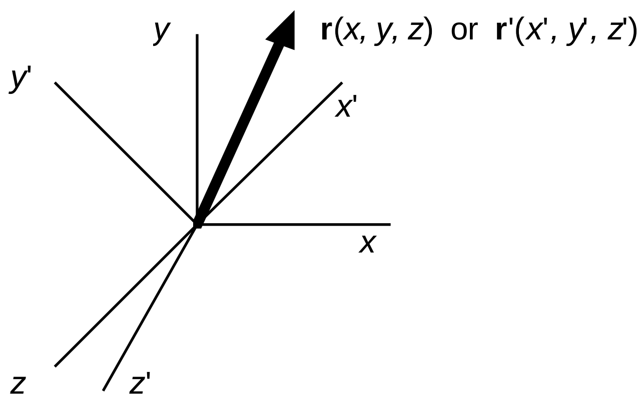
Lecture 11 - Rotating coordinate systems

Text: similar to Fowles and Cassiday, Chap. 5

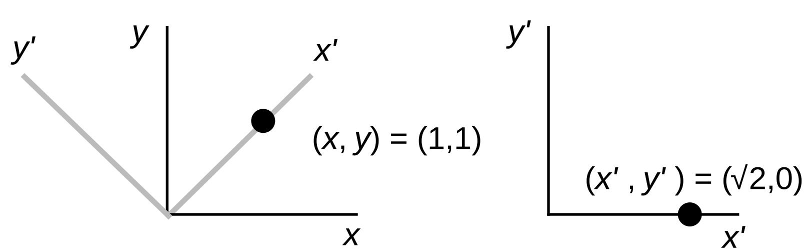
We start our discussion of rotating coordinate systems with the case of pure rotation about a common origin. The notation is as follows

> Stationary system: Cartesian unit vectors **i**, **j**, **k**
> Rotating system: Cartesian unit vectors **i**', **j**', **k**'

Thus a point *P* can alternately be described by the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or $\mathbf{r}' = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}'$, where (x, y, z) do not necessarily have the same numerical values as (x', y', z'):



The notation is subtle. The vectors **r** and **r**' represent the same point, and have the same magnitude $|\mathbf{r}| = |\mathbf{r}'|$, but the triple of points (x', y', z') do not have the same appearance in their respective frames. As a two-dimensional example, consider the point $(x, y) = (1,1)$ as seen in a frame rotated by 45º counter-clockwise:



Further, the velocities do not have the same magnitudes: $|\mathbf{v}| \neq |\mathbf{v}'|$, as we show later.

The moving system rotates about an axis with an angular velocity ω, defined by the usual convention that ω points towards the viewer when the motion down the rotational axis is counter-clockwise.

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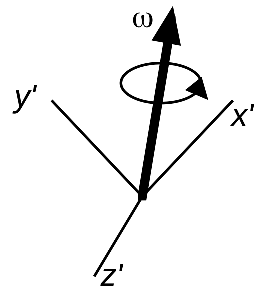
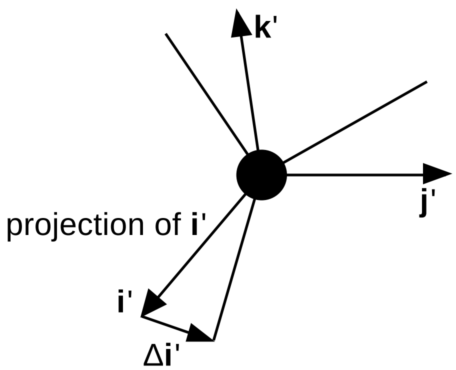
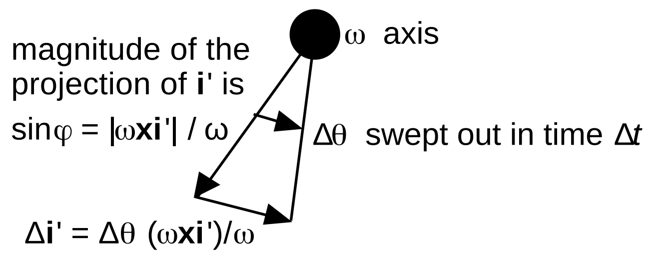


Let's examine how **i**', **j**', **k**' behave as seen by the stationary system. Since the coordinate system rotates, then clearly **i**', **j**', **k**' may be time-dependent. Hence, their time derivatives like $d\mathbf{i}'/dt$ may be non-zero.

As we discussed in Lecture 1 in a similar context, the change in **i**' in time Δt, defined as $\Delta\mathbf{i}'$, cannot be along **i**' since it is a unit vector. In fact, the change in **i**' must be perpendicular to the plane formed by **i**' and ω, and in the direction of $\omega \times \mathbf{i}'$ (note the order in the cross product).





If we look down the ω axis, then the projection of **i**' on a plane perpendicular to the ω-axis is $\sin\varphi$, where φ is the angle between ω and **i**'. Now $\Delta\mathbf{i}'$ equals the projection of **i**' (*i.e.*, $\sin\varphi$) times the angle $\Delta\theta$ that the **i**'-axis sweeps out in time Δt : But $\sin\varphi = |\omega \times \mathbf{i}'| / \omega$, so that

$$\Delta\mathbf{i}' = [(\omega \times \mathbf{i}') / \omega] \bullet \Delta\theta.$$

Dividing both side by Δt and using $\omega = \Delta\theta / \Delta t$, we find

$$\Delta\mathbf{i}' / \Delta t = [(\omega \times \mathbf{i}') / \omega] \bullet \Delta\theta / \Delta t = [(\omega \times \mathbf{i}') / \omega]\, \omega$$

or applying the infinitesimal limit

$$d\mathbf{i}' / dt = \omega \times \mathbf{i}' \qquad \text{(the order of the cross-product is important)}$$

Similar relationships apply to the other vectors as well

$$d\mathbf{j}' / dt = \omega \times \mathbf{j}' \qquad d\mathbf{k}' / dt = \omega \times \mathbf{k}'$$

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Next we determine how a velocity vector behaves in a rotating frame. We start with the position vector

$$\mathbf{r} = \mathbf{r}'$$

which in component language reads

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x'\,\mathbf{i}' + y'\,\mathbf{j}' + z'\,\mathbf{k}'$$

and take the derivative

$$(dx / dt)\,\mathbf{i} + (dy / dt)\,\mathbf{j} + (dz / dt)\,\mathbf{k}$$
$$= (dx' / dt)\,\mathbf{i}' + (dy' / dt)\,\mathbf{j}' + (dz' / dt)\,\mathbf{k}' + x'\,(d\mathbf{i}' / dt) + y'\,(d\mathbf{j}' / dt) + z'\,(d\mathbf{k}' / dt)$$

Substituting $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ and the definition $\mathbf{v}' = v_x'\,\mathbf{i}' + v_y'\,\mathbf{j}' + v_z'\,\mathbf{k}'$ gives

$$\mathbf{v} = \mathbf{v}' + x'\,(d\mathbf{i}' / dt) + y'\,(d\mathbf{j}' / dt) + z'\,(d\mathbf{k}' / dt).$$

Next, replace the time derivatives of the rotating basis vectors:

$$\mathbf{v} = \mathbf{v}' + x'\,(\omega\mathbf{x}\mathbf{i}') + y'\,(\omega\mathbf{x}\mathbf{j}') + z'\,(\omega\mathbf{x}\mathbf{k}')$$

and rearrange

$$\mathbf{v} = \mathbf{v}' + \omega\mathbf{x}(x'\,\mathbf{i}' + y'\,\mathbf{j}' + z'\,\mathbf{k}')$$
$$\mathbf{v} = \mathbf{v}' + \omega\mathbf{x}\mathbf{r}' \qquad (1)$$

Clearly, it's not just a matter of **v** being rotated with respect to **v**': they have completely different magnitudes.

One can obtain a relationship between the acceleration vectors by starting with $\mathbf{v} = \mathbf{v}' + \omega\mathbf{x}\mathbf{r}'$ and taking the time derivative:

$$d\mathbf{v} / dt = d\mathbf{v}' / dt + (d\omega / dt)\mathbf{x}\mathbf{r}' + \omega\mathbf{x}(d\mathbf{r}'/ dt) \qquad (2)$$

Now, $d\mathbf{v} / dt$ is just the acceleration **a**. But $d\mathbf{v}' / dt$ must be found in the same way as $d\mathbf{r}' / dt$ because of the rotating basis set:

$$d\mathbf{v}' / dt$$
$$= (dv_x' / dt)\,\mathbf{i}' + (dv_y' / dt)\,\mathbf{j}' + (dv_z' / dt)\,\mathbf{k}' + v_x'\,(d\mathbf{i}' / dt) + v_y'\,(d\mathbf{j}' / dt) + v_z'\,(d\mathbf{k}' / dt)$$

But, in analogy with the definition of **v**',

$$(dv_x' / dt)\,\mathbf{i}' + (dv_y' / dt)\,\mathbf{j}' + (dv_z' / dt)\,\mathbf{k}' = a_x'\mathbf{i}' + a_y'\mathbf{j}' + a_z'\mathbf{k}' = \mathbf{a}',$$

so, after substituting for the rotating basis vectors

$$d\mathbf{v}' / dt = \mathbf{a}' + \omega\mathbf{x}\mathbf{v}' \qquad (3)$$

Then Eq. (2) becomes

$$\mathbf{a} = \mathbf{a}' + \omega\mathbf{x}\mathbf{v}' + (d\omega / dt)\mathbf{x}\mathbf{r}' + \omega\mathbf{x}(d\mathbf{r}'/ dt)$$

Lastly, replace $d\mathbf{r}'/ dt = \mathbf{v}' + \omega\mathbf{x}\mathbf{r}'$ to obtain

$$\mathbf{a} = \mathbf{a}' + \omega\mathbf{x}\mathbf{v}' + (d\omega / dt)\mathbf{x}\mathbf{r}' + \omega\mathbf{x}\mathbf{v}' + \omega\mathbf{x}\,(\omega\mathbf{x}\mathbf{r}')$$

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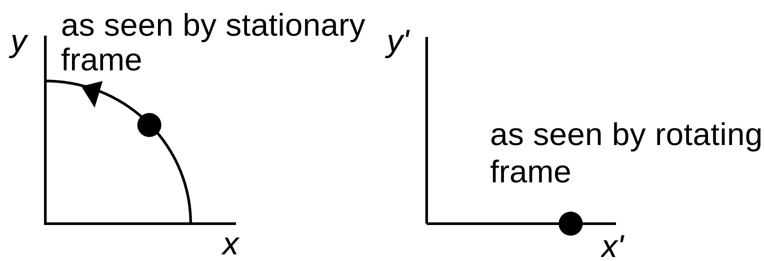
or

$$\mathbf{a} = \mathbf{a}' + 2\omega \mathbf{x} \mathbf{v}' + (d\omega / dt)\mathbf{x}\mathbf{r}' + \omega\mathbf{x}(\omega\mathbf{x}\mathbf{r}')$$

Summary of notation

- $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ are the usual kinematic quantities in the stationary frame
- (x', y', z') (v_x', v_y', v_z') are quantities observed in the rotating frame
- $\mathbf{r}'$, $\mathbf{v}'$, $\mathbf{a}'$ are vectors from the rotating frame
- $v_x' = dr_x' / dt$ and $a_x' = dv_x' / dt$ as expected.

Example Uniform circular motion in which frame O' is co-rotating, so $\mathbf{v}' = 0$



=> $\mathbf{v} = \mathbf{v}' + \omega\mathbf{x}\mathbf{R}$ --> $\mathbf{v} = \omega\mathbf{x}\mathbf{R}$ as expected

Since the motion is uniform, $(d\omega / dt) = 0$ and $\mathbf{a}' = d\mathbf{v}' / dt - \omega\mathbf{x}\mathbf{v}' = 0$. Hence

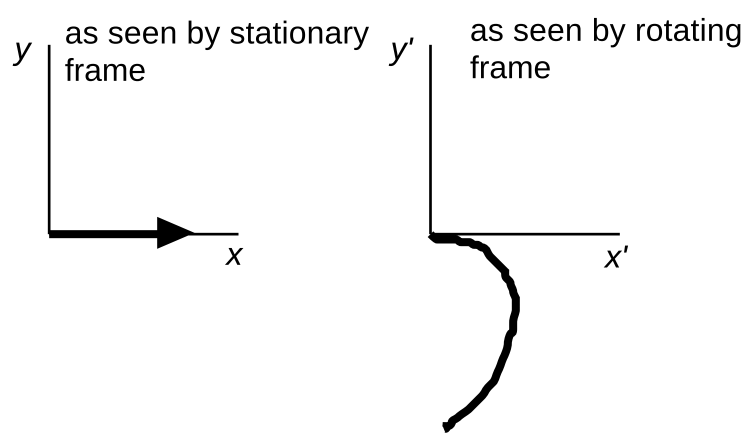
$$\mathbf{a} = \mathbf{a}' + 2\omega\mathbf{x}\mathbf{v}' + (d\omega / dt)\mathbf{x}\mathbf{r}' + \omega\mathbf{x}(\omega\mathbf{x}\mathbf{r}')$$

becomes

$$\mathbf{a} = 0 + 0 + 0 + \omega\mathbf{x}(\omega\mathbf{x}\mathbf{R}) = -\omega^2 R.$$

Thus, $\omega\mathbf{x}(\omega\mathbf{x}\mathbf{r}')$ is the centripetal acceleration.

Example Now let the particle move at constant speed in the stationary frame



In the stationary frame:

$v_x = v$ $v_y = 0$ $x = vt$ $y = 0.$

In the rotating frame

$x' = vt\cos(\omega t)$ $y' = -vt\sin(\omega t)$

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so that

$(dx' / dt) = v \cos(\omega t) - v\omega t \sin(\omega t)$ $\qquad$ $(dy' / dt) = -v \sin(\omega t) - v\omega t \cos(\omega t)$.

Then the magnitude of **v**' is

$$(dx' / dt)^2 + (dy' / dt)^2 = [v \cos(\omega t) - v\omega t \sin(\omega t)]^2 + [-v \sin(\omega t) - v\omega t \cos(\omega t)]^2$$
$$= v^2 + v^2 t^2 \omega^2$$

i.e.

$$\mathbf{v}'^2 = (dx' / dt)^2 + (dy' / dt)^2 = v^2 (1 + \omega^2 t^2).$$

Even though |**v**| is constant, |**v**'| grows with time, since the object moves away from the origin and the distance swept out in a turn of the coordinate system increases like *t*. This expression also can be obtained from $\mathbf{v} = \mathbf{v}' + \omega \mathbf{x} \mathbf{r}'$, whence

$$\mathbf{v}'^2 = (\mathbf{v} - \omega \mathbf{x} \mathbf{r}')^2 = \mathbf{v}^2 + (\omega \mathbf{x} \mathbf{r}')^2 = v^2 + \omega^2 r'^2.$$

The acceleration in the rotating frame has two components:

$-2\omega \mathbf{x} \mathbf{v}'$ is perpendicular to **v**' and increases with $v (1 + \omega^2 t^2)^{1/2}$

$-\omega \mathbf{x}(\omega \mathbf{x} \mathbf{r}')$ is radially outwards, and increases as *vt*.

Acceleration plus rotation

For the general expression for translating + rotating coordinate systems, simply add $\mathbf{V}_o$ and $\mathbf{A}_o$ to the expressions for rotating systems.

$$\mathbf{v} = \mathbf{v}' + \omega \mathbf{x} \mathbf{r}' + \mathbf{V}_o$$
$$\mathbf{a} = \mathbf{a}' + 2\omega \mathbf{x} \mathbf{v}' + (d\omega / dt) \mathbf{x} \mathbf{r}' + \omega \mathbf{x}(\omega \mathbf{x} \mathbf{r}') + \mathbf{A}_o$$

Forces in a rotating frame

With our expression for the acceleration, it is easy to relate the forces applicable in each frame. Transposing to separate the frame-dependent components:

$$m\mathbf{a}' = m\mathbf{a} - 2m\omega \mathbf{x} \mathbf{v}' - m(d\omega / dt) \mathbf{x} \mathbf{r}' - m\omega \mathbf{x}(\omega \mathbf{x} \mathbf{r}')$$
$$\mathbf{F}' = \mathbf{F} + \mathbf{F}'_{cor} + \mathbf{F}'_{trans} + \mathbf{F}'_{cen}$$

$\mathbf{F}'_{cor}$ = Coriolis force = $-2m\omega \mathbf{x} \mathbf{v}'$

$\mathbf{F}'_{trans}$ = transverse force = $-m(d\omega / dt) \mathbf{x} \mathbf{r}'$

$\mathbf{F}'_{cen}$ = centri*fugal* force = $-m\omega \mathbf{x}(\omega \mathbf{x} \mathbf{r}')$

The forces that apply in the rotating frame include several components that appear only because of the rotating coordinate system. If **F** = *m***a** were completely absent, then fictitious forces would still be needed in the rotating frame to explain why the object in question did not follow a straight line in that frame.

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