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COST MINIMIZATION

Profit Maximization and Cost Minimization

Remember that the firm's problem is maximizing profits by choosing the optimal quantities of inputs to employ and output to produce. We already know how to solve the firm's profit maximization problem in a competitive market environment. This direct approach is great but there are some very important insights especially on the cost side of the firm's problem that can be gained if we look at the firm's problem through a more indirect approach. The idea is to break down the profit maximization problem of the firm into two stages: first, we look at the problem of minimizing costs of producing any given amount of output, and second, we look at how to choose the most profitable output level.

These notes cover the first of these two stages, i.e. they answer the question: for a given amount of output, y (just some number, 10, 20, 100, which does not have to be necessarily the firm's optimal output that it will end up producing) what is the minimum cost of producing it. In other words, we ask given that we have to produce y units of output and given input prices w_1 and w_2, how much of each input should the firm employ to produce y in the least costly way. Notice that the amount of each input that will be used will depend on the output level that the firm wants to produce - if you want to produce 1000 units of outputs you would definitely need more workers and machines than to produce just 10 units.

Notice that cost minimization is a necessary condition for profit maximization in competitive markets. If, for a given level of output, one is not cost minimizing that means that he is also not profit maximizing. Why? For a given y revenue is fixed (taking p as given), so if there is a less costly way to produce this output level this will lead to higher profits.

The Cost Minimization Problem

Given the above discussion the firm wants to solve:

$$\min_{x_1,x_2} \ w_1x_1 + w_2x_2$$
$$s.t. \ f(x_1,x_2) = y$$

How is the above interpreted? The firm wants to minimize its costs ($w_1x_1 + w_2x_2$) of producing y units of output. The fact that the firm wants produce y units of output is given by the constrain $f(x_1,x_2) = y$. Remember that the production function, $f(x_1,x_2)$ corresponds to the maximum output that can be extracted from x_1 units of input 1 and x_2 units of input 2 - i.e. (since inputs are costly), using the production function we would use x_1 and x_2 most efficiently. The cost minimization is then done by choosing how much of each input to employ (x_1 and x_2) such that the costs of producing y using the production function are minimized.

How Do We Solve The Cost Minimization Problem?

In general the cost minimization problem is harder to solve that the consumer problem or the profit maximization problem since usually the constraint ($f(x_1, x_2) = y$) is a non-linear function of x_1 and x_2 so sometimes it may be hard to express x_2 in terms of x_1 from it and plug into the costs that we are minimizing. When this is the case there are two ways to proceed of which the first is simpler and recommended and the second is optional for those of you with more knowledge of math (you can use either one at the exams):

1. Solving the cost minimization problem using the optimality condition $TRS = -\frac{w_1}{w_2}$.

Remember that we showed in class graphically that at the optimal choice of input quantities (the optimum) we must have that the isocost line is tangent to the isoquant corresponding to output level y. The slope of the isoquant is given by the technical rate of substitution (TRS) which is a function of x_1 and x_2. Why? Because remember what the TRS is equal to: it equals minus the ratio of marginal products, i.e. $TRS = -\frac{MP_1(x_1,x_2)}{MP_2(x_1,x_2)}$ but the marginal products are functions of x_1 and x_2 since they are simply the partial derivatives of the production function which is a function of x_1 and x_2. The slope of the isocost line (remember from class) was $-\frac{w_1}{w_2}$ i.e. minus the ratio of the input prices (this is similar to the slope of the budget line, however note that the isocost curve does not serve the same purpose as the budget line). In the consumer problem we had a **fixed budget line** and we were trying to find the point on this budget line which would give us **maximum utility** - i.e. the point which lies on the **highest indifference curve**. In contrast, in the firm's problem we have a **fixed isoquant** and we are trying to find the point on this fixed isoquant which gives us **minimum costs**, i.e. the point lying on the **lowest isocost line**.

Given the above, we can solve the cost minimization problem by solving two equations for the two unknowns x_1 and x_2: the optimality condition and the constraint that y must be produced:

$$\begin{aligned} \frac{MP_1(x_1,x_2)}{MP_2(x_1,x_2)} &= \frac{w_1}{w_2} \text{ (optimality condition)} \\ f(x_1,x_2) &= y \text{ (output must be } y \text{ condition)} \end{aligned}$$

Notice that y is not an unknown! It is just a fixed level of output we want to produce (say 10, 20, 0 or whatever). We do not have to solve for y - it is given.

The solutions to the above problem $\hat{x}_1$ and $\hat{x}_2$ will be generally functions of everything that is given: the input prices, w_1 and w_2 and the output level y, i.e. we will have:

$$\begin{aligned} \hat{x}_1 &= \hat{x}_1(w_1, w_2, y) \\ \hat{x}_2 &= \hat{x}_2(w_1, w_2, y) \end{aligned}$$

These solutions (to the cost minimization problem of a firm facing input prices w_1 and w_2 which wants to produce y) are called **the conditional input demands** for producing y units of output at input rises w_1 and w_2. Why are they called conditional? Because they are conditioned (i.e. depend on) what level of output we want to produce - if we want to produce more output more inputs will be needed in general.

Difference between the conditional input demands from the cost minimization problem and the (unconditional) input demands from the profit maximization problem

It is important to understand that **the conditional input demands** coming from the cost minimization problem above are not the same thing as the (unconditional, as sometimes called) **input demands** resulting from the profit maximization problem (see the notes on profit maximization and input demands). Remember the profit maximizing input demands, $x_1^*(w_1, w_2, p)$ and $x_2^*(w_1, w_2, p)$ were functions of the input prices, w_1 and w_2 and the output price p. They are called unconditional (or simply not called conditional) because they do not depend on y - the output level in the firm's profit maximization problem is already optimized and this optimized output level is a function of the prices (w_1, w_2, p) that are taken as given by the firm - thus the (unconditional) input demands are only functions of what is given, i.e. the prices. In contrast, the cost minimization problem is being solved for a given output level which is not necessarily the optimized one that the firm will eventually choose. Why? Remember the cost minimization problem was only the first step in the indirect approach to solve the firm's problem and at that first step the output level was taken as given as we were only interested in how to produce any given output level at minimum cost. Thus the conditional input demands depend only on what is given in the cost minimization problem - i.e. the input prices, w_1 and w_2 and the output level y. One more time: the (unconditional) input demands depend on **output price,** p, the conditional input demands depend on **output level**, y. Both of them depend on input prices, w_1 and w_2.

2. Solving the cost minimization problem using the Lagrange method (OPTIONAL! NOT REQUIRED FOR THE EXAM!)

For those of you interested in math and knowing what the Lagrange method of solving optimization problems is, the cost minimization problem of the firm stated above can be solved also using this method. What we do is write the Lagrangean:

$$\Lambda(x_1, x_2, \lambda) = w_1 x_1 + w_2 x_2 - \lambda(f(x_1, x_2) - y)$$

where λ is the Lagrange multiplier on the constraint. Then we have to take the partial derivatives of the Lagrangean with respect to its three arguments, x_1, x_2 and λ and set them equal to zero:

$$\begin{aligned} w_1 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_1} &= 0 \\ w_2 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_2} &= 0 \\ f(x_1, x_2) - y &= 0 \end{aligned}$$

where $\frac{\partial f(x_1,x_2)}{\partial x_1}$ and $\frac{\partial f(x_1,x_2)}{\partial x_2}$ are the first partial derivatives of the production function with respect to x_1 and x_2. The last equation is simply our constraint. We can take the first two equations above to get:

$$w_1 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_1}$$

$$w_2 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_2}$$

Divide them through (both sides) to get:

$$\frac{w_1}{w_2} = \frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}}$$

But what are the partial derivatives of the production - remember they are simply the marginal products of each input, $MP_1(x_1, x_2)$ and $MP_2(x_1, x_2)$. Thus we get:

$$\frac{w_1}{w_2} = \frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)}$$

which is what we had before! Then this equation, together with the constraint $f(x_1, x_2) = y$ can be used to solve for the conditional input demands $\hat{x}_1(w_1, w_2, y)$ and $\hat{x}_2(w_1, w_2, y)$.

An Example With A Specific Production Function:

Take $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$ and let input prices be $w_1 = 1$, $w_2 = 2$. We want to solve the firm's cost minimization problem of producing y units of output. We will use all three methods discussed above and obtain the same results:

– substitution from the constraint
– using the optimality condition $\frac{MP_1(x_1,x_2)}{MP_2(x_1,x_2)} = \frac{w_1}{w_2}$
– using the Lagrange method (optional)

Let us first write down the problem:

$$\min_{x_1,x_2} \ 1x_1 + 2x_2$$
$$\text{s.t. } x_1^{1/3} x_2^{1/3} = y$$

Method 1 (Direct Substitution)

1. We first have to express x_2 in terms of x_1 from the technology constraint $f(x_1, x_2) = y$. The easiest way to to it is get rid of the exponents first by raising the constraint to power 3 - we get:

$$x_1 x_2 = y^3$$

thus $x_2 = \frac{y^3}{x_1}$.

2. Now let's plug the expression for x_2 obtained above in our cost function to get the following minimization problem:

$$\min_{x_1} \ x_1 + 2\frac{y^3}{x_1}$$

3. We solve the above problem of just one unknown (x_1) by taking the first derivative of the function being minimized and setting it to zero:

$$1 + 2y^3(-\frac{1}{x_1^2}) = 0$$

(we use that the derivative of $\frac{1}{x_1}$ is $-\frac{1}{x_1^2}$). Remember y is given, i.e. treated as if it were a number!. From the above equation we can solve for the optimal x_1 as function of y (in general it is also a function of w_1 and w_2 but remember we plugged numbers for these already):

$$\frac{1}{x_1^2} = \frac{1}{2y^3} \text{ or } x_1^2 = 2y^3, \text{ or } \hat{x}_1(y) = \sqrt{2}y^{3/2}$$

which is the **conditional input demand for input 1**.

4. Now how do we find the optimal quantity demanded of input 2 (i.e. its conditional input demand, $\hat{x}_2$)? Remember we know that $\hat{x}_2 = \frac{y^3}{\hat{x}_1}$ so we have:

$$\hat{x}_2(y) = \frac{y^3}{\sqrt{2}y^{3/2}} = \frac{y^{3/2}}{\sqrt{2}}$$

which is the **conditional input demand for input 2**. Notice that both conditional input demands depend on the given level of output, y that the firm wants to produce.

5. The **minimum cost of producing y units of output** then can be found by simply substituting the conditional input demands obtained above into the cost function:

$$\hat{c}(y) = 1\hat{x}_1(y) + 2\hat{x}_2(y) = \sqrt{2}y^{3/2} + \frac{2}{\sqrt{2}}y^{3/2} = 2\sqrt{2}y^{3/2}$$

Method 2 (using the optimality condition)

This method works even if it is not obvious how to express x_2 in terms of x_1 from the constraint. However, it relies on tangency and interiority of the solution.

1. To be able to write down the optimality condition for cost minimization we need to compute the marginal products of each input, i.e. the partial derivatives of the production function with respect to x_1 and x_2. We have:

$$\begin{aligned} MP_1(x_1, x_2) &= \frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{1}{3}x_1^{-2/3}x_2^{1/3} \\ MP_2(x_1, x_2) &= \frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{1}{3}x_1^{1/3}x_2^{-2/3} \end{aligned}$$

(remember when taking partial derivatives you hold all variables that you are not taking the derivative with respect to as constant).

2. Thus the optimality condition for our particular production function and input prices is (cancelling the 1/3s):

$$\frac{x_1^{-2/3}x_2^{1/3}}{x_1^{1/3}x_2^{-2/3}} = \frac{1}{2} \text{ or collecting the exponents, } \frac{x_2}{x_1} = \frac{1}{2}$$

(please remember how to manipulate fractions of numbers raised to different powers!). From above we get: $x_2 = \frac{1}{2}x_1$.

3. We can use the above relationship between x_2 and x_1 and plug it in the second equation that they must satisfy, namely that output should be y :

$$x_1^{1/3}(\frac{1}{2}x_1)^{1/3} = y$$

Again, raising both sides to power 3 helps:

$$\frac{1}{2}x_1^2 = y^3 \text{ or, } x_1^2 = 2y^3, \text{ so } \hat{x}_1(y) = \sqrt{2}y^{3/2}$$

or the conditional input demand for input one is the same thing as before once again. Then

$$\hat{x}_2(y) = \frac{1}{2}\hat{x}_1(y) = \frac{y^{3/2}}{\sqrt{2}}$$

again same as before. Clearly then the minimized costs will be also the same.

Method 3 (using Lagrange, OPTIONAL)

1. We write the Lagrangean for our specific problem:

$$\Lambda(x_1, x_2, \lambda) = 1x_1 + 2x_2 - \lambda(x_1^{1/3}x_2^{1/3} - y)$$

2. Taking the partial derivatives with respect to the arguments of the Lagrangean we get:

$$\begin{aligned} 1 - \lambda\frac{1}{3}x_1^{-2/3}x_2^{1/3} &= 0 \\ 2 - \lambda\frac{1}{3}x_1^{1/3}x_2^{-2/3} &= 0 \\ x_1^{1/3}x_2^{1/3} - y &= 0 \end{aligned}$$

3. Re-arrange the first two equations as described in the general example above and divide them through to get:

$$\frac{1}{2} = \frac{x_1^{-2/3}x_2^{1/3}}{x_1^{1/3}x_2^{-2/3}}$$

which is the same what we had in Method 2. Thus you can use the rest of the steps in Method 2 to finish the solution.