Lecture 6 - Damped harmonic motion

Text: Fowles and Cassiday, Chap. 3

Simple harmonic motion is an idealization of most physical systems: in general there is some dissipative force present that robs the system of energy and reduces the amplitude of vibration. Here, we consider the damping effects of a drag force that is linear in velocity, which should be applicable at low speeds. We add $-c dx / dt$ to the force in $\mathbf{F} = m\mathbf{a}$ to obtain

$$m d^2x / dt^2 = -c(dx / dt) - kx$$

or

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Given that d^2x / dt^2 represents the successive operation $(d/dt)(d/dt)x$, we can regard the differential equation as

$$\left(\frac{d^2}{dt^2} + \frac{c}{m}\frac{d}{dt} + \frac{k}{m}\right)x = 0$$

This expression can be factored by introducing a constant q in the form

$$\left(\frac{d}{dt} + \frac{c}{2m} + q\right)\cdot\left(\frac{d}{dt} + \frac{c}{2m} - q\right)x(t) = 0$$

We immediately make the following replacement to simplify the notation,

$$\gamma \equiv \frac{c}{2m}$$

To regain the original expression, the product of the constant terms must satisfy

$$(\gamma + q) \bullet (\gamma - q) = k / m$$
$$-q^2 + \gamma^2 = k / m$$
$$\Rightarrow q = [\gamma^2 - k / m]^{1/2} \qquad \text{(take the positive root)}$$

The differential equation we must solve is thus

$$(d/dt + \gamma + q) \bullet (d/dt + \gamma - q)x = 0$$

Since each $(d/dt + \alpha)x = 0$ has the solution $A\exp(-\alpha t)$, then

$$x(t) = A_1\exp(-[\gamma + q]t) + A_2\exp(-[\gamma - q]t)$$

is an appropriate trial solution. The proof is by explicit substitution:

$$(d/dt + \gamma + q) \bullet (d/dt + \gamma - q) \bullet \{A_1\exp(-[\gamma + q]t) + A_2\exp(-[\gamma - q]t)\}$$
$$= (d/dt + \gamma + q)\{-[\gamma + q]A_1\exp(-[\gamma + q]t) + (\gamma - q)A_1\exp(-[\gamma + q]t)$$
$$- [\gamma - q]A_2\exp(-[\gamma - q]t) + [\gamma - q]A_2\exp(-[\gamma - q]t)\}$$
$$= (d/dt + \gamma + q)\{-2qA_1\exp(-[\gamma + q]t)\}$$

$= -2qA_1 \{-[\gamma + q]\exp(-[\gamma + q]t) + (\gamma + q)\exp(-[\gamma + q]t)\} = 0$

Now, depending on the magnitudes of k, m and c, the combination represented by q could be real or imaginary. There is nothing in principle (like conservation of energy) preventing this possibility. Three situations arise, each with a specific name

q is real and > 0	overdamping
$q = 0$	critical damping
q is imaginary	underdamping.

We take each of these in turn.

I. q is real and > 0 : overdamping

Since $\gamma > q$ for q real (since $\gamma^2 = q^2 + k/m$) then both $\gamma - q$ and $\gamma + q$ are positive, so that $\exp[-(\gamma + q)t]$ and $\exp[-(\gamma - q)t]$ both decay with time. There is no oscillatory motion.

II. $q = 0$: critical damping

The solution that we have obtained becomes $x(t) = (A_1 + A_2)\exp(-\gamma t)$. While this is still valid, it is not the most general solution of $(d/dt + \gamma)(d/dt + \gamma)x = 0$, since application of the first $(d/dt + \gamma)$ to $x(t)$ immediately yields zero. We proceed by noting that the entire $(d/dt + \gamma)x(t)$ in the equation

$$(d/dt + \gamma)(d/dt + \gamma)x(t) = 0$$

must be of the form $B\exp(-\gamma t)$ to satisfy the leftmost $(d/dt + \gamma)$. Thus, we have

$$(d/dt + \gamma)x(t) = B\exp(-\gamma t). \quad (1)$$

This can be solved via:

$$\exp(+\gamma t)\bullet(d/dt + \gamma)x(t) = B \quad (2)$$

Now the product $x(t)\exp(+\gamma t)$ behaves like

$$d/dt\,[x(t)\exp(+\gamma t)] = \exp(+\gamma t)\,(dx/dt) + \gamma x\exp(+\gamma t) = \exp(+\gamma t)\,(d/dt + \gamma)\,x(t) \quad (3)$$

So (2) + (3) implies

$$d/dt\,[x(t)\exp(+\gamma t)] = B$$

Since $d/dt\,[f(t)] = B$ has the solution $f(t) = A + Bt$, then

$$x(t) = (A + Bt)\exp(-\gamma t)$$

The special solution with $B = 0$ is what we obtained previously for $q = 0$. The values of A and B are set by the initial position (A) and initial velocity B-γt (found from d/dt ú$_{t=0}$). As with the overdamped case, this solution decays to $x = 0$ as $t \rightarrow \infty$ without oscillating.

III. q imaginary: underdamping

If q is imaginary, the solution for $x(t)$ is still valid, and becomes oscillatory since $\exp(i\theta) = \cos\theta + i\sin\theta$. The oscillations are not simple harmonic motion because:

- the oscillations are damped:

$$\exp[-(\gamma - q)t] \rightarrow \exp(-\gamma t)\exp(qt)$$
$$\rightarrow \exp(-\gamma t)(\cos\theta + i\sin\theta)$$

- the period of oscillation is lengthened because of the damping force.

To see how the solution behaves, we perform a number of changes of variables. First, change q to $i\omega_d$, where ω_d is real:

$$q = (\gamma^2 - k/m)^{1/2} = i\,(k/m - \gamma^2)^{1/2} = i\,(\omega_o^2 - \gamma^2)^{1/2} = i\,\omega_d$$

ω_o is just the SHM result $\omega_o = (k/m)^{1/2}$. The solution is thus

$$x(t) = A_1 \exp(-\gamma t)\exp(-i\omega_d t) + A_2 \exp(-\gamma t)\exp(i\omega_d t)$$
$$= \exp(-\gamma t) \bullet [A_1 \exp(-i\omega_d t) + A_2 \exp(i\omega_d t)]$$

Now, A_1 and A_2 may be complex. In fact, if x is to be real, then A_1 and A_2 are complex conjugates:

$$x(t) = \exp(-\gamma t) \bullet [A_1 \exp(-i\omega_d t) + A_2 \exp(i\omega_d t)]$$
$$x^*(t) = \exp(-\gamma t) \bullet [A_2^* \exp(-i\omega_d t) + A_1^* \exp(i\omega_d t)]$$

$$\Rightarrow A_1 = A_2^*$$

Now we can express a complex number in terms of two real numbers by writing

$$A_1 = (A/2)\exp(-i\delta) \qquad A_2 = (A/2)\exp(i\delta)$$

so that

$$x(t) = \exp(-\gamma t)\,\{[A/2]\exp[-i(\omega_d t + \delta)] + [A/2]\exp[i(\omega_d t + \delta)]\}$$

Expanding the exponentials as cos + $i$sin, the imaginary terms all cancel, and one has

$$x(t) = \exp(-\gamma t)\; 2\,(A/2)\cos(\omega_d t + \delta)$$
$$= \exp(-\gamma t)\,A\cos(\omega_d t + \delta)$$

This expression now looks like SHM except:

(i) the amplitude is damped like $\exp(-\gamma t)\,A$

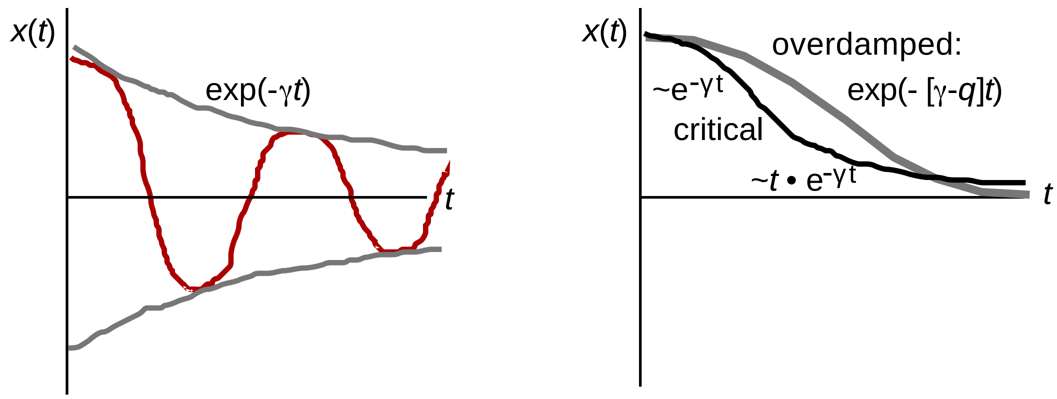
(ii) the angular frequency is reduced from ω_o to ω_d since $\omega_d^2 = \omega_o^2 - \gamma^2$.

Physically, we expect ω_d to be less than ω_o because friction slows down the oscillator.

<u>Summary</u>

In total, the behavior of a damped harmonic oscillator can be described by

$c = 0$	q imaginary $c^2 / 4m^2 < k / m$	$q = 0$ $c^2 / 4m^2 = k / m$	$q > 0$ (real) $c^2 / 4m^2 > k / m$
SHM	underdamped	critically damped	overdamped



For a car suspension, you want $q \sim 0$:

if q is too large, the ride is stiff and responds harshly to every bump
if q is imaginary, the car oscillates over every pothole.

Quality factor or Q (Chap. 3 Quality Factor)

The rate of energy loss of an oscillator is an important characteristic. Sometimes, one wants a high energy loss (say in the suspension of a car) while other times one wants a minimal energy loss (as in the crystal of a watch). The quality factor is

$$Q = 2\pi \text{ (Energy stored in oscillator / Energy lost per period)}$$

After some work, one can show that

$$Q = \omega_d / 2\gamma \quad \omega_o / 2\gamma.$$

The larger γ is, the faster the amplitude of oscillation dies away and the smaller is the value of Q. Some examples from the text:

System	Q
Earth (*i.e.*, earthquakes)	$\sim 10^3$
Piano string	3000
Neutron star	10^{12}