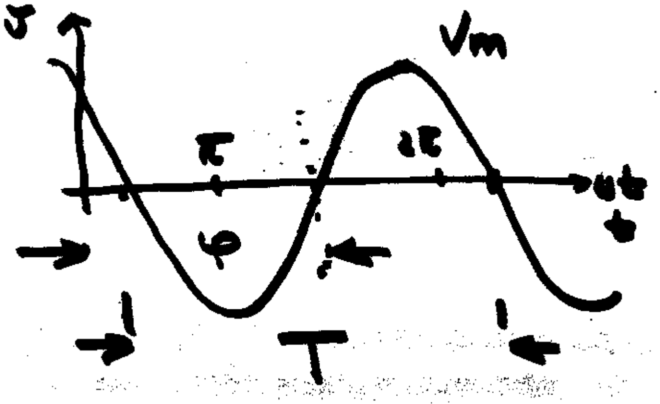
What would be a constant value of $i = I$ that would produce the same power dissipation:

$$W = R \cdot I^2 \cdot T$$

Hence: $$RI^2T = \int_{t_1}^{t_1+T} R \cdot i^2(x)dx$$

$$I = \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} i^2(x)dx}$$ <u>rms</u> value

Sinusoidal waveforms.



✓ ✓ phase

$$v(t) = V_m \cdot \sin(\omega t + \varphi)$$

amplitude ✓ angular frequency
radian frequency

$$\omega = 2\pi f \; ; \; = 2\pi \cdot \frac{1}{T} \qquad \text{rad/sec}$$

$$f = \frac{1}{T} \checkmark \; ; \; T = 1/f \qquad f = \frac{1}{\text{sec}} \checkmark \text{Hz}$$

Average: Full-cycle : 0

$\frac{1}{2}$ cycle: $\frac{1}{T/2}\int_0^{T/2} V_m \sin\omega t \cdot dt$.

$$= \underbrace{0.637}_{2/\pi} V_m$$

RMS:

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cdot \cos^2 \omega t \, dt}$$

↑ Does it matter is sin or cos ?

$$V_{rms} = \frac{V_m}{\sqrt{2}} \; ; \; = 0.707 V_m \leftarrow$$

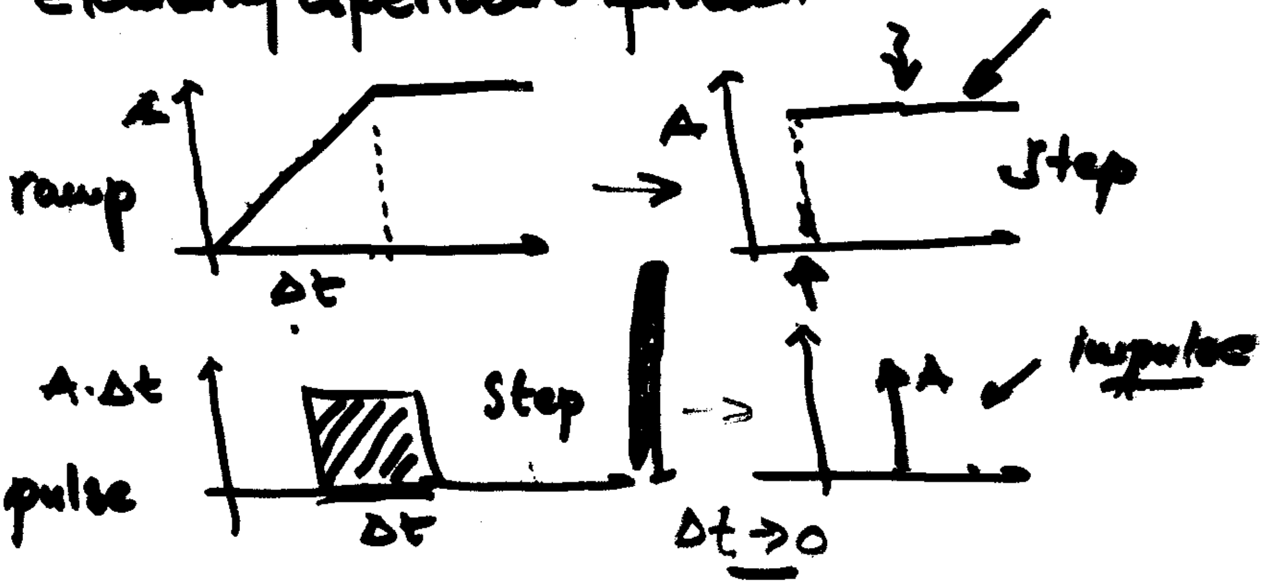
Recall:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

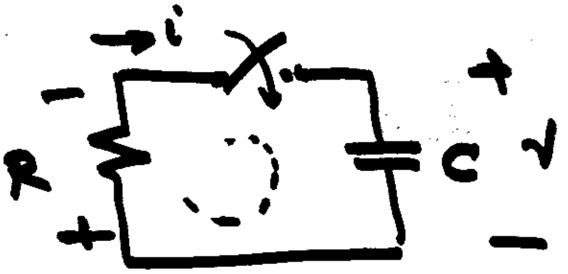
$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Elementary aperiodic funct:



RC circuits :



Ri

$$i = C \cdot \frac{dv}{dt}$$

$$C \frac{dv}{dt} + \frac{1}{R} \cdot v = 0$$

v

$\frac{dv}{dt}$

$$\rightarrow \frac{dv}{dt} + \frac{1}{RC} \cdot v = 0$$

DE
Linear
constant coefficient
$\downarrow$
$\frac{1}{RC}$
homogeneous
0 RHS

$$\int_{v_0}^{v(t)} \frac{dv}{v} = -\int_0^t \frac{dt}{RC}$$

$$\ln v \Big|_{v_0}^{v(t)} = -\frac{t}{RC}$$

$$\ln v(t) - \ln v_0 = -t/RC \qquad \frac{v(t)}{v_0} = e^{-t/RC}$$

$$\rightarrow \underline{v(t) = v_0 \, e^{-t/RC}}$$

$$v(t) = V_0 e^{-t/\boxed{RC}}$$

exponentia

initia value of the capacit voltage

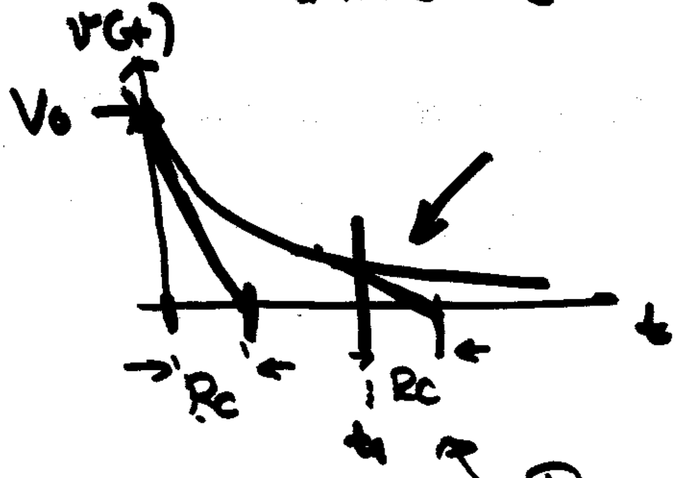
Note: $t = 0$

$$v(0) = V_0 e^{-0/RC}$$

$$v(0) = \underline{V_0}$$

RC = time constant

< dimension: time
units: seconds >



$\underline{RC}$

Prove! $\forall t_1$

Natural response

<another method for finding solution>

Suppose $v(t) = \underline{\underline{K}}e^{st}$ ←

is a solution to

$$\frac{dv}{dt} + \frac{1}{RC}v = 0$$

If yes, then:

$$\rightarrow k \cdot s e^{st} + \frac{1}{RC} \cdot k e^{st} = 0$$

or

Natural freq.!

$$\rightarrow \underset{\times}{k\underline{\underline{e^{st}}}} \cdot \left(\underbrace{s + \frac{1}{RC}}\right) = 0 \Rightarrow \boxed{\underline{s} = -\frac{1}{RC}}$$

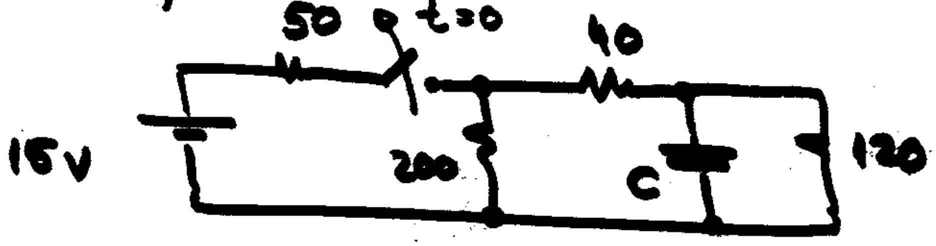
Hence

$$\rightarrow v(t) = Ke^{-t/RC}$$

K is defined by knowing $v(t)$ at <u>some</u> t.

If $v(0) = V_0 \Rightarrow \underline{K = V_0}$ and $v(t) = V_0 e^{-t/RC}$ ↖

Example:



Switch opens at t=0

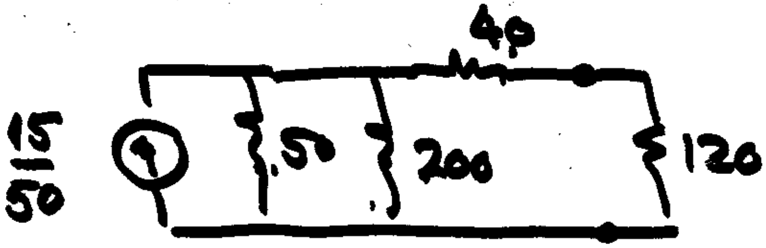
<It was closed for a very long time>

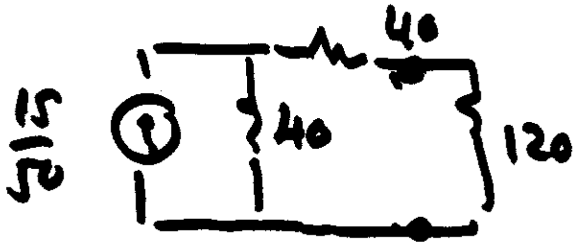
If closed for a long time:

$$\frac{dV}{dt}=0 \Rightarrow i_c=0$$

no current flows through $\underline{C}$

<DC battery!>



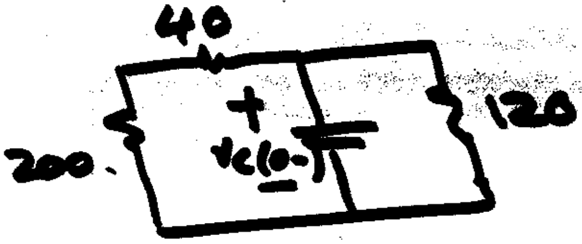


$V_c(0_-)=7.5$

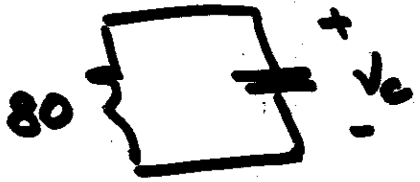
just before switch opens

Hence, initial voltage across C is: $\underline{7.5V}$

At $t=0$, switch opens:



or



$C = 12.5\mu F$

$$V_c(t) = \underline{V_c(0)}\, e^{-t/RC}$$

$RC = 80 \times 1.25 \times 10^{-6}$

$RC = 1$ msec

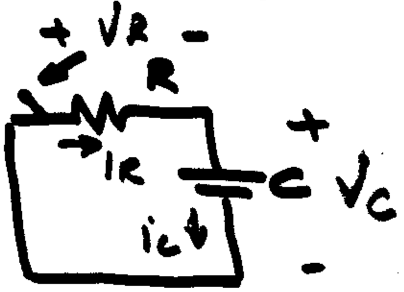
$$\rightarrow V_c(t) = 7.5\, e^{-1000t}$$

$\nwarrow V_c(0) = V_c(0+)$ after switch opens

$= V_c(0-)$ before.

$\rightarrow V_c(0+) = V_c(0-)$ Capacitor voltage has <u>No</u> jump

RC circuit without input <again>



$i_R - i_C = 0$

$V_R = R i$

$V_C = \frac{1}{C}\int i_C(x)\,dx$

$\downarrow$

$i_C = C \cdot \frac{dV_C}{dt}$

Hence:

$$\frac{V_R}{R} - C \cdot \frac{dV_C}{dt} = 0$$

KVL: $V_R + V_C = 0 \Rightarrow V_R = -V_C$

① $-\frac{V_C}{R} - C \cdot \frac{dV_C}{dt} = 0 \Rightarrow \frac{dV_C}{dt} + \frac{1}{RC} \cdot V_C = 0$

② $\frac{V_R}{R} - C \cdot \frac{d}{dt}(-V_R) = 0 \Rightarrow \frac{dV_R}{dt} + \frac{1}{RC} \cdot V_R = 0$

③ $V_R + V_C = 0$

All $\underline{DE}$: $\underline{1^{st} \text{ order}}$

$R \cdot i_R + \frac{1}{C}\int i_C(x)\,dx = 0$;

$i_R = i_C$

$R \cdot i_C + \frac{1}{C}\int i_C(x)\,dx = 0 \Rightarrow \frac{di_C}{dt} + \frac{1}{RC} \cdot i_C = 0$

What is different, if all these equations have the same form of the solution.

$$y = Ae^{-t/RC}$$

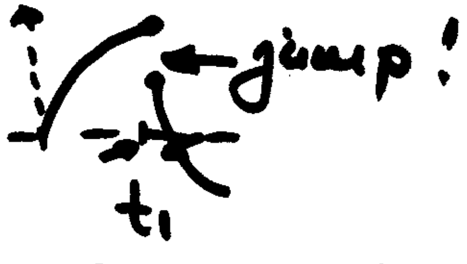
! depends on which variable : v_C
v_R
i_C
i_R !

What do we know about their values at some instance t_1:

(Because then: $y_{(t_1)} = A \cdot e^{-\frac{t_1}{RC}}$

and $A = y(t_1)e^{t_1/RC}$

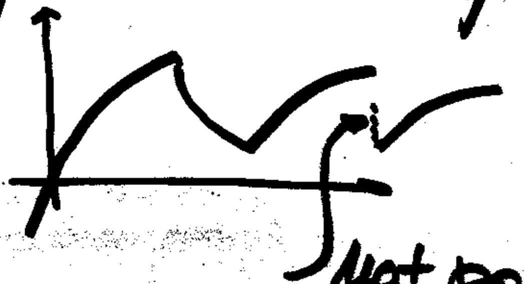
Slope of i_C i_R ?



Since $\underline{v_R} = R \cdot i_R$: v_R also can be discontinuous function

One voltage that has to be a continuous function of time is:

Voltage across the capacitor!



Not possible!
Otherwise

$i_C = C \cdot \frac{dv_C}{dt}$

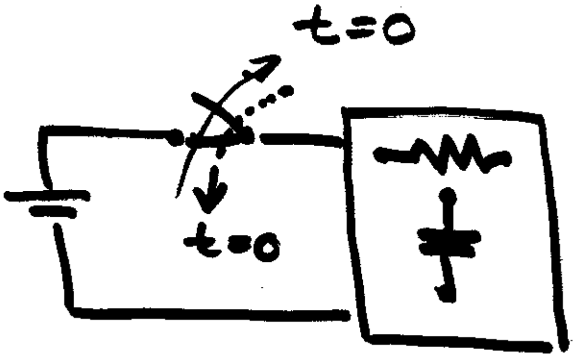
$i_C = \infty$

in a circuit (model) that faithfully represents a physical circuit in a lab!

Two consequences:

1. Write your DE in terms of v_C!
2. Use one known value of v_C to find the constant in: $v_C(t) \overset{\rightarrow}{=} \underset{\uparrow}{V_c}\, e^{-t/RC}$

Usually <but not always>: $v_C(0)$ is known.



Two cases: ① S closes at $t=0$

$V_c(0_-) = 0$

(just before closing)

What is $V_c(0_+) = 0$

(just after closing)

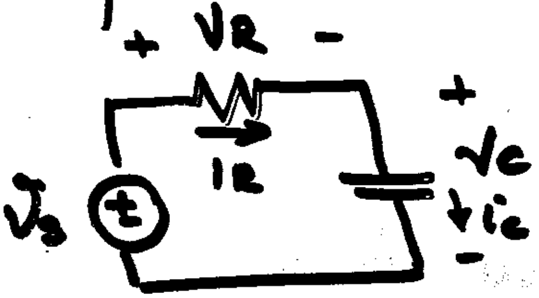
② S opens at $t=0$

$V_c(0_-) = V_c$

As a consequence

$V_c(0_+) = V_c$

Resistance - Capacitance circuits with input



$v_s - v_R - v_c = 0$

$v_R = R \cdot i_R$

$i_c = C \frac{dv_c}{dt}$

$i_c = i_R$

$v_R + v_c = v_s$

$R \cdot i_R + v_c = v_s$

$R \cdot i_c + v_c = v_s$

$R \cdot C \cdot \frac{dv_c}{dt} + v_c = v_s$ ← Non homogeneous!

← $v_s(t)$

← function of time

$$RC\cdot\frac{dv_c}{dt}+v_c=0$$

↑ homogeneous

(As $v_s(t)\equiv 0$)

DE: 1st order
linear
constant coefficients

$v_c(t) = v_h(t) + v_p(t)$ ← particular solution that depends on $v_s(t)$

↑ solution to the homogeneous DE

Proof: $RC\cdot\frac{d}{dt}(v_h+v_p)+(v_h+v_p)=v_s(t)$

$$\underbrace{\left(RC\cdot\frac{dv_h}{dt}+v_h\right)}_{\substack{\text{homog}\\ =0}}+\underbrace{\left(RC\cdot\frac{dv_p}{dt}+v_p-v_s(t)\right)}_{\substack{\text{non-homog}\\ =0}}=0$$

We already know that solutions to:

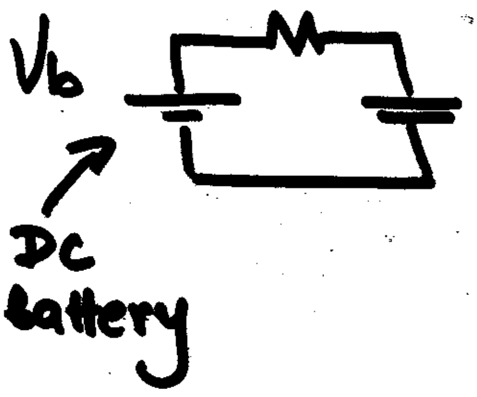
$$RC \cdot \frac{dv_h}{dt} + v_h = 0$$

is $v_h(t) = Ke^{-\frac{t}{RC}}$

What is $v_p(t) = ?$

The form of $v_p(t)$ depends on $v_s(t)$

Assume : $v_s(t) = V_b$



$v_p(t) = ?$

Could it be a constant? If yes:

$$RC \cdot \underbrace{\frac{dv_p}{dt}}_{0} + v_p - V_b = 0 \quad v_p = V_b \;!$$

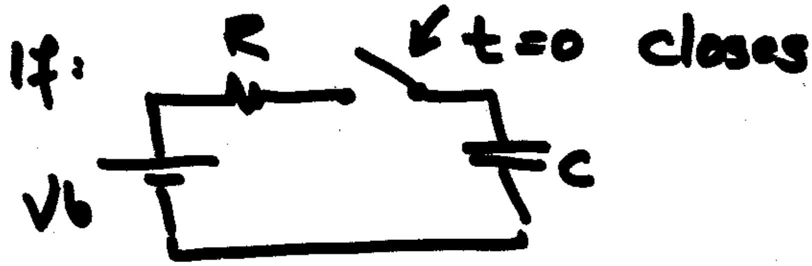
Hence $v_c(t) = V_b + Ke^{-t/RC}$

How about constant K ?

We need to know at least one solution to v_c (at some time t_1)

Usually : $t_1 = 0$ ← initial condition

If:



then $V_c(t) = V_b + Ke^{-t/RC}$

$V_c(0-) = 0$ (no charge on the capacitor)

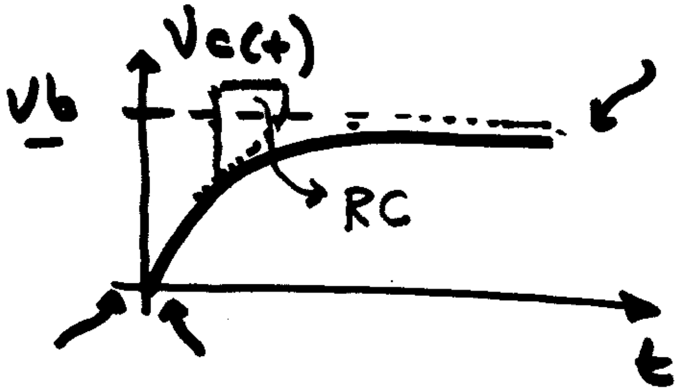
$V_c(0+) = 0$

From $V_c(t) = V_b + Ke^{-t/RC}$

$V_c(0+) = V_b + K$

$V_b + K = 0 \Rightarrow K = -V_b$ ↙

and, finally $\underline{V_c(t) = V_b\left(1 - e^{-t/RC}\right)}$

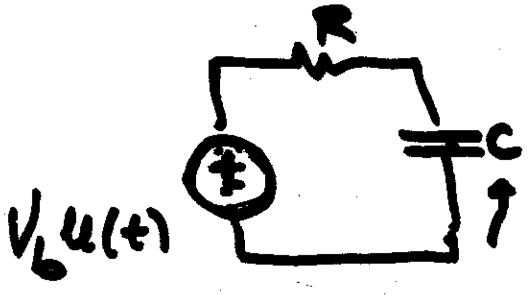


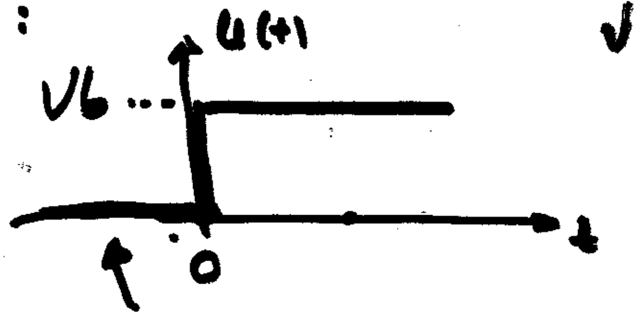
exponential function of time:

RC : time constant

$s = + 1/RC$: natural frequency

Another circuit :





$v_c(0^-) = v_c(0^+) = 0$

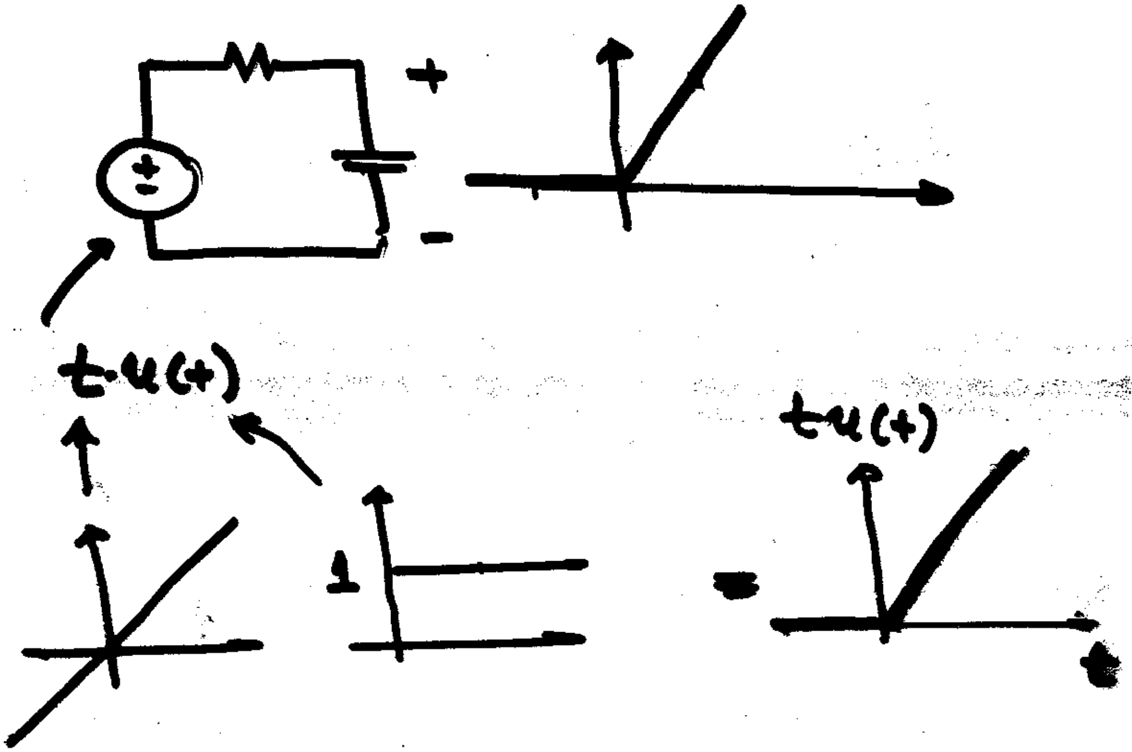
For $t > 0$: same circuit as before

$t < 0$: <u>Not</u> : in this circuit $u(t) = 0$

C cannot be charged

in the previous circuit, S was open and C could have maintained the charge!

What about the ramp function:



$$v_c(t) = Ke^{-t/RC} + v_p(t)$$

$v_p(t) = ?$ $\quad RC\cdot\frac{dv_c}{dt} + v_c(t) = v_s(t)$

$$RC\cdot\frac{dv_c}{dt} + v_c(t) = t\cdot u(t)$$

Can it be:

$$v_p(t) = (\underline{A + B\underline{t}})\cdot u(t) \quad ?$$

Substitute predicted solution (only a candidate) into DE:

$$RC\cdot B + (A + Bt) = t \qquad t>0$$

$$B = 1 \checkmark$$

$$RC\cdot B + A = 0$$

$$\hookrightarrow A = -RC \checkmark$$

Hence

0+

0−

$$v_p(t) = (-RC + t)\cdot u(t) \leftarrow t > 0$$

$$v_c(t) = \Big[\underbrace{K\cdot e^{-t/RC}}_{v_h(t)} + \underbrace{(-RC + t)\Big]u(t)}_{v_p(t)}$$

at $t=0$: $v_c(0-) = 0$

Hence: $v_c(0+) = 0$ or: $K - RC = 0$

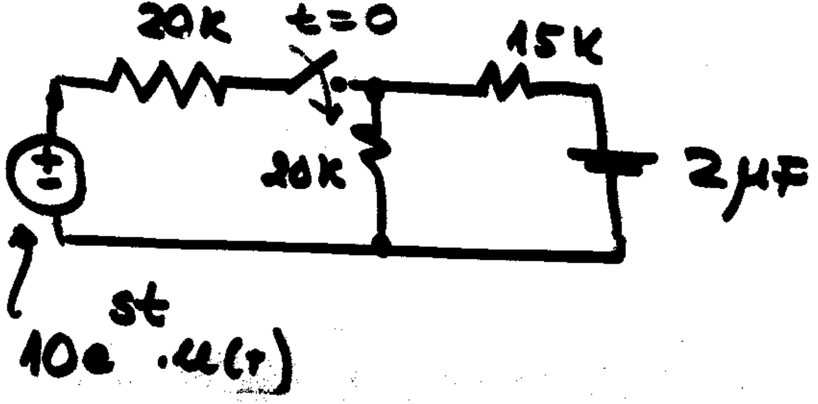
$$K = RC \leftarrow$$

Ans

$$v_c(t) = (t - RC)\,\underline{u(t)} + RC\,e^{-t/RC}\cdot\underline{u(t)}$$

to emphasize that for $t<0$ $v_c(t) \equiv 0$

RC circuit with an exponential input.



$u(t) = 0 \quad t<0$

$= 1 \quad t>0$

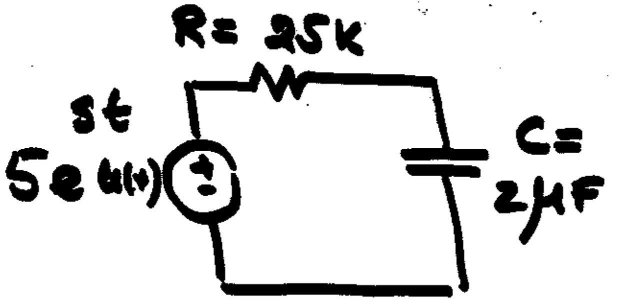
undefined at $t=0$

$u(0_-) = 0$

$u(0_+) = 1$

S = switch. S open for a long time. $V_c(0_-)=0$

S closes at $t=0$



$v_s(t) = R\cdot i + v_c$

$i = C\cdot\frac{dv_c}{dt}$

$RC\cdot\frac{dv_c}{dt} + v_c = v_s(t)$

$$\frac{dv_c}{dt} + \frac{1}{RC}\cdot v_c = \frac{1}{RC}\cdot v_s(t)$$

DE:

$$\frac{dv_c}{dt} + 20v_c = 100\, e^{st}\, u(t)$$

↑ error in the text

R = 25k
C = 2μF
$RC = 50 \cdot 10^{3} \cdot 10^{-6}$ sec
RC = 50 msec
$\frac{1}{RC} = \frac{1}{50 \cdot 10^{-3}}$
$\frac{1}{RC} = 20$

What is $v_c(t)$?

RHS: exponential function

Linear DE with constant coefficients:

$$v_c(t) = v_{c_{homogeneous}} + v_{c_{particul.}}$$

$$\uparrow \; \frac{dv_c}{dt} + 20v_c = 0 \Rightarrow$$

$$v_{c_h}(t) = K e^{-20t}$$

Conjecture: $v_{c_{partic}} = A \cdot e^{st}$

Does it satisfy DE ?

$$A \cdot s e^{st} + 20 \cdot A \cdot e^{st} = 100\, e^{st} \quad A(s+20) = 10$$

Ⓢ HW4

$A(s+20) = 100$

$\langle 20 \rangle \leftarrow \frac{1}{RC} = 20$

Only if $s \neq -20$ $\quad A = \dfrac{100}{s+20}$

$\uparrow$ from the source! $\quad$ let $s = -\underline{\underline{10}}$

$$V_c(t) = \{ K e^{-20t} + 10 e^{-10t} \} u(t)$$

How to find K:

$V_c(0-) = 0$

$V_c(0+) = K + 10 \implies V_c(0+) = 0$

$K = -10$

$$V_c(t) = 10 (e^{-10t} - e^{-20t}) u(t)$$

$t = 0$

$V_c(0) = 0$

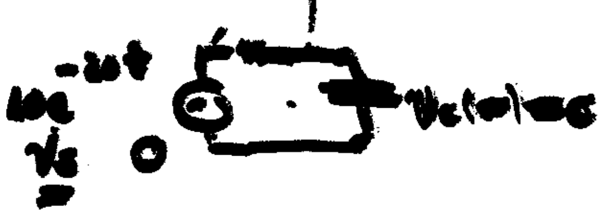
$V_c(0-) = 0$

$t = \infty$

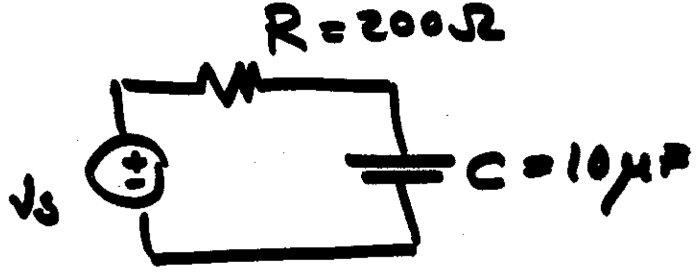
$V_c(\infty) = 0$

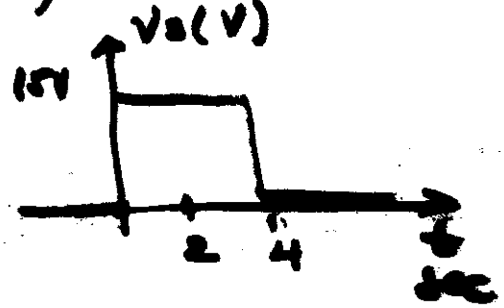
Exercise $\langle$p. 266$\rangle$
10

$V_s(t) = A \cos(\omega t + \theta)$



RC circuit with a pulse input





$RC = 2\text{ sec}$

$Ri + V_c = V_s$

$RC\frac{dV_c}{dt} + V_c = V_s$

$\longrightarrow 2\cdot\frac{dV_c}{dt} + V_c = V_s$

$0 < t \leq 4 :\quad 2\frac{dV_c}{dt} + V_c = 15$ ①

$t \geq 4 \quad 2\frac{dV_c}{dt} + V_c = 0$ ②

① $V_c(t) = \underbrace{K\cdot e^{-t/2}} + A$

because $V_s = 15$

$2\cdot\underset{0}{\frac{dA}{dt}} + A = 15$

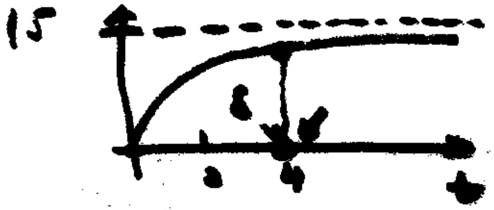
$A = 15$

$V(t) = Ke^{-t/2} + 15$

$V_c(0-) = 0$

$V_c(0+) = K + 15 \Rightarrow K = -15$

$$V_C(t) = 15(1-e^{-t/2}) \qquad 0<t<4$$



② At t=4 sec.

$$V_C(4_-) = 15\cdot(1-e^{-2}) \approx 13V$$

Then at $t=4_+$

$$2\frac{dV_C}{dt}+V_C=0$$

A) $V_C(t') = Ke^{-t'/2}$ But now $t' = \underline{t-4}$ shifted

$$V_C(t') = ke^{-t'/2} \qquad t'\geq 0$$

$V_C(0_-) = 13$, $V_C(0_+) = 13e^{-0/2}$

$V_C(t') = 13e^{-t'/2}$ $\underline{t'=t-4}$

$$V_C(t) = 13e^{-\frac{t-4}{2}} \qquad t\geq 4 \text{ sec}$$

B : $2\frac{dv_c}{dt} + v_c = 0$ $\quad t \geq 4$

$v_c = K e^{-t/2}$

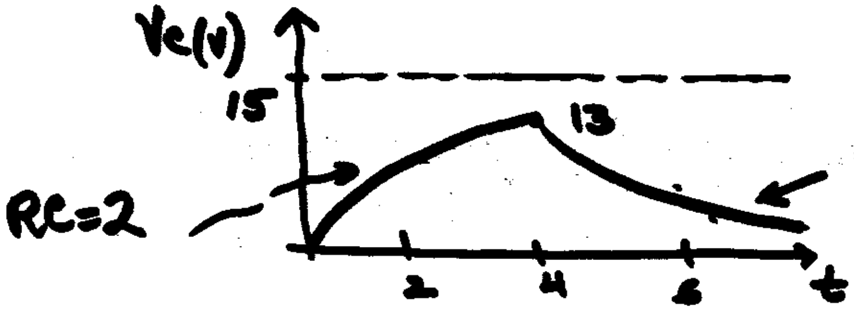
$V_c(4_-) = 13$ ✓

$v_c(4_+) = K e^{-2} \Rightarrow K = 13e^{2}$ ✓

$V_c(t) = 13 \cdot e^{2} \cdot e^{-t/2}$

$V_c(t) = 13\, e^{-\frac{t-4}{2}}$ $\quad t \geq 4$

< Same answer! >



How about i_c : $\quad i_c = C\frac{dv_c}{dt}$

✓ $0 < t \leq 4$ $\quad v_c(t) = 15(1 - e^{-t/2})$

$t \geq 4$ $\quad v_c(t) = 13\, e^{-\frac{t-4}{2}}$

i_c ↗ ↑ ? is this correct ?

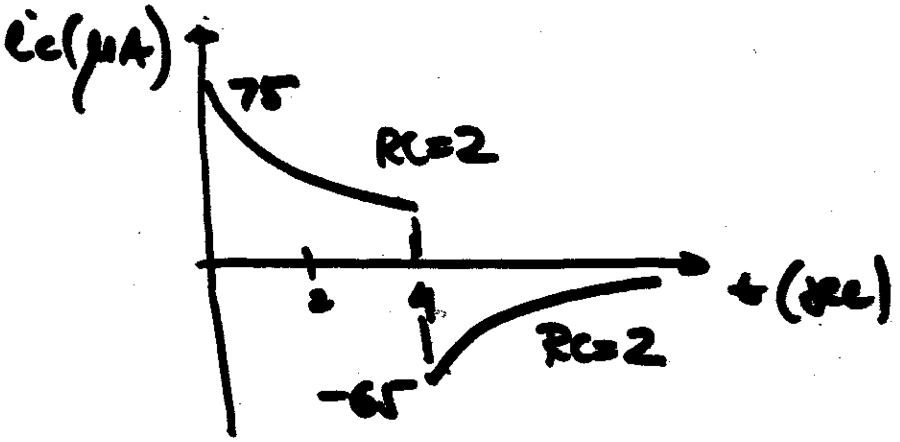
$C = 10\mu F$

$i_c = 10 \cdot 15 \cdot \frac{1}{2} e^{-t/2}$ μA $\quad 0 \leq t < 4$

$i_c = 75 e^{-t/2} \quad 0 \leq t < 4$

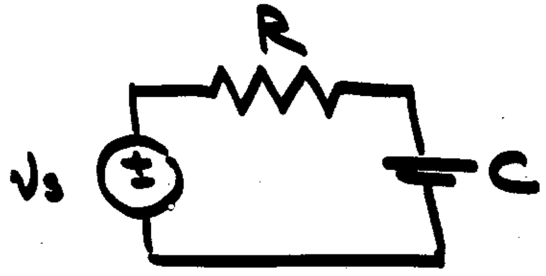
For $t > 4$: $\quad i_c = 10 \cdot 13 \cdot \left(-\frac{1}{2}\right) e^{-\frac{t-4}{2}}$

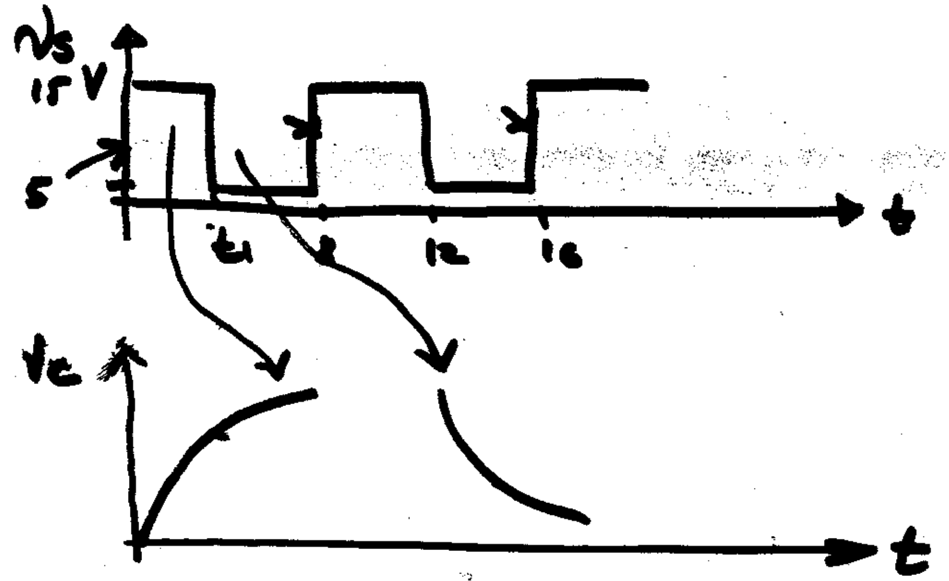
$$i_c = -65 e^{-\frac{t-4}{2}}$$



$i_R = i_c$
$v_R = R \cdot i_R$ $\Big\}$ all have the same shape

How about RC with a periodic wave?





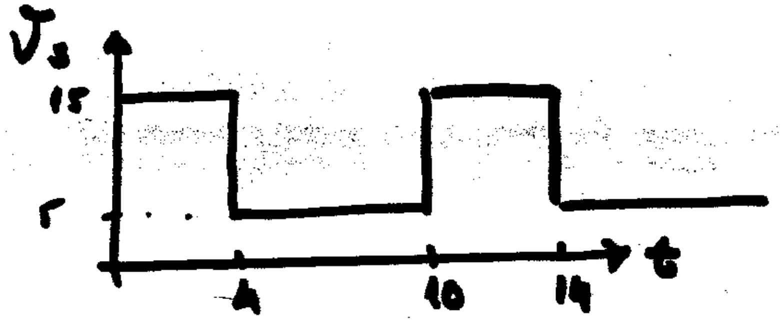
How can we calculate the "steady-state" response?

<periodic in most systems>

Aperiodic steady-state : chaotic systems

$0 \le t \le 4$ $RC \cdot \frac{dv_c}{dt} + V_c = 15$

$4 \le t \le 10$ $RC \cdot \frac{dV_c}{dt} + Vc = \overset{5}{\cancel{15}}$



Solutions:

$0 \le t \le 4$: $V_c(t) = 15 + K_1 e^{-t/RC}$

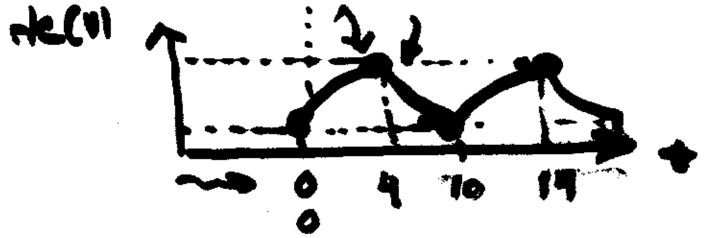
$4 \le t \le 10$ $V_c(t) = 5 + K_2 e^{-t/RC}$

at $\underline{4_-}$: $15 + \check{K}_1 e^{-4/RC} = 5 + \check{K}_2 e^{-4/RC}$ ← K_1 $K_=$

at $\underline{4+}$

at $\underline{10_-}$: $5 + \underline{K_2} e^{-10/RC} = 15 + \check{K}_3 e^{-10/RC}$

at $\underline{10+}$

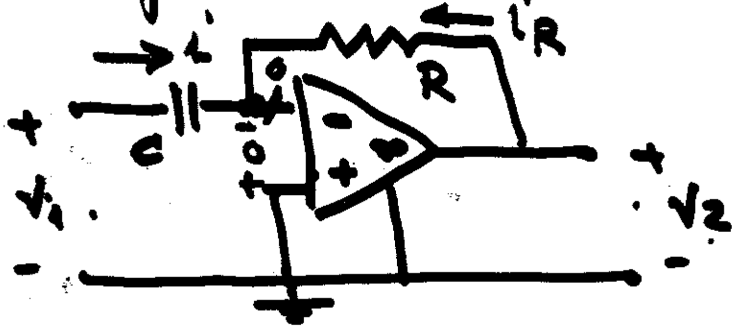


RC circuits with Op-amps

→ Source integrator R_{Load} ←

interference and affects the integration

To separate Load :



$i = C \cdot \frac{dv_1}{dt}$ ←

$v_2 = R \cdot i_R$ ←

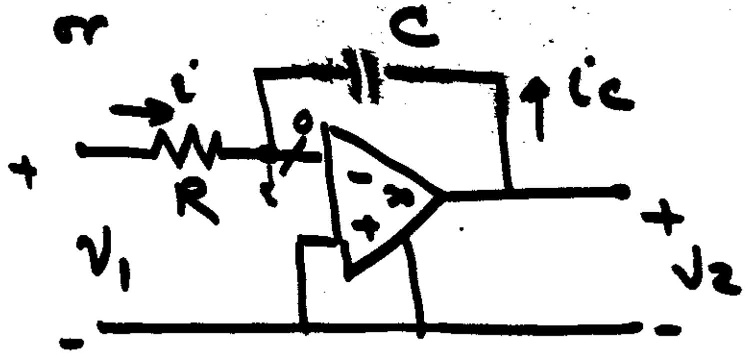
$i + i_R = 0$

$C \frac{dv_1}{dt} + \frac{v_2}{R} = 0$

$\rightarrow v_2 = -RC \frac{dv_1}{dt}$

Diffrent.

or



$v_1 = R \cdot i$ $\quad i + i_c = 0$

$i_c = C \cdot \frac{dv_2}{dt} \quad \frac{v_1}{R} + C \frac{dv_2}{dt} = 0 \quad \frac{dv_2}{dt} = -\frac{1}{RC} \cdot v_1$

integrator

Saturation:

$$V_{out} = V_{sat}$$

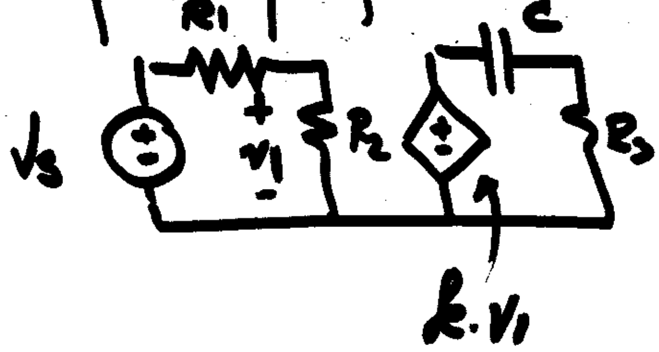
or $V_2 = V_{sat}$

Differentiat: $\left| RC \cdot \frac{dv_1}{dt} \right| \leqslant V_{sat}$

Integral: $\left| \frac{1}{RC} \int v_1(x)dx \right| \leqslant v_{sat}$

Circuits with controlled Sources:

Simple amplifier:



$$v_1 = \frac{R_2}{R_1+R_2} \cdot v_s$$

$$k \cdot V_1 - v_c - R_3 \cdot i = 0 \leftarrow$$

$$i = C \cdot \frac{dv_c}{dt} \leftarrow$$

$$v_c + R_2 C \cdot \frac{dv_c}{dt} = k \cdot \frac{R_2}{R_1+R_2} \cdot v_s$$

↗ DE

RL circuits

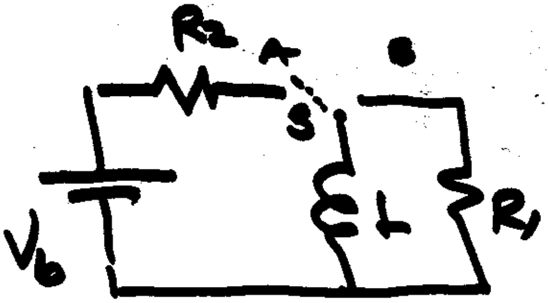
Duality: i_L vs. v_C

↑

continuous function of time —"—

1st order DE, linear, with constant coeff.

$i_L(0-) = i_L(0+)$ ← condition used to find unknown constant when solving DE



S : A for a long time

$v_L = 0$

$i_L = \frac{V_b}{R_2}$

L : looks like a short circuit!

S: At $t=0$ switch moves to position B

$$v_L = L\frac{di_L}{dt}$$

$$V_R = R_1 \cdot i_R$$

$$i_R + i_L = 0 \quad ; \quad v_L = v_R$$

Let $i_L = i^{\circ}$
$\quad i_R = -i^{\circ}$

$$L \cdot \frac{di}{dt} = R \cdot (-i) \quad \text{or:}$$

$$\searrow L\frac{di}{dt} + Ri = 0 \qquad 1^{st} \text{ order DE}$$

$$\frac{di}{i} = -\frac{R}{L} \cdot dt$$

$$\int \frac{di}{i} = -\frac{R}{L}\int dt$$

$$\ln i = -\frac{R}{L} \cdot t + \ln k$$

$$i = K e^{-\frac{R}{L} \cdot t} \leftarrow$$

$i(0+) = K$ Time constant: $\frac{L}{R}$

$i(0-) = \frac{V_b}{R}$ Hence: $i = \frac{V_b}{R} e^{-\frac{R}{L} \cdot t} \leftarrow$

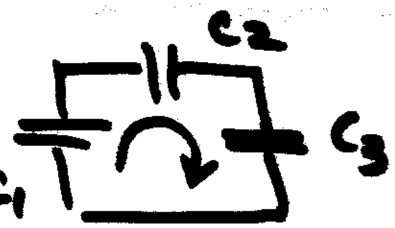
7.

Second order circuits

Number of C's and L's will define the order of the DE

Usually: # C's + # L's = # DE

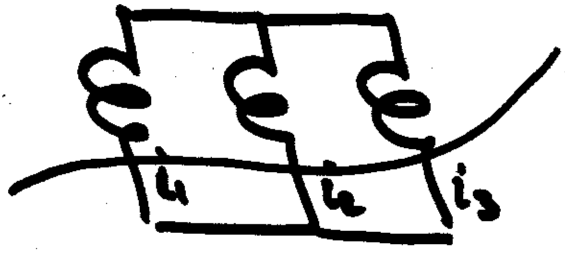
< Not always!



$v_{C_1} + v_{C_2} + v_{C_3} = 0$

Algebraic equation

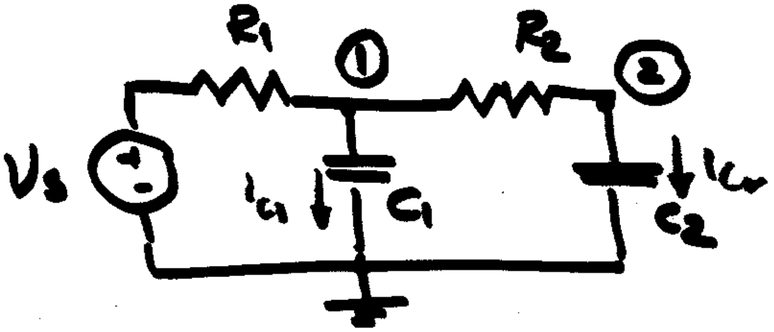
or:



$i_{L_1} + i_{L_2} + i_{L_3} = 0$

A circuit with C_1 and C_2:

2nd order DE : Linear with constant coefficients.



Nodal equations:

$$(v_1 - v_s)\cdot\frac{1}{R_1} + i_{c_1} + \frac{v_1 - v_2}{R_2} = 0$$

$$\frac{v_2 - v_1}{R_2} + i_{c_2} = 0$$

$$i_{c_1} = C_1\cdot\frac{dv_1}{dt} \quad ; \quad i_{c_2} = C_2\cdot\frac{dv_2}{dt}$$

$$\left.\begin{aligned} &v_1\cdot\frac{1}{R_1} + C_1\cdot\frac{dv_1}{dt} + \frac{v_1}{R_2} - \frac{1}{R_2}\cdot v_2 = \frac{v_s}{R_1} \\ &\frac{v_2}{R_2} - \frac{1}{R_2}\cdot v_1 + C_2\cdot\frac{dv_2}{dt} = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} &v_1\cdot\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + C_1\cdot\frac{dv_1}{dt} - \frac{1}{R_2}\cdot v_2 = \frac{v_s}{R_1} \\ &-\frac{1}{R_2}\cdot v_1 + \frac{1}{R_2}\cdot v_2 + C_2\cdot\frac{dv_2}{dt} = 0 \end{aligned}\right\}$$

$$C_1 \cdot \frac{dv_1}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_1 - \frac{1}{R_2}v_2 = \frac{1}{R_1} \cdot v_S$$

$$-\frac{1}{R_2} \cdot v_1 + C_2 \cdot \frac{dv_2}{dt} + \frac{1}{R_2} \cdot v_2 = 0$$

Nodal equations.

Re-arranged:

$$\left.\begin{aligned} C_1 \cdot \frac{dv_1}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_1 - \frac{1}{R_2}v_2 &= \frac{1}{R_1} \cdot v_S \\ C_2 \cdot \frac{dv_2}{dt} - \frac{1}{R_2}v_1 + \frac{1}{R_2}v_2 &= 0 \end{aligned}\right\}$$

Matrix:

$$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & 1/R_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/R_1 \\ 0 \end{pmatrix} \cdot v_S$$

State-equation.

2 equat: for 2 capacitors

v_1, v_2 : capacitor's voltages

Solutions:

2 DE → 1 DE by substitution!
1st order   2nd order

Nodal equations:

From 2nd equation:

$$V_1 = V_2 + R_2C_2 \cdot \frac{dV_2}{dt}$$

Substitute it into 1st:

$$C_1 \cdot \left\{ \frac{dV_2}{dt} + R_2C_2 \frac{d^2V_2}{dt^2} \right\} + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \cdot \left(V_2 + R_2C_2 \frac{dV_2}{dt} \right) - \frac{1}{R_2} V_2 = \frac{1}{R_1} \cdot V_S$$

Or:

$$\left. \begin{aligned} \frac{d^2V_2}{dt^2} + \left(\frac{G_1+G_2}{C_1} + \frac{G_2}{C_2} \right) \frac{dV_2}{dt} + \frac{G_1G_2}{C_1C_2} V_2 \\ = \frac{G_1G_2}{C_1C_2} \cdot V_S \end{aligned} \right\}$$

$$\left(G_i = 1/R_i \ , \ i = 1, 2 \right)$$

How about V_1? Find the DE.

How to solve it:

$$v_2 = v_{2_{\text{homogenen}}} + v_{2_{\text{particular}}}$$

Homogeneous:

$$V_{2_h} = K e^{st}$$ ← Conjecture!

Natural response

Substitute it into the DE:

$$s^2 + \left(\frac{G_1 + G_2}{C_1} + \frac{G_2}{C_2} \right) \cdot s + \frac{G_1 \cdot G_2}{C_1 C_2} = 0$$

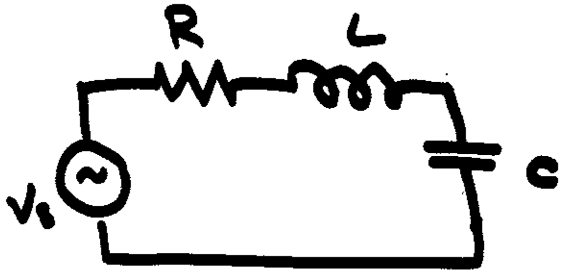
$s_{1/2}$ Two natural frequencies
Two time constants

Characteristic equation

Solutions: two real, distinct,
negative numbers!

$$v_{2_h} = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

RLC Circuits



$$v_s = Ri + L\cdot\frac{di}{dt} + v_c$$

$$i = C\cdot\frac{dv_c}{dt}$$

$$v_s = Ri + L\cdot C\cdot\frac{d^2v_c}{dt^2} + v_c$$

$$RC\cdot\frac{dv_c}{dt} + LC\cdot\frac{d^2v_c}{dt^2} + v_c = v_s$$

$$LC\,\frac{d^2v_c}{dt^2} + RC\cdot\frac{dv_c}{dt} + v_c = v_s$$

2nd order DE : linear
constant coefficients

$$v_c = v_{c\,\text{homogenen}} + v_{c\,\text{particular}}$$

Homogeneous solution:

$$LC\frac{d^2V_C}{dt^2} + RC\cdot\frac{dV_C}{dt} + V_C = 0$$

Homogeneous: RHS = 0

Assume $V_C(t) = K\cdot e^{st}$

then: $\frac{dV_C}{dt} = Kse^{st}$

$$\frac{d^2V_C}{dt^2} = Ks^2\cdot e^{st}$$

Substitute into DE:

$$LC\cdot Ks^2e^{st} + RC\cdot K\cdot se^{st} + K\cdot e^{st} = 0$$

$$(LCs^2 + RCs + 1)\cdot \underbrace{Ke^{st}}_{\neq 0} = 0$$

$LCs^2 + RCs + 1 = 0$ Characteristic equation.

$$s_{1/2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$ 2 solutions!

Characteristic equation is quadratic:

$$S_{1/2} = \underset{\substack{\uparrow \\ \text{always} < 0}}{-\frac{R}{2L}} \pm \sqrt{\underset{\substack{\nwarrow \\ >0 \\ \text{or} \\ <0}}{\frac{R^2}{4L^2} - \frac{1}{LC}}}$$

<u>Case 1:</u> $\frac{R^2}{4L^2} - \frac{1}{LC} > 0 \Rightarrow R^2 > \frac{4L}{C}$

$$R > 2\sqrt{\frac{L}{C}}$$

$S_1 =$
$S_2 =$ Real and both <u>Negative</u>

<u>Case 2:</u> $\frac{R^2}{4L^2} = \frac{1}{LC} \Rightarrow R = 2\sqrt{\frac{L}{C}}$

$S_1 = S_2$ (identical)
Both real and Negative.

$S_1 = S_2$; $S_1 = -R/2L$

Case 3: $$\frac{R^2}{4L^2} - \frac{1}{LC} < 0$$

$$R < 2\sqrt{\frac{L}{C}}$$

$$s_{1/2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

imaginary : $j^2 = -1$

$s_{1/2}$ complex conjugate numbers

Real part of $s_{1/2} = -\frac{R}{2L}$

Always negative

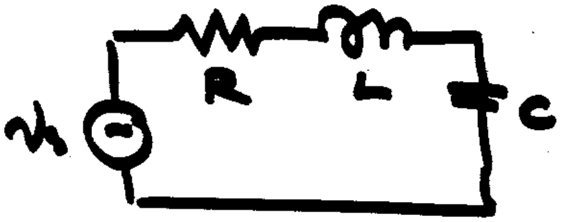
Note: always in pairs.

$-\alpha + j\omega$

$-\alpha - j\omega$

$\pm j\omega$: because all R, L, C are real numbers.

Another way of writing the equation:



$$v_s = R i_L + L \frac{di_L}{dt} + v_C$$

$$i_L = C \frac{dv_C}{dt}$$

or:
$$\frac{di_L}{dt} = -\frac{R}{L} \cdot i_L - \frac{1}{L} \cdot v_C + \frac{1}{L} \cdot v_s$$

$$\frac{dv_C}{dt} = \frac{1}{C} \cdot i_L$$

Matrix form:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_c \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} v_s$$

i : current through L

V : voltage across capacit C

↑ State variables

let $x = \begin{pmatrix} i_L \\ v_C \end{pmatrix}$

$$\frac{dx}{dt} = \underset{\uparrow}{A}x + b \leftarrow \begin{pmatrix} -1/L \\ 0 \end{pmatrix} \cdot V_S$$

$$\begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}$$

b: depends on the type of the source connected to the circuit

A: depends only on circuit topology and elements

Homogeneous equation:

$$\frac{dx}{dt} = Ax$$ (Matrix DE homogeneous.)

Characteristic equation:

$x = \begin{pmatrix} i_L \\ v_C \end{pmatrix}$ Solutions: $K_1 e^{st} \leftarrow i_L$

$K_2 e^{st} \leftarrow v_C$

Hence, suppose $x = \begin{pmatrix} k_1 e^{st} \\ k_2 e^{st} \end{pmatrix}$ is the solution:

$$s \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{st} = A \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{st}$$

$$e^{st} \neq 0 \qquad \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq 0_{\nwarrow \text{matrix.}}$$

Hence:

$$(s . \underset{\uparrow}{I} - A) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

identity matrix

(also denoted as $\mathcal{U}$) ↖ text

(*) $\det(s I - A) = 0$ is the characteristic equation we have found earlier:

$$A = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}$$

Prove that this is indeed the case!

Substitute A into (*) and expand the determinant.

let us consider again the three cases:

$$s_{1/2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

$$s_{1/2} = -\alpha \pm j\omega$$

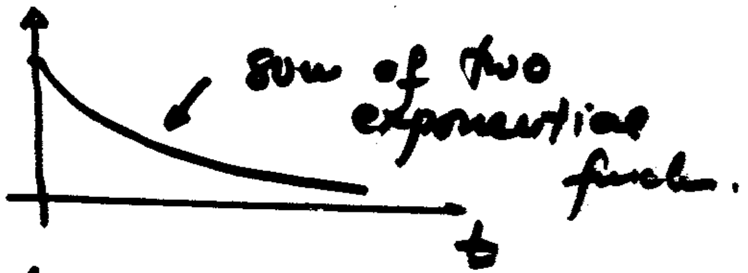
$$\alpha = R/2L \;;\; \omega = \sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$$

Case 1: $s_{1/2}$ Real and distinct $\quad v_{c_{hom}} = Ke^{st}$

$$\rightarrow v_{c_h} = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

s_1
s_2 Real, negative...

Assume $v_c(0) \neq 0$



v_c : sum of two exponential functions

Case 2: $s_1 = s_2 := -\alpha$

$\alpha = R/2L$

then

$$v_c(t) = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t}$$

homogeneous solution

We have to assume this form in case $s_1 = s_2$

Since:

$K_1 e^{s_1 t} + K_2 e^{s_2 t}$

is not general enough if $s_1 = s_2$:

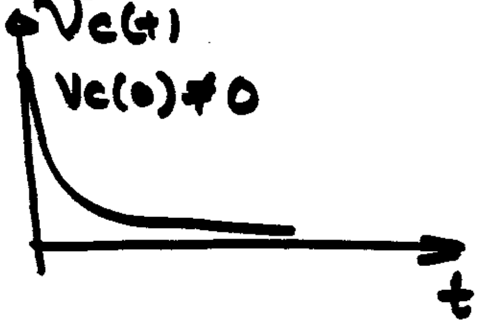
$(K_1 + K_2) e^{s_1 t}$

$= \underline{K e^{s_1 t}}$

is too simplified!

$\neq K_1 e^{s_1 t}$

$K_1 e^{s_1 t} + K_2 e^{s_2 t}$



as $t \to \infty \Rightarrow v_c(\infty) \to 0$ homogeneous soluti

Case 3 : $S_{1/2} = -\alpha \pm j\omega$

$j^2 = -1$

ω : Real

$\Downarrow$

$S_{1/2}$ complex conjugate

$$v_o(t) = k_1 e^{S_1 t} + k_2 e^{S_2 t}$$

$$v_o(t) = k_1 e^{(-\alpha + j\omega)t} + k_2 e^{(-\alpha - j\omega)t}$$

$$v_c(t) = e^{-\alpha t} \cdot \left(k_1 e^{j\omega t} + k_2 e^{-j\omega t}\right)$$

Euler's formulae: $e^{j\omega t} = \cos\omega t + j\sin\omega t$

$e^{-j\omega t} = \cos\omega t - j\sin\omega t$

Hence:

$$k_1 e^{j\omega t} + k_2 e^{-j\omega t} = (k_1 + k_2)\cos\omega t + j(k_1 - k_2)\sin\omega t$$

$$= A\cos\omega t + B\sin\omega t$$

$$= \underline{\underline{V}} \cdot \sin(\omega t + \underline{\theta})$$

If:

$$A\cos\omega t + B\sin\omega t = V\sin(\omega t + \theta)$$

Find $\begin{matrix} V \\ \theta \end{matrix}$ in terms of $\begin{matrix} A \\ B \end{matrix}$

Or: Since $\begin{matrix} K_1 \\ K_2 \end{matrix}$ are unknown constants to be determined from "initial conditions"

$\begin{matrix} A \\ B \end{matrix}$ are also unknown

$\begin{matrix} V \\ \theta \end{matrix}$ are also unknown

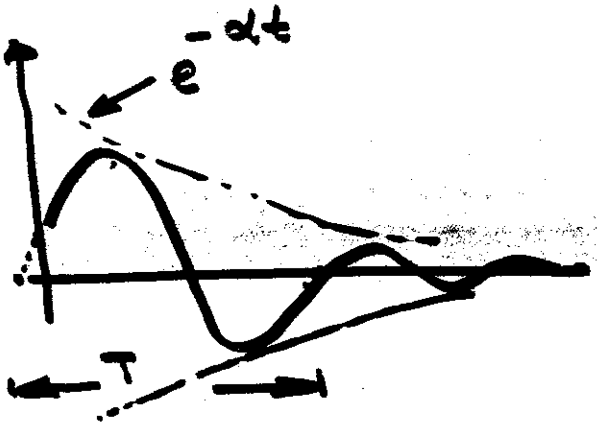
Hence, it is advantageous to assume:

$$V_C(t) = e^{-\alpha t} \cdot A \sin(\omega t + \theta)$$

unknown constants to be determined in the same way as $\begin{matrix} K_1 \\ K_2 \end{matrix}$ or $\begin{matrix} A \\ B \end{matrix}$ would have been from the initial conditions.

Response for the case:

s_1, s_2 complex conjugate:



$T = 2\pi f$

$\omega = 1/f$

$$v_s(t) = V e^{-\alpha t} \cdot \sin(\omega t + \Theta)$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Natural response

<homogeneous solution>

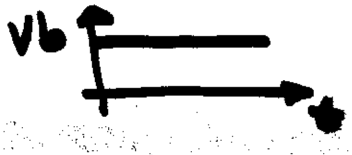
Hence: Natural response of an RLC circuit depends on its natural frequencies:

- sum of two exponents
- exponent + t · exponent
- decaying sinusoidal

What if $V_s(t)$ = Step function:

$$LC\frac{d^2v_c}{dt^2} + RC\cdot\frac{dv_c}{dt} + v_c = v_s(t)$$

$V_b \cdot u(t)$

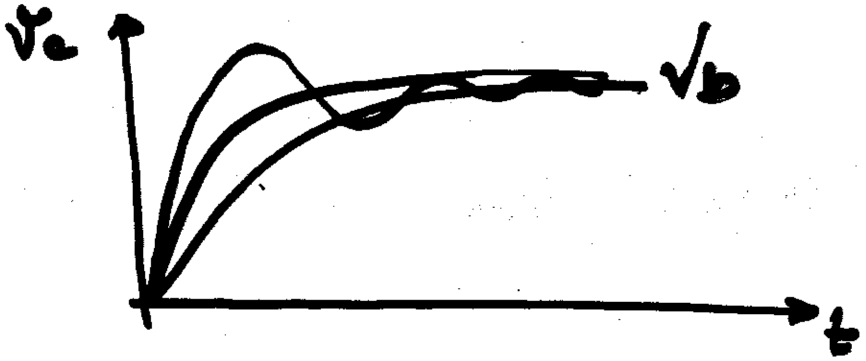


$v_{c\,particular} = K$

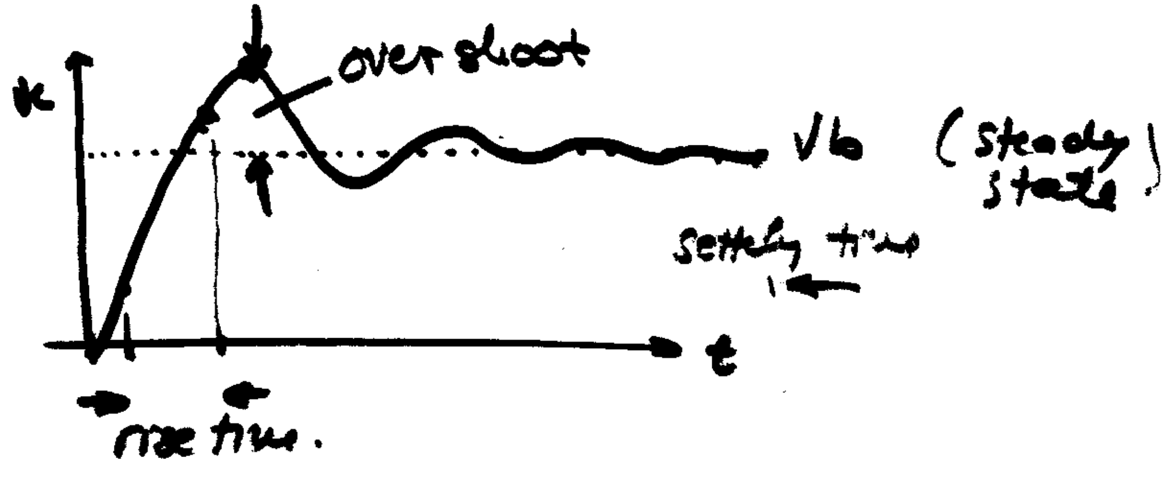
$K = V_b$

$v_c(t) = v_h(t) + V_b$

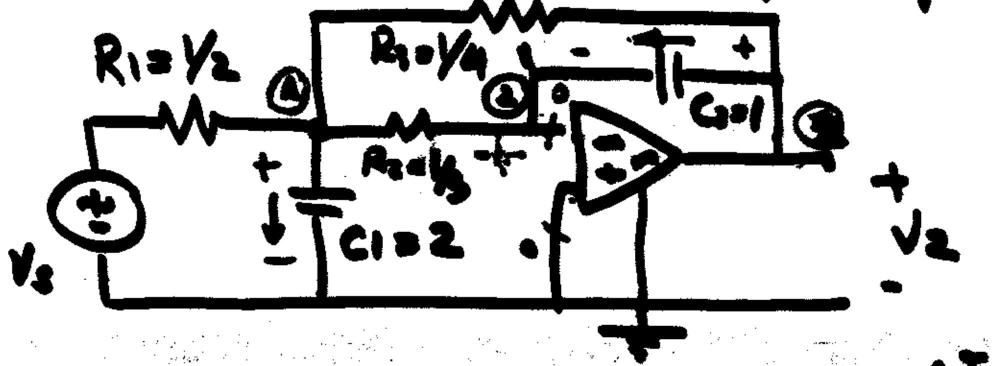
Voltage across the capacitor.



Lab #4:



RC circuit with an op-amp.



Note $\langle R_2=\frac{1}{3}\rangle$

$v_2' = v_-$

$v_{c1}=v_1$

$v_{c2}=v_2-\underset{\displaystyle 0}{\underset{\downarrow}{v_-}}$

- Ideal op-amp:

Two capacitors : 2^{nd} order DE

$$① \quad (v_1-v_s)\cdot\frac{1}{R_1} + (v_1-v_-)\cdot\frac{1}{R_2} + i_{c1} + (v_1-v_2)\cdot\frac{1}{R_3} = 0$$

KCL ①

$$②' \quad (v_- - v_1)\cdot\frac{1}{R_2} - i_{c2} = 0 \qquad v_- = 0$$

Also: $i_{c1}=C_1\cdot\frac{dv_{c1}}{dt}$ or $i_{c1}=C_1\cdot\frac{dv_1}{dt}$

$i_{c1}=2\cdot\frac{dv_1}{dt}$

$i_{c2}=C_2\frac{dv_{c2}}{dt}$ or $i_{c2}=\frac{dv_2}{dt}$ $\qquad v_{c2}=v_2 \equiv v$

Hence:

$$2(v_1-v_s) + 3v_1 + 2\frac{dv_1}{dt} + 4(v_1-v_2) = 0$$

$$-3v_1 - \frac{dv_2}{dt} = 0$$

And:

$$\left.\begin{aligned} 2\frac{dv_1}{dt} + 9v_1 - 4v_2 &= 2v_s \\ \frac{dv_2}{dt} + 3v_1 &= 0 \leftarrow \end{aligned}\right\}$$

Two approach.

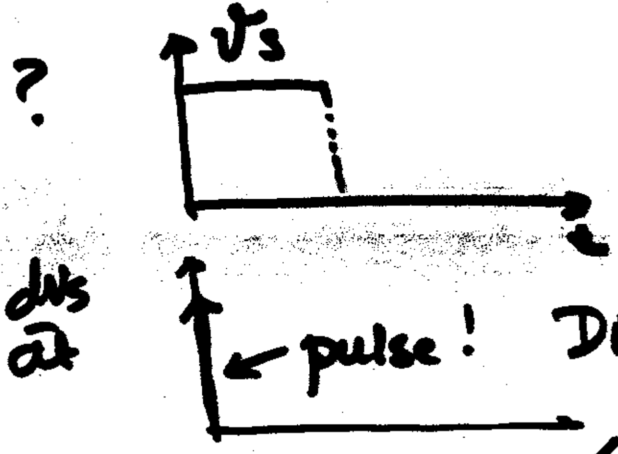
1) Substitution: Final DE equation is of 2^{nd} order

2) Matrix form:

Method 1: Finding 2^{nd} order DE

*) Avoid taking derivative of $v_s(t)$

Why?



Dirac's function

<difficult to deal with if you do not know how>

let us find derivative of the 2^{nd} equation:

$$\frac{d^2v_2}{dt^2} + 3\cdot\frac{dv_1}{dt} = 0$$

From the 1^{st} equation: $\frac{dv_1}{dt} = -4.5\cdot v_1 + 2v_2 + v_s$

$$\downarrow$$

$$v_1 = -\frac{1}{3}\frac{dv_2}{dt}$$

Hence

$$\frac{d^2V_2}{dt^2} + 3\cdot\left\{\underset{\downarrow\; -\frac{1}{3}\cdot\frac{dV_2}{dt}}{-4.5\,V_1} + 2V_2 + V_3\right\} = 0$$

Hence:

$$\frac{d^2V_2}{dt^2} + 3\cdot\left\{1.5\cdot\frac{dV_2}{dt} + 2V_2 + V_3\right\} = 0$$

or:

$$\frac{d^2V_2}{dt^2} + 4.5\cdot\frac{dV_2}{dt} + 6V_2 = -3V_3$$

DE: 2nd order

Characteristic equation:

$$s^2 + 4.5\,s + 6 = 0$$

Natural frequencies:

$$s_{1/2} = -2.25 \pm \sqrt{2.25^2 - 6}$$

or

$$s_{1/2} = -2.25 \pm j\,0.97$$

! Complex conjugate !

<u>RC</u> network

Why should you be surprised that natural frequencies of an RC network are complex numbers?

RC_1 : $s = -1/RC$

RC_1C_2 : $s_{1/2}$: real and negative

RC_1C_2 and op-amp : $s_{1/2}$: complex!

because of the op-amp!

Not a passive element

It could even make s_1 s_2 to have positive Real part.

Nevertheless: if complex natural frequencies they will always appear as complex conjugate pairs.

2) Matrix form

< State equations: $v_1 = v_{c1}$

$v_2 = v_{c2}$

$\uparrow$

State variables

$$\left.\begin{array}{l} 2\frac{dv_1}{dt} + 9v_1 - 4v_2 = 2v_s \\ \frac{dv_2}{dt} + 3v_1 = 0 \end{array}\right\}$$

or: $$\frac{dv_1}{dt} = -4.5v_1 + 2v_2 + v_s$$

$$\frac{dv_2}{dt} = -3v_1$$

Matrix form:

$$\frac{d}{dt}\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -4.5 & 2 \\ -3 & 0 \end{bmatrix}}_{A} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} v_s \\ 0 \end{bmatrix}}_{b}$$

$$\frac{dx}{dt} = Ax + b$$

$$A = \begin{bmatrix} -4.5 & 2 \\ -3 & 0 \end{bmatrix}$$

How can we find natural frequencies:

Assume: $x_1 = K_1 e^{st}$ ← v_1

$x_2 = K_2 e^{st}$ ← v_2

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\underbrace{(s \cdot I - A)}_{} \underset{\nwarrow \text{ vectors}}{K} = 0,$$

Has nontrivial solutions only if:

$$\det(\underset{\overset{\uparrow}{2\times2}}{sI} - \underset{\overset{\nwarrow}{2\times2}}{A}) = 0$$

↖ Another notation:

$$sU - A = 0$$

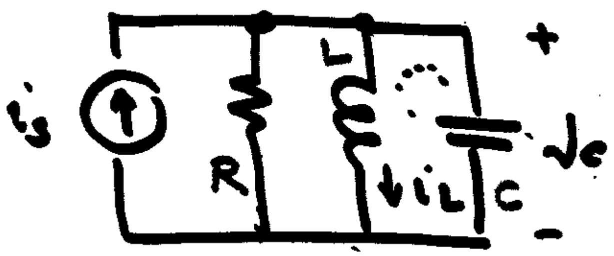
$$\det\left\{ s \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -4.5 & 2 \\ -3 & 0 \end{pmatrix} \right\} = 0$$

$$\det \begin{pmatrix} s+4.5 & -2 \\ 3 & s \end{pmatrix} = 0$$

or: $(s+4.5) \cdot s + 6 = 0$

$s^2 + 4.5s + 6 = 0$ ← identical to the characteristic eq. found by method 1).

RLC parallel circuit



① $i_S = i_R + i_L + i_C$: KCL

Ohm → $i_R = v_c/R$

$\left.\begin{array}{l} i_R = v_c/R \\ i_C = C\dfrac{dv_c}{dt} \end{array}\right\}$

$v_L = v_c$; $v_L = L\dfrac{di_L}{dt}$ ←

② $L\dfrac{di_L}{dt} - v_c = 0$

Hence:

$$\left.\begin{array}{l} C\cdot\dfrac{dv_c}{dt} + \dfrac{1}{R}\cdot v_c + i_L = i_S \\ L\dfrac{di_L}{dt} - v_c = 0 \end{array}\right\}$$

2nd order
or
2 : 1st order
DE

State equations form:

$$\begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix} \cdot \frac{d}{dt} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 1/R & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_c \\ i_L \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} i_s$$

Let $x = \begin{bmatrix} v_c \\ i_L \end{bmatrix}$ (vector: 2×1)

$$\begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix} \cdot \frac{dx}{dt} + \left(\begin{matrix} +1/R & 1 \\ -1 & 0 \end{matrix} \right] \cdot x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot i_s$$

Or:

$$\frac{dx}{dt} = - \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1/R & 1 \\ -1 & 0 \end{bmatrix} \cdot x$$

$$+ \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} i_s$$

$$\frac{dx}{dt} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \cdot x \underset{\uparrow \; \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}^{-1}}{+} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot i_s$$

Form: $\frac{dx}{dt} = Ax + b$

Characteristic equation:

Homogeneous DE:

$$\frac{dx}{dt} = Ax$$

Solution $\underset{\uparrow}{x} = \underset{\uparrow}{K}\cdot e^{st}$ (Vectors!)

$$s\cdot K\cdot e^{s} = AKe^{s}$$

$$(s\cdot \underset{\uparrow\text{Vector}}{I} - A)\cdot Ke^{s} = 0$$

Nontrivial solutions only if:

(sI - A) is a singular matrix

or: $\det(sI - A) = 0$

$$\det\left\{ s\cdot\underset{\uparrow I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} - \begin{pmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ 1/L & 0 \end{pmatrix}\right\} = 0$$

$$\det\left\{\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1/RC & -\frac{1}{C} \\ 1/L & 0 \end{pmatrix}\right\} = 0$$

$$\det\begin{pmatrix} s + 1/RC & 1/C \\ -1/L & s \end{pmatrix} = 0$$

$$s\cdot\left(s + 1/RC\right) + \frac{1}{LC} = 0$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Compare with the characteristic of RLC in series:
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Duality

$R \leftarrow G$

$L \leftarrow C$

$C \leftarrow L$

Same equation

Natural frequencies:

$$s_{1/2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

3 cases: $s_{1/2}$: Real dist.

: -"- ident

: complex

Natural response:

Finding coefficients k_1 and k_2

$$v_{c_{\text{homog.}}}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$i_{L_{\text{homog}}}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

How to find $\begin{matrix} K_1, K_2 \\ A_1, A_2 \end{matrix}$?

From initial conditions:

$$\left.\begin{matrix} v_c(0-) = v_c(0+) \\ i_L(0-) = i_L(0+) \end{matrix}\right. \quad \text{conservation laws!}$$

Recall:

$$\left.\begin{matrix} C \cdot \frac{dv_c}{dt} + \frac{1}{R} \cdot v_c + i_L = 0 \leftarrow \text{Homogeneous case} \\ L \frac{di_L}{dt} - v_c = 0 \end{matrix}\right\} \text{Our DE.}$$

Let us find K_1 and K_2 :

$v_{C_a}(0+) = K_1 + K_2 \Rightarrow K_1 + K_2 = V_b$

↑ assume C was charged with V_b.

What if $v_C(0-) = 0$

$\Rightarrow K_1 + K_2 = 0$

If, in addition, $i_L(0-) = 0$

TRIVIAL solutions ($\underline{\text{all } 0}$) only!

$$\frac{dv_c}{dt} = K_1 \cdot s_1 e^{s_1 t} + K_2 s_2 e^{s_2 t}$$

$$\left(\frac{dv_c}{dt}\right)_{0+} = K_1 s_1 + K_2 s_2$$

↑ How do we find $\frac{dv_c}{dt}$ at $t = 0+$!

From DE equations and by knowing $v_C(0-)$ and $i_L(0-)$!

From DE:

$$C\cdot\frac{dv_c}{dt} + \frac{1}{R}\cdot v_c + i_L = 0 \qquad 1)$$

$$L\frac{di_L}{dt} - v_c = 0 \qquad 2)$$

From 1)

$$C\cdot\frac{dv_c}{dt} = -\frac{1}{R}\cdot v_c - i_L$$

At $t=0+$:

$$C\cdot\left.\frac{dv_c}{dt}\right|_{0+} = -\frac{1}{R}\underset{v_C(0-)}{\underset{\downarrow}{v_c(0+)}} - \underset{i_L(0-)}{\underset{\downarrow}{i_L(0+)}}$$

Hence:

$$\left(\frac{dv_c}{dt}\right)_{0+} = -\frac{1}{RC}\cdot V_b - \frac{I_b}{C}$$

assume a value or $\underline{0}$

Hence:

$$K_1 + K_2 = v_c(0-)$$

$$\underline{K_1S_1} + K_2S_2 = -\frac{1}{RC}\cdot v_c(0-) - \frac{i_L}{C}(0-)$$

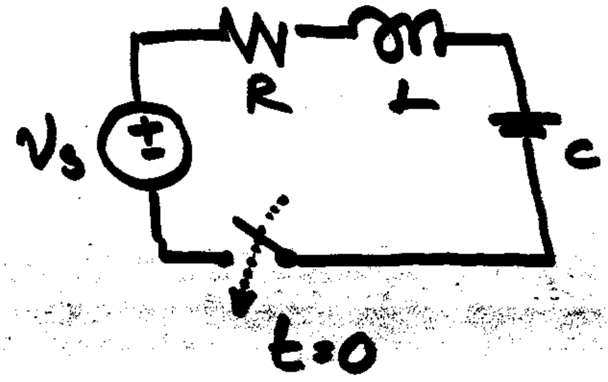
2 equation with K_1 and K_2 unknown

Solve it for K_1 and K_2

<Similar procedure for findin A_1 and A_2 : i_L>

Complete response:

RLC ⟨series⟩:



S: closes at t=0
It was open for
a long time
⟨<u>equilibrium</u>⟩

DE:
$$\left.\begin{aligned} \frac{di_L}{dt} &= -\frac{R}{L}\cdot i_L - \frac{1}{2}\cdot v_c + \frac{1}{L}\cdot v_s \\ \frac{dv_c}{dt} &= \frac{1}{C}\cdot i_L \end{aligned}\right\rangle$$

State equation: $\frac{dx}{dt} = Ax + b$

include $v_s(t)$

Often written as:

$\to \grave{b}\cdot v_s(t)$

to emphasize time function of the source!

Procedure:

1. Find solution to the homogeneous DE
   two natural frequencies : 2nd ord. circuit

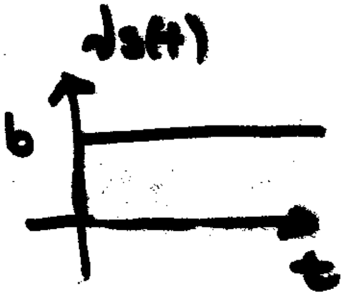
2. Find the particular solution

3. Add them ⇒ complete response

4. Find initial conditions and calculate constants.

RLC circuit :

<series connection>



Step response: $v_s(t) = V_b \cdot u(t)$

$$\text{DE: } \left.\begin{aligned} \frac{di_L}{dt} &= -\frac{R}{L} \cdot i_L - \frac{1}{L} \cdot v_c + \frac{1}{L} \cdot V_b \cdot u(t) \\ \frac{dv_c}{dt} &= \frac{1}{C} \cdot i_L \end{aligned}\right\}$$

①

$$v_c(t) = \underbrace{K_1 e^{s_1 t} + K_2 e^{s_2 t}}_{v_{c_h}} + \underbrace{V_b}_{v_{c_p}}$$

Why? ① 2nd order DE
② Set of 1st order DE's

S ②

① $$LC\underset{\nwarrow 0}{\frac{d^2v_c}{dt^2}} + RC\cdot\underset{\nearrow 0}{\frac{dv_c}{dt}} + v_c = \underset{\nwarrow V_b u(t)}{V_s}$$

In equillibrium
⟨steady-state⟩

Hence $\underbrace{v_c = V_b}_{v_{c_p}}$

or

② $$\overset{\swarrow 0}{\frac{di_L}{dt}} = -\frac{R}{L}\cdot \overset{0}{i_L} - \underbrace{\frac{1}{L}\cdot v_c + \frac{1}{L}V_b u(t)}_{\substack{v_c = V_b \\ \langle v_{c_p} \rangle}}$$

particular solution.

$$\underset{\substack{\downarrow \\ 0}}{\frac{dv_c}{dt}} = \frac{1}{C}\cdot i_L = 0$$

Look at the circuit !

s ③

$$v_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + V_b \qquad ①$$

s ④ Find unspecified constants from the initial conditions:

$v_C(0+) = v_C(0-)$

$i_L(0+) = i_L(0-)$

Assume $v_C(0-) = 0$, $i_L(0-) = 0$ } Switch open for a long time.

<see text p. 319 for non-zero case>

① $K_1 + K_2 + V_b = 0$

② ? From DE: $\rightarrow \dfrac{dv_c}{dt} = \dfrac{1}{C} \cdot i_L$ 0

$$s_1 \cdot K_1 e^{s_1 t} + s_2 \cdot K_2 e^{s_2 t} \rightarrow \frac{dv_c}{dt} = \frac{1}{C} \cdot i_L$$

at $t=0+$: $s_1 K_1 + s_2 K_2 = \dfrac{1}{C} \cdot \underset{i_L(0-)}{i_L(0+)}$

Hence:

$K_1 + K_2 + V_b = 0$

$\rightarrow K_1 s_1 + K_2 s_2 = 0$ } Solve them:

Complete solution:

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + V_b \leftarrow$$

$$\rightarrow i_L(t) = C \cdot \frac{dv_C}{dt}$$

$$\downarrow$$

$$i_L(t) = C\left(K_1 s_1 \cdot e^{s_1 t} + K_2 s_2 e^{s_2 t}\right)$$

As $t \to \infty$ $\quad v_C(t) \to V_b$

$i_L(t) \to 0$ $\quad$ As expected in the circuit }

2nd Order circuits: example

Prob #28 p. 334 text

$$v_s(t) = 10\left[u(t) - u(t-10^{-4})\right]$$

Find $v_2(t)$?

$v_{c1} = v_1$ $\qquad i_{c1} = 10^{-6} \cdot \frac{dv_{c1}}{dt}$

$v_{c2} = v_1 - v_2$ $\qquad i_{c2} = 5\times10^{-6} \cdot \frac{dv_{c2}}{dt}$

Hence: $v_2 = v_{c1} - v_{c2}$ ←

If we write state equations, we can find

v_{c1}
v_{c2} => $v_2 = v_{c1} - v_{c2}$ (algebraic, no DE)

"quazi" nodal equations:

$$\begin{cases} \frac{v_{c1}-v_s}{50}+i_{c1}+i_{c2}=0 & \text{KCL } ① \\ -i_{c2}+0.02v_1+\frac{v_2}{500}=0 & \text{KCL } ② \end{cases}$$

$$\begin{cases} \frac{v_{c1}}{50}+C_1\cdot\frac{dv_{c1}}{dt}+C_2\cdot\frac{dv_{c2}}{dt}=\frac{v_s}{50} & ① \\ -C_2\cdot\frac{dv_{c2}}{dt}+0.02\cdot\underset{=v_1}{\underset{\downarrow}{v_{c1}}}+\frac{1}{500}\underbrace{(v_{c1}-v_{c2})}_{=v_2}=0 & ② \end{cases}$$

or:

$$10^{-6}\cdot\frac{dv_{c1}}{dt}+5\times10^{-6}\cdot\frac{dv_{c2}}{dt}+0.02\cdot v_{c1}=\frac{v_s}{50} \quad ①$$

$$-5\times10^{-6}\cdot\frac{dv_{c2}}{dt}+0.02v_{c1}+0.002(v_{c1}-v_{c2})=0 \quad ②$$

$$5\times10^{-6}\cdot\frac{dv_{c2}}{dt}-0.022v_{c1}+0.002v_{c2}=0 \quad ②$$

$$\searrow 5\times10^{-6}\frac{dv_{c2}}{dt}=0.022v_{c1}-0.002v_{c2} \quad ②$$

Substitute into ①

$$10^{-6}\cdot\frac{dV_{C1}}{dt} + 0.022V_{C1} - 0.002V_{C2} + 0.02V_{C1} = \frac{V_S}{10}$$

Hence:

$$\left.\begin{aligned} 10^{-6}\cdot\frac{dV_{C1}}{dt} + 0.042V_{C1} - 0.002V_{C2} &= \frac{V_S}{10} \\ 5\times10^{-6}\cdot\frac{dV_{C2}}{dt} - 0.022V_{C1} + 0.002V_{C2} &= 0 \end{aligned}\right\}$$

Let $x = \begin{pmatrix} V_{C1} \\ V_{C2} \end{pmatrix}$

Then

$$\begin{pmatrix} 10^{-6} & 0 \\ 0 & 5\times10^{-6} \end{pmatrix}\frac{dx}{dt} + \begin{pmatrix} 0.042 & -0.002 \\ -0.022 & 0.002 \end{pmatrix} x = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}$$

Let $x = \underset{\nwarrow \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}}{K} e^{s}$

$$\det\left\{10^{-6}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\cdot \overset{\downarrow}{s} + \begin{pmatrix} 0.042 & -0.002 \\ -0.022 & 0.002 \end{pmatrix}\right\} = 0$$

$$\det \left\{ \begin{matrix} 10^{-6}s+0.042 & -0.002 \\ -0.022 & 5\times10^{-6}s+0.002 \end{matrix} \right\} = 0$$

$$(10^{-6}s+0.042)\,(5\times10^{-6}s+0.002)-0.002\times0.022=0$$

$$\downarrow\ s_{1/2}$$

$$5\times10^{-12}s^2+0.212\times10^{-6}s+40\times10^{-6}=0$$

$$5s^2+0.212\times10^{6}s+40\times10^{6}=0$$

$$s_{1/2}=-\frac{0.212}{10}\times10^{6}\pm\sqrt{(0.212\times10^{6})^2-8\cdot10^{6}}$$

$$s_1 \doteq -42210\ ;\ s_2=-190$$

then: $v_2=v_{c1}-v_{c2} \swarrow$

$$\longrightarrow v_2 = K_1e^{s_1t}+K_2e^{s_2t}+v_{2\underline{p}}$$ Complete solution

Note that:

$$v_2(0+)=v_{c1}(0+)-v_{c2}(0+)$$

$$v_2(0+)=v_{c1}(0-)-v_{c2}(0-)$$

$$v_2(0+)=0$$

Hence: $v_2(0+)=0$ (but only because $v_{c1}(0+)=0$, $v_{c2}(0+)=0$)

We also need

$$\left.\frac{dv_2}{dt}\right|_{0+} = ? \qquad \text{From: } v_2 = v_{c1} - v_{c2}$$

$$\left.\frac{dv_2}{dt}\right|_{0+} = \left.\frac{dv_{c1}}{dt}\right|_{0+} - \left.\frac{dv_{c2}}{dt}\right|_{0+}$$

From DE:

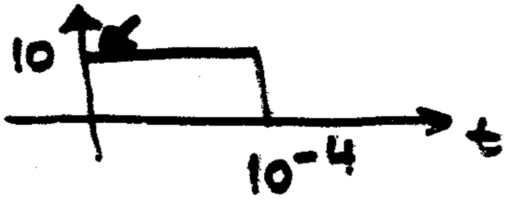
$$10^{-6}\frac{dv_{c1}}{dt} + 0.042 \cdot v_{c1} - 0.002 v_{c2} = 0.2$$

$$5\times10^{-6}\cdot\frac{dv_{c2}}{dt} - 0.022\, v_{c1} + 0.002\, v_{c2} = 0$$

At $t=0+$: $v_{c1}(0+) = v_{c2}(0+) = 0$

Recall: $v_s(t) = 10[u(t) - u(t-10^{-4})]$

$v_s(0+) = 10$



Hence: $$10^{-6} \cdot \frac{dv_{c1}}{dt}\bigg|_{0+} = 0.2$$

$$5\times10^{-6} \cdot \frac{dv_{c2}}{dt}\bigg|_{0+} = 0$$

And: $$\frac{dv_2}{dt} = \frac{dv_{c1}}{dt} - \frac{dv_{c2}}{dt} \quad \forall t$$ (for any t)

At t=0+: $$\frac{dv_2}{dt}\bigg|_{0+} = 0.2\times10^{6}$$

Again: $$v_2 = K_1 e^{s_1 t} + K_2 e^{s_2 t} + v_p$$

Recall: $$v_2(0+) = 0$$

$$\frac{dv_2}{dt}\bigg|_{0+} = .2\times10^{6}$$

And: $$K_1 + K_2 + v_p = 0$$

$$K_1 S_1 + K_2 S_2 + \frac{dv_p}{dt} = 0.2\times10^{6}$$

We now need to find the particular solution for v_2:

Since the source is constant (DC) and equal to 10V for $t>0$ (and $t<10^{-4}$) assume that

$$V_{C1\,part.} = \text{constant} \quad (A)$$

$$V_{C2\,part.} = \text{constant} \quad (B)$$

$$V_{2\,part} = \text{constant} \quad (C)$$

From DE:

$$10^{-6}\cdot\overset{\swarrow 0}{\frac{dV_{C1}}{dt}} + 0.042\cdot V_{C1} - 0.002\,V_{C2} = 0.2$$

$$5\times10^{-6}\cdot\underset{\overset{\uparrow}{0}}{\frac{dV_{C2}}{dt}} - 0.022\,V_{C1} + 0.002\,V_{C2} = 0$$

Particular solutions satisfy:

$$0.042\cdot V_{C1p} - 0.002\,V_{C2p} = 0.2$$

$$-0.022\,V_{C1p} + 0.002\,V_{C2p} = 0$$

Find V_{C1p} and V_{C2p}: then $V_{2p} = V_{C1p} - V_{C2p}$

$V_{c1p} = 10$

$V_{c2p} = 110$

Here $v_{2p} = -100$

We can now find constants K_1 and K_2.

$$K_1 + K_2 - 100 = 0$$

$$K_1 S_1 + K_2 S_2 = 0.2 \times 10^6$$

$K_1 = -5.211$; $K_2 = 105.211$

Here

$$v_2(t) = -5.211\, e^{-42210t} + 105.211\, e^{-190t} - 100$$

complete solution for

$0 < t < 10^{-4}$ sec.

at $t = 10^{-4}$ sec: $v_2 \big|_{at\ t = 10^{-4}\,sec} = \underline{3.154}$

Even though we were interested in finding v_2, it is advantageous to find v_{c1}

v_{c2}

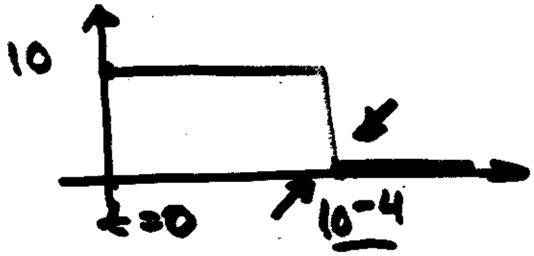
From the same DE, complete solution for two "state" voltages are:

$$v_{c1} = -4.714e^{-4220t} - 5.286e^{-190t} + 10$$

$$v_{c2} = 0.497e^{-4220t} - 110.497e^{-190t} + 110$$

They can be found from the same DE equations, using the same approach as for finding v_2.

At $t = 10^{-4}$, $v_s(t)$ changes



then: DE:

$$10^{-6} \cdot \frac{dv_{c1}}{dt} + 0.042 v_{c1} - 0.002 v_{c2} = \overset{v_s}{0}$$

$$5\times10^{-6} \cdot \frac{dv_{c2}}{dt} - 0.022 v_{c1} + 0.002 v_{c2} = 0$$

The system has not changed:

$s_{1/2}$ are the same

But now; the particular solution is zero. the system of DE's is homogeneous.

the easiest approach is to find

v_{c1}
v_{c2} then: $v_2 = v_{c1} - v_{c2}$

From:

$$v_{c1} = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$v_{c2} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

particular solution $\equiv 0$

DE are homogeneous

Note

$$v_{c1}(10^{-4}+) = v_{c1}(10^{-4}-)$$

$$v_{c2}(10^{-4}+) = v_{c2}(10^{-4}-)$$

We already have expressions for v_{c1}, v_{c2} for $0<t<10^{-4}$

Substitute 10^{-4} in these

$$v_{c1}(10^{-4}_-) = 4.344v$$

$$v_{c2}(10^{-4}_-) = 1.590$$

From DE:

$$\frac{dv_{c1}}{dt} = 10^{6} \times (-0.042 v_{c1} + 0.002 v_{c2})$$

$$\frac{dv_{c2}}{dt} = 10^{6} \times \frac{1}{5} \cdot (0.022 v_{c1} - 0.002 v_{c2})$$

Hence, we can find

$$\frac{dv_{C1}}{dt} \text{ and } \frac{dv_{C2}}{dt} \quad \text{at } t = 10^{-4}_{+}$$

$$\left(\text{Recall } v_{C1}(10^{-4}_{-}) = v_{C1}(10^{-4}_{+}) \right.$$
$$\left. v_{C2}(10^{-4}_{-}) = v_{C2}(10^{-4}_{+}) \right)$$

Then: $K_1 e^{-4220\cdot10^{-4}} + K_2 e^{-190\cdot10^{-4}} = 0$

$$K_1(-4220\, e^{-4220\cdot10^{-4}}) + K_2(-190\, e^{-190\cdot10^{-4}}) = \left.\frac{dv_{C1}}{dt}\right|_{t=10^{-4}+}$$

Repeat the same for finding constant

A_1 and A_2 in

$$v_2 = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

gives

$$v_{C1} = 4.645 e^{-4220t} + 0.099 e^{-190t}$$
$$v_{C2} = -0.489 e^{-4220t} + 2.079 e^{-190t}$$

and $v_2 = v_{C1} - v_{C2}$

$$v_2 = 5.134 e^{-4220t} - 1.98 e^{-190t}$$

Note that once we found

v_{c1}

v_{c2}

V_2 was found from an "algebraic" relationship:

$$v_2 = V_{c1} - V_{c2}$$

In addition, all other circuit variables (voltages and currents) can be found from simple KCL and KVL equations, or by finding derivatives

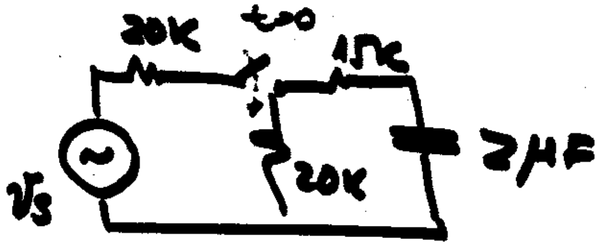
Example:

$$i_{c1} = c_1 \frac{dv_{c1}}{dt}$$

$$i_{c2} = c_2 \frac{dv_{c2}}{dt}$$

Complete response: examples

<p.265>



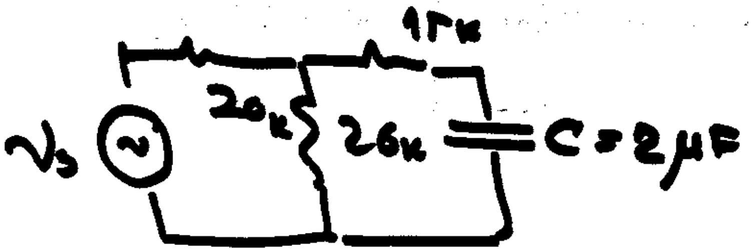
$v_C(0-)=0$ (No charge stored at $t=0$.)

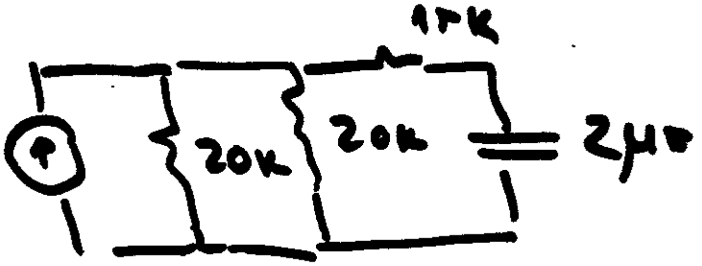
Hence: $v_C(0+)=0$

1st order circuit: One capacitor

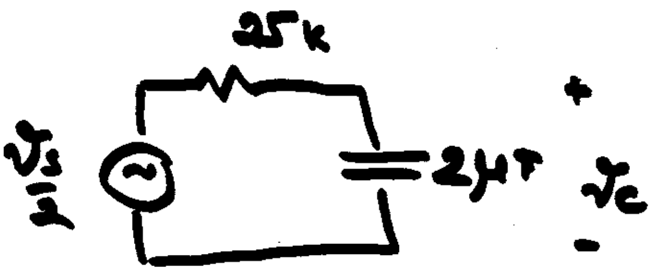
Switch closes at $t=0$,

Hence, for $t \geq 0$:





$i_s = \frac{v_s}{20}$



$$\frac{v_s}{2} - 25\times10^{3}\cdot i - V_c = 0$$

$$i = 2\cdot10^{-6}\cdot\frac{dV_c}{dt}$$

$$50\times10^{-3}\cdot\frac{dV_c}{dt} + V_c = \frac{v_s}{2}$$

$$\frac{dV_c}{dt} + 20\cdot V_c = \frac{1}{100\times10^{-3}}\cdot v_s$$

$$\frac{dV_c}{dt} + 20V_c = 10\cdot v_s$$

Let $v_s = V_m \sin(\omega t)$ Sinusoidal source

$$v_c(t) = V_{c_h}(t) + V_{cp}(t)$$

$$V_{c_h}(t) = K\cdot e^{st} \Rightarrow \frac{dV_c}{dt} + 20\cdot V_c = 0$$

$$K\cdot s + 20\cdot K = 0$$

$$s = -20$$

$$V_c(t) = Ke^{-20t} + V_{cp}(t)$$

$v_{cp}(t) = V_{cm} \cdot \sin(\omega t + \theta)$

$\frac{dv_c}{dt} + 20 v_c = 10 \cdot V_s$

$V_c \cdot \omega \cdot \cos(\omega t + \theta) + 20 \cdot V_c \cdot \sin(\omega t + \theta)$

$\downarrow$

$= 10 \cdot V_m \cdot \sin \omega t$

$\forall t$

For simpericity

let $V_c = V_{cm}$

$\uparrow$

cap.

$V_c \cdot \omega \cdot (\cos\omega t \cos\theta - \sin\omega t \cdot \sin\theta)$

$+ 20 \cdot V_c \cdot (\sin\omega t \cdot \cos\theta + \cos\omega t \cdot \sin\theta)$

$= 10 V_m \cdot \sin \omega t$

Hence:

$V_c \cdot \omega \cdot \cos\theta + 20 V_c \cdot \sin\theta = 0$ (1)

$-V_c \cdot \omega \cdot \sin\theta + 20 \cdot V_c \cdot \cos\theta = 10 V_m$ (2)

Find V_c and θ :

From (1): $\omega \cos\theta = -20 \sin\theta \Rightarrow \tan\theta = -\frac{\omega}{20}$

$$\theta = -\arctan\frac{\omega}{20}$$

From (2)

$$-V_c \cdot \omega \cdot \sin\theta + 20 V_c \cos\theta = 10 V_m$$

$$V_c = \frac{10 V_m}{20\cos\theta - \omega\sin\theta}$$

Recall: $\omega\cos\theta = -20\sin\theta$

$$V_c = \frac{10 V_m}{20 \cdot \frac{-20}{\omega}\sin\theta - \omega\sin\theta}$$

$$V_c = \frac{10 V_m}{\left(\frac{-400}{\omega} - \omega\right)\sin\theta}$$

$$V_c = \frac{-10 V_m \cdot}{\frac{400+\omega^2}{\omega} \cdot \sin\theta}$$

$\nwarrow$ $\tan\theta = -\frac{\omega}{20}$

$$\sin\theta = \frac{-\omega/20}{\sqrt{1+\frac{\omega^2}{400}}}$$

$$V_c = \frac{10 V_m}{\sqrt{400+\omega^2}}$$

$$\sin\theta = \frac{-\omega}{\sqrt{400+\omega^2}}$$

Hence:

$$v_c(t) = K e^{-20t} + \frac{10 V_m}{\sqrt{400+\omega^2}} \cdot \sin(\omega t - \arctan \omega/20)$$

$$v_c(t) = Ke^{-20t} + \frac{1}{2}\cdot\frac{20V_m}{\sqrt{400+\omega^2}}\cdot\sin\left(\omega t - \arctan\frac{\omega}{20}\right)$$

to be determined from $v_c(0+) = v_c(0-)$

ω from the source

If $v_c(0-) = 0$

$$K + \frac{1}{2}\cdot\frac{20V_m}{\sqrt{400+\omega^2}}\cdot\sin\left(-\arctan\frac{\omega}{20}\right) = 0$$

$$\tan\theta = -\frac{\omega}{20}$$

$$\theta = -\arctan\frac{\omega}{20}$$

$$\sin\theta = \frac{-\omega/20}{\sqrt{1+\frac{\omega^2}{400}}}$$

$$\sin\theta = \frac{-\omega}{\sqrt{400+\omega^2}}$$

$$K + \frac{10V_m}{\sqrt{400+\omega^2}}\cdot\frac{-\omega}{\sqrt{400+\omega^2}} = 0$$

$$K = \frac{10V_m\cdot\omega}{400+\omega^2}$$

From:

$$v_c(t) = Ke^{-20t} + \frac{1}{2}\cdot\frac{20V_m}{\sqrt{400+\omega^2}}\cdot\sin\left(\omega t - \arctan\frac{\omega}{20}\right)$$

Recall: $R = 25K$

$C = 2\mu F$

$RC = 50\times10^{-3}$ sec

$\frac{1}{RC} = 20$

$$v_c(t) = Ke^{-20t} + \frac{1}{2}\cdot\frac{20V_m}{\sqrt{20^2+\omega^2}}\sin\left(\omega t - \text{arctg}\,\frac{\omega}{20}\right)$$

$$v_c(t) = Ke^{-\frac{t}{RC}} + \frac{1}{2}\cdot\underbrace{\frac{1}{RC}\cdot\frac{V_m}{\sqrt{\left(\frac{1}{RC}\right)^2+\omega^2}}}_{\frac{1}{\omega C}\cdot\frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}}\sin(\omega t - \text{arctg}\,\omega RC$$

The sinusoidal steady state:

As $t \to \infty$

$$K e^{-t/RC} \to 0$$

Hence:

$$v_c(t) = v_p(t)$$

$$v_c(t) = \frac{1}{2} \cdot \frac{1}{\omega C} \cdot \frac{V_m}{\sqrt{R^2 + (1/\omega C)^2}} \cdot \sin(\omega t - \theta)$$

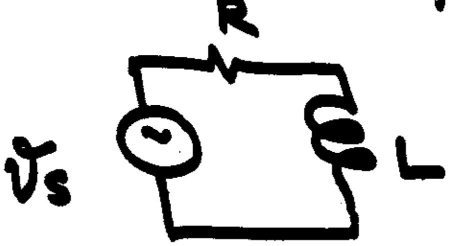
$$\theta = \arctan \omega RC$$

Periodic function:

frequency ω (the same as ω of the independent source)

angle θ: Lagging $\theta = \arctan \omega RC$

Another simple circuit:



$$L\frac{di}{dt}+Ri=v_s(t)$$

$V_s=V_m\cos\omega t$

$$L\frac{di}{dt}+Ri=V_m\cos\omega t$$

$$L\frac{di}{dt}+Ri=0 \Rightarrow i=Ke^{st}$$

$$L\cdot K\cdot s\,e^{st}+R\cdot Ke^{st}=0$$

$$Ls+R=0 \quad s=-\frac{R}{L}$$

$$i(t)=Ke^{-\frac{R}{L}t}+i_p(t)$$

$$i_p(t)=I_m\cos(\omega t+\varphi)$$

Method ①: substitute $i_p(t)$ into DE and find $\left.\begin{matrix}I_m\\ \varphi\end{matrix}\right\}$

2):

$$\frac{di_p}{dt} = -I_m \omega \sin(\omega t + \varphi)$$

$$-L \cdot i_m \cdot \omega \cdot \sin(\omega t + \varphi) + R \cdot i_m \cdot \cos(\omega t + \varphi)$$
$$= V_m \cos \omega t$$

<Same method as in the RC case>

$$-L \cdot i_m \cdot \omega \cdot (\sin \omega t \cos \varphi + \cos \omega t \sin \varphi)$$
$$+ R \cdot i_m \cdot (\cos \omega t \cos \varphi - \sin \omega t \sin \varphi) = V_m \cos \omega t$$

$$-L \cdot i_m \cdot \omega \cdot \cos \varphi - R \cdot i_m \sin \varphi = 0 \qquad (1)$$

$$-L \cdot i_m \cdot \omega \cdot \sin \varphi + R \cdot i_m \cdot \cos \varphi = V_m \qquad (2)$$

From (1): $-L\omega \cos \varphi - R \sin \varphi = 0$

$$\tan \varphi = -\frac{\omega L}{R} \qquad \varphi = \arctan \frac{-\omega L}{R}$$

$$\varphi = -\arctan \frac{\omega L}{R}$$

$$\varphi = -\tan^{-1} \omega L/R$$

$$I_m = \frac{V_m}{R \cos \varphi - \omega L \cdot \sin \varphi}$$

if $\tan\varphi = \frac{-\omega L}{R}$

then:

$$\sin\varphi = \frac{-\frac{\omega L}{R}}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \Rightarrow \sin\varphi = -\frac{\omega L}{\sqrt{R^2+\omega L^2}}$$

$$\cos\varphi = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \Rightarrow \cos\varphi = \frac{R}{\sqrt{R^2+(\omega L)^2}}$$

Hence:

$$I_m = \frac{V_m}{R\cdot\frac{R}{\sqrt{R^2+(\omega L)^2}} - \omega L\cdot\frac{-\omega L}{\sqrt{R^2+(\omega L)^2}}}$$

$$I_m = \frac{V_m}{\sqrt{R^2+(\omega L)^2}}$$

Hence:

$$i_p(t) = \frac{V_m}{\sqrt{R^2+(\omega L)^2}}\cdot\cos\left(\omega t - \arctan\frac{\omega L}{R}\right)$$

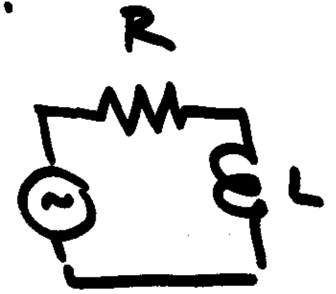
Lags behind $\cos\omega t$

Complete response.

$\langle$ periodic steady . state $\rangle$

$$i(t) = \underset{\rightarrow 0 \text{ as } t\rightarrow\infty}{Ke^{-R/Lt}} + \frac{V_m}{\sqrt{R^2+(\omega L)^2}}\cdot\cos\left(\omega t - \arctan\frac{\omega L}{R}\right)$$

Method ②:



$v_s = V_m \cos\omega t$

Find the steady-state solution
<or: the particular solution>

$$L\frac{di}{dt} + Ri = V_m \cos\omega t$$

Assume:

$$i(t) = I_m(\cos\omega t + \varphi)$$

Recall that we found by solving DE:

$$i_p(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cdot \cos\left(\omega t - \arctan\frac{\omega L}{R}\right)$$

<lengthy, but predictable, process>

⟨Phasors⟩ or. complex variables

Euler's identity:

$$e^{jx} = \cos x + j\sin x$$

$$e^{-jx} = \cos x - j\sin x$$

$$\cos x = Re\{e^{jx}\}$$

$$\sin x = Im\{e^{jx}\}$$

Complex:

$$z = \underset{\underset{Re\{z\}}{\uparrow}}{a} + j\underset{\underset{Im\{z\}}{\nwarrow}}{b}$$

Our assumed solution:

$$i_p(t) = I_{pm}\cos(\omega t + \varphi) \leftarrow$$

$$i_p(t) = I_{pm}\cdot Re\{e^{j(\omega t+\varphi)}\}$$

$$= Re\{I_{pm}\, e^{j(\omega t+\varphi)}\}$$

$i_p(t)$ has to satisfy DE:

$$L\frac{di}{dt} + Ri = V_m \cos\omega t$$

Steady-state:

$$i(t) = Re\left\{I_m e^{j(\omega t+\varphi)}\right\}$$

Hence:

$$\frac{di(t)}{dt} = \frac{d}{dt}\left\{Re\left\{I_m e^{j(\omega t+\varphi)}\right\}\right\}$$

exchange:

$$\frac{di(t)}{dt} = Re\left\{\frac{d}{dt}\left(I_m e^{j(\omega t+\varphi)}\right)\right\}$$

$$= Re\left\{I_m j\omega e^{j(\omega t+\varphi)}\right\}$$

DE:

$$L\cdot Re\left\{I_m j\omega e^{j(\omega t+\varphi)}\right\} + R\cdot Re\left\{I_m e^{j(\omega t+\varphi)}\right\}$$

$$= Re\left\{V_m e^{j\omega t}\right\}$$

or: $$Re\left\{j\omega L\cdot I_m e^{j(\omega t+\varphi)} + R\cdot I_m e^{j(\omega t+\varphi)}\right\}$$

$$= Re\left\{V_m e^{j\omega t}\right\}$$

$$Re\left\{(j\omega L \cdot I_m e^{j\varphi} + R \cdot I_m e^{j\varphi})e^{j\omega t}\right\}$$

$$= Re\left\{V_m e^{j\omega t}\right\}$$

or:

$$Re\left\{\left[(j\omega L + R) \cdot I_m e^{j\varphi} - V_m\right]e^{j\omega t}\right\} = 0$$

$\forall t$ (For any t)

Only if:

$$(j\omega L + R) I_m e^{j\varphi} - V_m = 0$$

or

$$I_m e^{j\varphi} = \frac{V_m}{\underset{\text{Resistor}}{R} + \underset{\text{inductor}}{j\omega L}}$$

$$\nearrow I_m = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \qquad \nearrow \varphi = -\arctan\frac{\omega L}{R}$$

And

$$i(t) = \underset{\checkmark}{I_m}\cos(\omega t + \underset{\checkmark}{\varphi})$$

$\frac{I_m}{\sqrt{2}} e^{j\varphi}$ $\nwarrow$ phasor

Magnitude of the pasor: $\frac{\text{Max of the } \underline{\text{sin}}}{\sqrt{2}}$

< Recall ≡ root mean square value! >

angle of the pasor : angle of the $\underline{\text{sin}}$.

Example: $x(t) = \sqrt{2}\cdot X_{rms}\cos(\omega t + \alpha)$

$$\Downarrow$$

$$\underline{X} = X_{rms}\, e^{j\alpha} \leftarrow \text{phasor}$$

$v(t) = 20\, \underline{\sin}(\omega t + \pi/3)$

$= 20 \cdot \cos(\omega t + \underbrace{\pi/3 - \pi/2})$

$V = \underset{\uparrow}{\frac{20}{\sqrt{2}}}\, e^{-j\pi/6}$

< Note: there would be nothing wrong in assuming

$\underline{V} = V_m e^{j\varphi}$

$\nwarrow$ max >

Phasors are a convenient way to calculate steady-state solutions for sinusoidal <periodic> inputs

Only valid for:

- Linear time-invariant systems

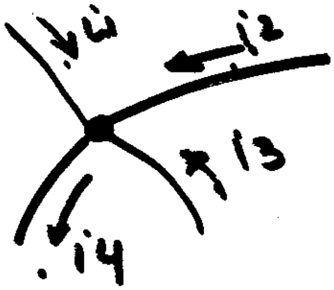
  R
  L : constants
  C

- sinusoidal sources

- periodic steady-state solutions <particular solutions of DEs>

KCL and KVL revisited

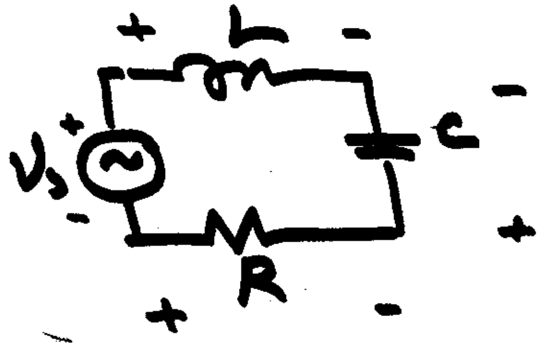
$\sum \underline{I} = 0$ algebraic sum of all phasor-currents leaving (or entering) a node = 0



$\underline{I}_1 + \underline{I}_2 + \underline{I}_3 - \underline{I}_4 = 0$ ←

phasor corresponds to $i_1(t)$

Same for KVL



$$\underline{V}_S - \underline{V}_L + \underline{V}_C + \underline{V}_R = 0$$

↑ phasor associated with $v_s(t) = V_{s_m}\cos\omega t$

Ohm's Law:

$$v(t) = R\cdot i(t)$$

$$Re\{\underline{V}e^{j\omega t}\} = R\cdot Re\{\underline{I}e^{j\omega t}\}$$

$$\underline{V} = R\cdot\underline{I}$$

Recall: $\underline{V} = V_{rms}\, e^{j\varphi}$

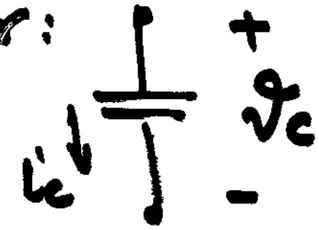
$$\longrightarrow v(t) = Re\{\sqrt{2}\cdot\underline{V}\, e^{j\omega t}\}$$

$$\downarrow$$

$$\sqrt{2}\cdot V_{rms}\cdot \underbrace{e^{j\varphi}\cdot e^{j\omega t}}_{\cos(\omega t+\varphi)+j\sin(\omega t+\varphi)}$$

$$v(t) = \sqrt{2}\cdot V_{rms}\cos(\omega t+\varphi)$$

Capacitor:



$\rightarrow i_c = C \frac{dv_c}{dt}$

$i_c = Re\{\sqrt{2}\cdot \underline{I_c}\cdot e^{j\omega t}\}$ ← Function of time

$i_c(t) = \sqrt{2}\cdot I_c \cos(\omega t + \alpha)$

$\underline{I_c} = I_c \cdot e^{j\alpha}$

$Re\{\sqrt{2}\cdot \underline{I_c}\, e^{j\omega t}\} = C\cdot \frac{d}{dt}\{Re\{\sqrt{2}\cdot \underline{V_c}\, e^{j\omega t}\}\}$

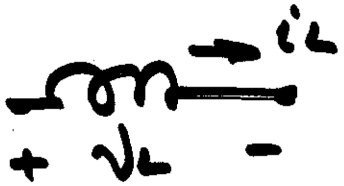
exchange

$\sqrt{2}$ gets canceled:

$Re\{\underline{I_c}\, e^{j\omega t} - C\cdot \underline{V_c}\cdot j\omega\, e^{j\omega t}\} = 0$

$\underbrace{\underline{I_c} = j\omega C\cdot \underline{V_c}}$; $\underline{V_c} = \frac{\underline{I_c}}{j\omega C}$

Inductor:



$$v_L = L\frac{di_L}{dt}$$

$$Re\{\underset{\checkmark}{\sqrt{2}}\cdot\underline{V_L}\cdot e^{j\omega t}\} = L\underbrace{\frac{d}{dt}Re}_{\equiv Re\frac{d}{dt}}\{\underset{\checkmark}{\sqrt{2}}\,\underline{I_L}\,e^{j\omega t}\}$$

Hence:

$$Re\{\underline{V_L}\,e^{j\omega t} - L\cdot j\omega\cdot\underline{I_L}\,e^{j\omega t}\} = 0$$

$$\underline{V_L} = j\omega L\cdot\underline{I_L}$$

General: $\underline{V} = Z\cdot\underline{I}$

← impedance
complex number

$\underline{Z} = R + jX$ ← reactance

↑ resistance

$\underline{Y} = \frac{1}{Z}$
← admittance

$\underline{Y} = G + jB$ ← susceptance

↑ conductance

Resistr: $V = R \cdot I$

Capacitr: $V = \frac{1}{j\omega C} \cdot I$

impedance

$Z = -j \cdot \frac{1}{\omega C}$

reactance: $-\frac{1}{\omega C}$

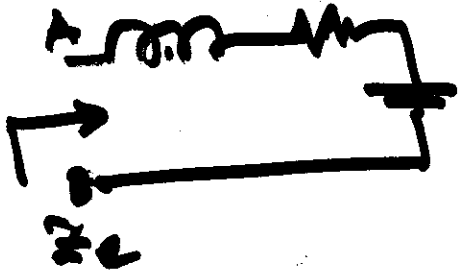
inductr: $V = j\omega L \cdot I$

impedance

$Z = j\omega L$

reactance: ωL

Phasor calculus:



$Z_e = j\overset{\vee}{\omega}L + R + \frac{1}{j\omega C}$

$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$

Resonance: $W_0 = \frac{1}{\sqrt{LC}}$

$\underline{V} = \underline{Z_e}\,\underline{I}$

$Z_{at\ \omega_0} = R$!

Time – Phasors – Time transform

$$\begin{tikzcd} \text{Time} \arrow[d] & \text{Phasors} \arrow[d] \\ x(t) \arrow[r] & \underline{X} \\ & \underline{X} \cdot \underline{Z} \\ & \underline{I} \arrow[r] & y(t) \end{tikzcd}$$

$$\text{Incorrect}: \underset{\text{periodic}: \omega}{\underset{\uparrow}{y(t)}} = \underset{\text{periodic}: \omega}{\underset{\uparrow}{x(t)}} \cdot \underset{\omega!}{\underset{!!}{\underset{\nwarrow}{z(t)}}})$$

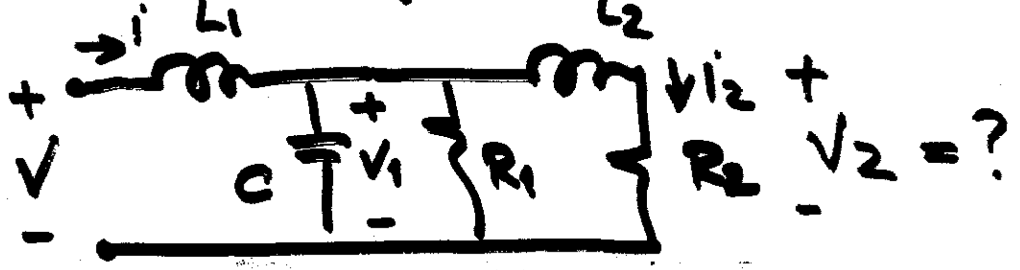
product of two periodic function with radian freq $\omega \Rightarrow$ p.f. : 2ω

Recall:

$$\cos x \cdot \cos y \to \cos(\overset{2\omega}{\overset{\nearrow}{x+y}}) + \cos(x-y)$$

$x = y$:

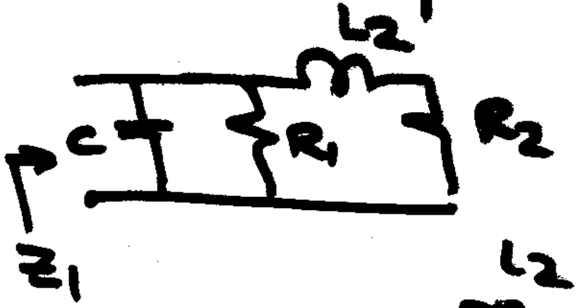
All the rules we learned when dealing with Resistive circuits, now apply to phasor quantities:

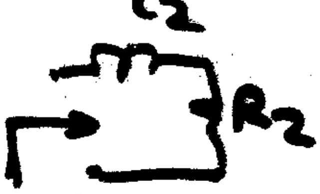


Assumption: v ← sinusoidal input

We are only interested in the steady-state solution:

$Z = j\omega L_1 + Z_1$



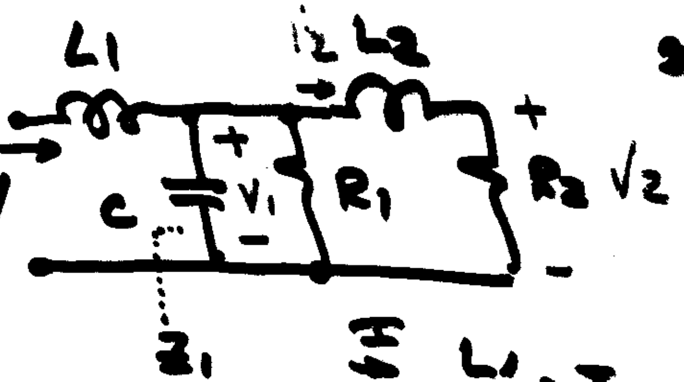


$Y_1 = \frac{1}{Z_1}$

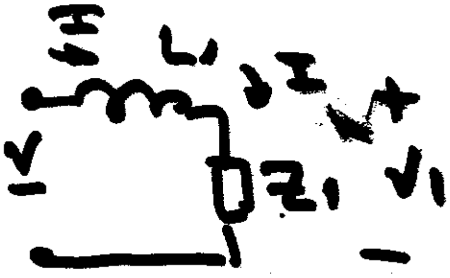
$Y_1 = j\omega C + G_1 + \frac{1}{R_2 + j\omega L_2}$

$$Z = j\omega L_1 + \frac{1}{j\omega C + \frac{1}{R_1} + \frac{1}{R_2 + j\omega L_2}}$$

$$I = \frac{V}{Z}$$

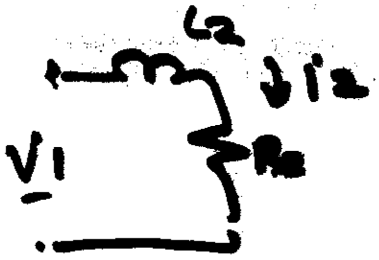


$$\nearrow V_1 = \underset{\checkmark}{I} \cdot Z_1 \checkmark$$



$$\rightarrow i_2 = \frac{V_1}{R_2 + j\omega L_2}$$

$$V_2 = R_2 i_2$$



then:

$$V_2 = R_2 \cdot \frac{V_1}{R_2 + j\omega L_2}$$

$$V_2 = \frac{R_2}{R_2 + j\omega L_2} \cdot Z_1 \cdot I$$

$$V_2 = \frac{R_2}{R_2 + j\omega L_2} \cdot Z_1 \cdot \frac{V}{j\omega L_1 + Z_1}$$

$$\frac{V_2}{V} = \frac{R_2}{R_2 + j\omega L_2} \cdot \frac{1}{j\omega L_1 Y_1 + 1}$$

$$\frac{V_2}{V} = \frac{R_2}{R_2 + j\omega L_2} \cdot \frac{1}{j\omega L_1 \cdot \left(j\omega C + G_1 + \frac{1}{R_2 + j\omega L_2}\right) + 1}$$

$$\frac{V_2}{V} = \frac{R_2}{j\omega L_1 \left[(j\omega C + G_1)(R_2 + j\omega L_2) \right] + R_2 + j\omega L_2}$$

in terms of phasors, we can also write
nodal / mesh eq. or find Thevenin's / Norton's equival.

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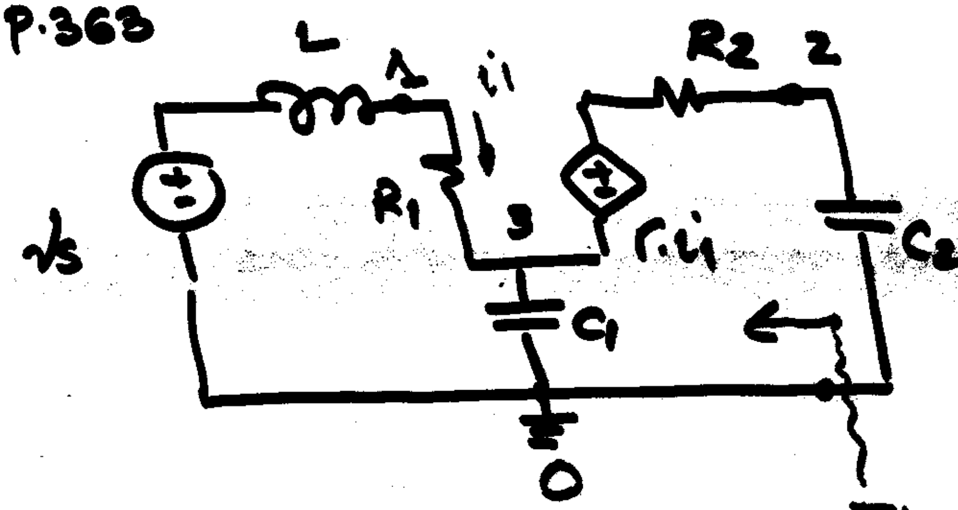
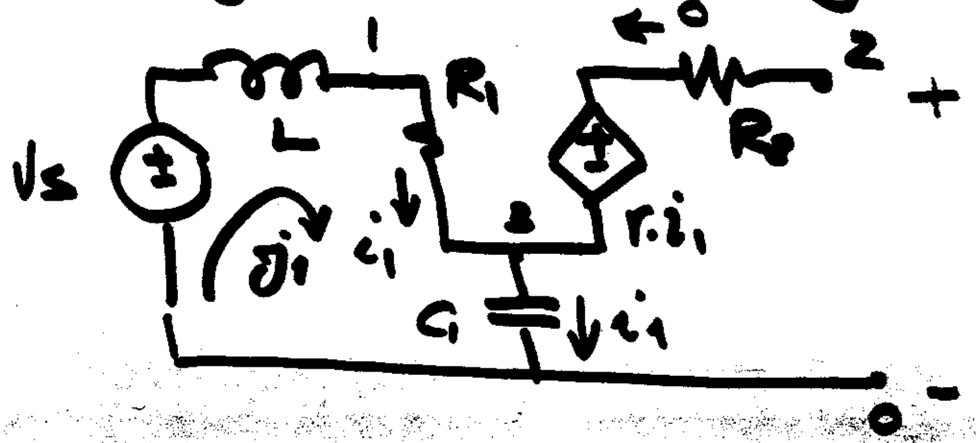


Find Thevenin's equivalent between 2 and 0:

1. Thevenin's voltage (phasor): V_{Th}
2. impedance: Z_{th}

Find the circuits from which you can determine $\underline{V_{Th}}$ $\underline{Z_{Th}}$

1) Finding Thevenin's voltage.



2-0 : open circuited: $\underline{V_{Th}}$ (phasor)

$$\underline{V_{Th}} = r\cdot\underline{I_1} + \frac{1}{j\omega C_1}\cdot\underline{I_1}$$

$$\underline{V_{Th}} = \left(r + \frac{1}{j\omega C_1}\right)\cdot\underline{I_1}$$

Mesh: $$j\omega L\cdot\underline{J_1} + R_1\cdot\underline{J_1} + \frac{\Delta}{j\omega C_1}\cdot\underline{J_1} = \underline{V_s}$$

$$\underline{J_1} = \frac{\underline{V_s}}{R_1 + j\omega L + \frac{1}{j\omega C_1}}$$

Note: $\underline{I_1} = \underline{J_1}$

$$\underline{I_1} = \frac{\underline{V_s}.}{R_1 + j\omega L + \frac{1}{j\omega C_1}}$$

Hence:

$$\underline{V}_{Th} = \left(r + \frac{1}{j\omega C_1}\right)\cdot \frac{\underline{V}_S}{R_1 + j\omega L + \frac{1}{j\omega C_1}}$$

$$\underline{V}_{Th} = \frac{1 + j\omega r\cdot C_1}{1-\omega^2 LC_1 + j\omega R_1C_1}\cdot \underline{V}_S$$

We often assume $v_S(t) = V_S\cdot\cos(\omega t)$

↑ phase $= 0$

$\underline{V}_S = V_S$ ← real number

then:

$$\underline{V}_{Th} = \underbrace{\frac{1 + j\omega r\cdot C_1}{1-\omega^2 LC_1 + j\omega R_1C_1}}_{\text{complex \#}}\cdot V_S \leftarrow *$$

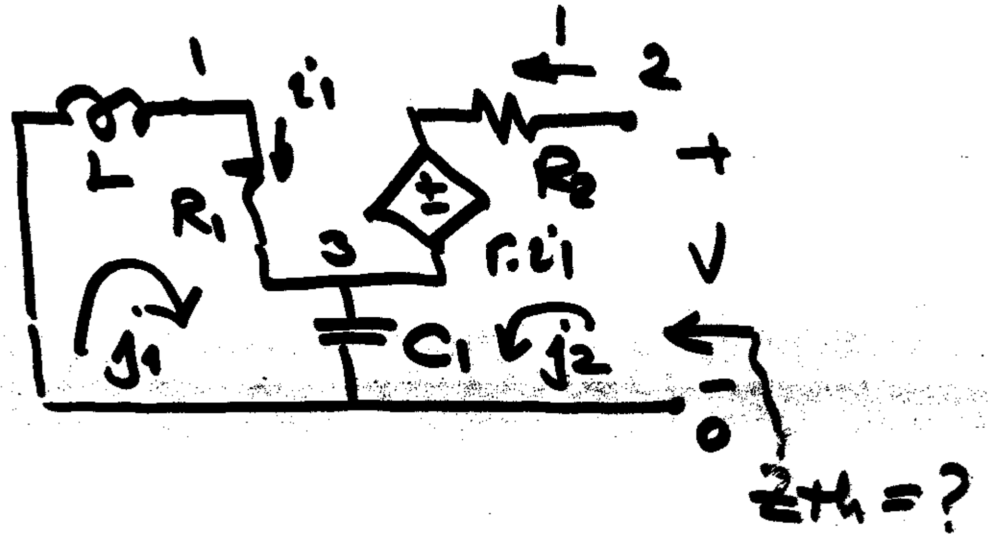
$$\Downarrow$$

$$v_{Th}(t) = \mathrm{Re}\left\{\underline{V}_{Th}\cdot\sqrt{2}\cdot e^{j\omega t}\right\}$$

↑ if V_S^* was rms.

2. What is $\underline{Z}_{th}$

Circuit:



Dependent source : hence we need to write equation to find V/i

imagine V to be a voltage source connect between 2 and 0:

Write mesh equations:

$$j\omega L \cdot J_1 + R_1 \cdot J_1 + \frac{1}{j\omega C_1} \cdot (J_1 + J_2) = 0$$

$$R_2 \cdot J_2 + r I_1 + \frac{1}{j\omega C_1} \cdot (J_2 + J_1) = \underline{V}$$

Note: $J_1 = J_1$

Hence: $$\left(R_1 + j\omega L + \frac{1}{j\omega C_1}\right) J_1 + \frac{1}{j\omega C_1} \cdot J_2 = 0$$

$$\left(r + \frac{1}{j\omega C_1}\right) \cdot J_1 + \left(R_2 + \frac{1}{j\omega C_1}\right) J_2 = \underline{V}$$

Find $\underline{J_2}$

then $\underline{I} = \underline{J_2}$

And $\frac{\underline{V}}{\underline{I}} = \underline{Z_{th}}$ ← complex #

From: $\underline{V_{Th}} \leftarrow v_{th}(t)$

$Z_{th} = R_{th} + j \cdot X_{th}$