Swarm Dynamics on the Sphere

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Objective

The integro-differential equation on the sphere:

$$\rho_t + \nabla \cdot (\rho v) = 0,$$
$$v = -\nabla K * \rho,$$

where ρ denotes density and v denotes velocity.
Goal: Look for a potential K that gives a steady state of even density distribution over the sphere.

Motivation for the choice of K

In $\mathbb{R}^2$, let r be the Euclidean distance between two points and G the Green's function to the Laplacian given by

$$G(r) = -\frac{1}{2\pi} \log r.$$

It is shown in [1] that, the desired interaction potential is a modification of the Green's function:

$$K(r) = -\frac{1}{2\pi} \log r + \frac{1}{2} r^2.$$

Sketch of idea: The key observation is that K satisfies

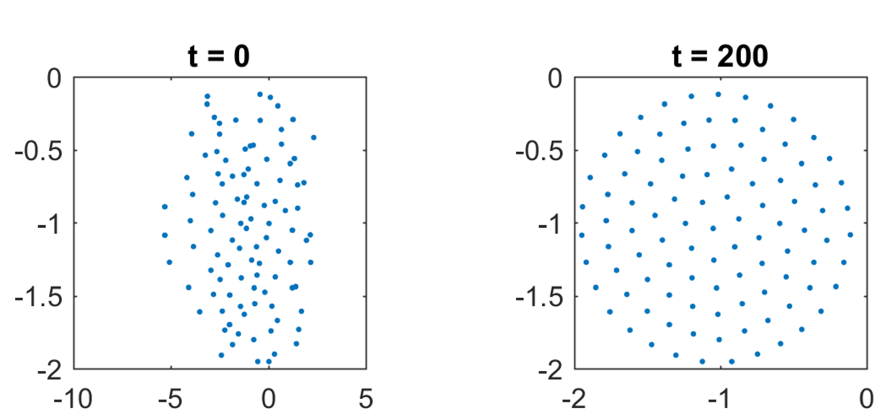
$$\boxed{\Delta K = -\delta + 2.}$$

Along the characteristic path $X(\alpha, t)$ of a particle with α as its initial position, $\rho(X(\alpha,t),t)$ satisfies

$$\begin{aligned} \frac{D}{Dt}\rho &= \nabla \rho \cdot v + \rho_t \\ &= -\rho \nabla \cdot v \\ &= -\rho(-\Delta K * \rho) \\ &= -\rho(\rho - 2M), \end{aligned}$$

where M denotes the constant total mass in 2D free space. The global attractor for this ODE is then $\rho_0 = 2M$.

2D Case



Dynamics on the sphere

Motivated by the $\mathbb{R}^2$ case, we look for an interaction potential K on the sphere such that it satisfies

$$\boxed{\Delta_S K = -\delta + C,}$$

where Δ_S is the Laplace-Beltrami operator on the sphere. A natural choice is the generalized Green's function of Δ_S:

Choice of K

Let θ be the angle between two points on the sphere, then the interaction potential K and its Laplace-Beltrami are:

$$K(\theta) = -\frac{1}{2\pi} \log \sin \frac{\theta}{2}, \quad (1)$$

$$\Delta K = -\delta + \frac{1}{4\pi}. \quad (2)$$

Main Result: Using K in (1), we show that:

- the uniform density distribution $\rho_0 = \frac{1}{4\pi}$ over the whole sphere is an equilibrium state.
- ρ_0 is the global attractor, that is, any initial state will converge to ρ_0 as $t \to \infty$.

Sketch of idea: Similarly, along the characteristic path $X(\alpha, t)$, we have

$$\frac{D}{Dt}\rho = -\rho(\rho - \frac{1}{4\pi}).$$

By conservation of mass, the density is uniform over the whole sphere.

Numerics

Rewrite $v = -\nabla K * \rho$ in terms of particle paths:

$$\frac{d\vec{\mathbf{X}}_i}{dt} = -\frac{1}{N} \sum_{\substack{j=1...N \\ j \neq i}} \nabla_i K(\vec{\mathbf{X}}_i, \vec{\mathbf{X}}_j).$$

Expand these terms in spherical basis:

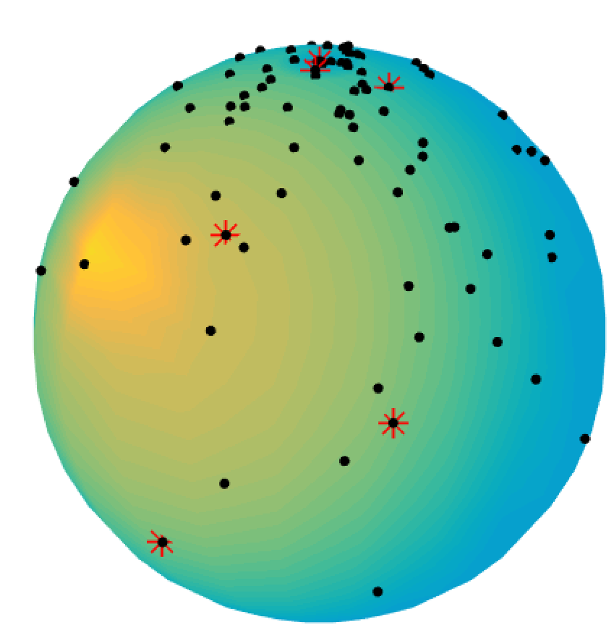
$$\begin{aligned} \frac{d\vec{\mathbf{X}}_i}{dt} &= \frac{\partial \vec{\mathbf{X}}_i}{\partial \theta_i}\frac{d\theta_i}{dt} + \frac{\partial \vec{\mathbf{X}}_i}{\partial \phi_i}\frac{d\phi_i}{dt} \\ &= R\frac{d\theta_i}{dt}\vec{\mathbf{e}}_{\theta_i} + R\sin\theta_i \frac{d\phi_i}{dt}\vec{\mathbf{e}}_{\phi_i} \end{aligned}$$

$$\nabla_i K(\vec{\mathbf{X}}_i, \vec{\mathbf{X}}_j) = \frac{1}{R}\frac{\partial K}{\partial \theta_i}\vec{\mathbf{e}}_{\theta_i} + \frac{1}{R\sin\theta_i}\frac{\partial K}{\partial \phi_i}\vec{\mathbf{e}}_{\phi_i}$$

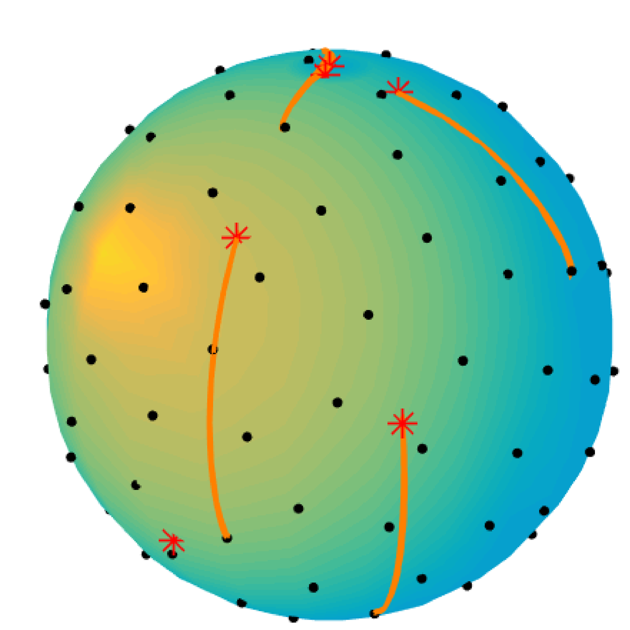
Compare coefficients of the spherical basis to obtain the ODE equations:

$$\frac{d\theta_i}{dt} = -\frac{1}{N} \sum_{\substack{j=1...N \\ j \neq i}} \frac{1}{R^2}\frac{\partial K}{\partial \theta_i}$$

$$\frac{d\phi_i}{dt} = -\frac{1}{N} \sum_{\substack{j=1...N \\ j \neq i}} \frac{1}{R^2 \sin^2\theta_i}\frac{\partial K}{\partial \phi_i}$$



(a) Initial State



(b) Steady State, particle paths traced

Extension: with an obstacle

If we add a barrier at $\theta = \theta_0$ and redefine the velocity at the boundary as

$$v = \text{Proj}(-\nabla K * \rho),$$

then numerics show that new equilibrium state is:

- constant density $\rho_0 = \frac{1}{4\pi}$ in the interior.
- another constant density depending on θ_0 on the boundary, with zero projected velocity.

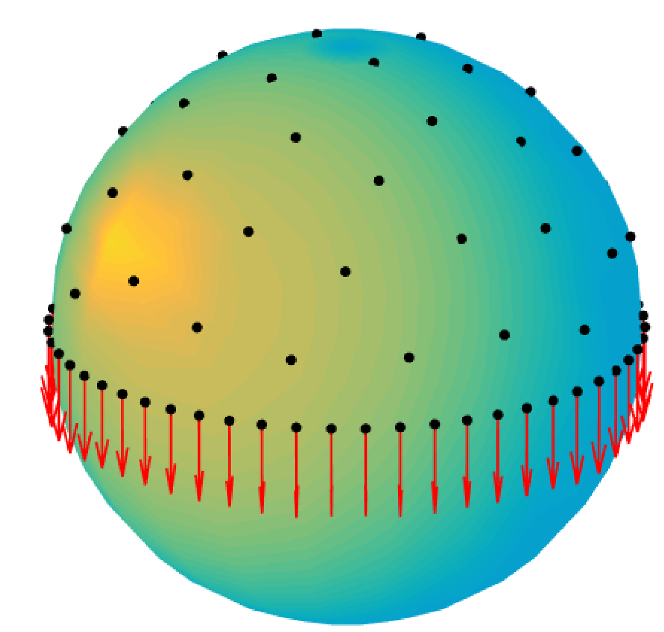


Figure: Equilibrium state with $\theta_0 = \pi$

Future Work

- Understand analytically the approach to equilibrium in the case with an obstacle.
- Study dynamics on more general surfaces and develop effective numerical schemes.

References

[1] R. C. Fetecau, Y. Huang and T. Kolokolnikov, Swarm dynamics and equilibria for a nonlocal aggregation model , Nonlinearity, Vol. 24, No. 10, 2681-2716 (2011)

[2] Yoshifumi Kimura, Vortex motion on surfaces with constant curvature, Proc. R. Soc. Lond. A (1999) 455, 245-259