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Number concepts with *Number Worlds*: thickening understandings

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This study focuses on the nature of preservice elementary school teachers' understandings of several concepts in elementary number theory that are evoked by a computer-based microworld called *Number Worlds*. In particular, the focus is on the concepts of factor, multiple and prime number. The notion of 'thickness' is examined with respect to understanding mathematical concepts and alludes to various attributes of *Number Worlds* that may be responsible for provoking different layers and depths of understanding, termed 'thick understanding'.

1. Introduction

Mathematical concepts can be understood in many different ways, through definitions, visual representations, properties, connections with other concepts, and even theorems. One may not be able to articulate or exhaustively list all these ways, but a given problem situation will often trigger a different assortment of these ways of understanding. For example, Thurston [1] provides over a dozen ways of understanding the derivative. He is able to reconcile each with the others, and thus see how each way of understanding relates to *the same* concept. However, in problem-solving contexts he must think *with* the concept of the derivative. That is, certain triggers may cause him to ignore some aspects of his understanding of the derivative while highlighting others. For example, if presented with a graph, he might think of the derivative as a rate: the instantaneous speed of $f(t)$ when t is time. If presented with an equation, he might think of the derivative symbolically, recalling some of the derivatives he knows: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, etc. [2]. We say that Thurston has a very 'thick' understanding of the derivative, one that is likely coloured by the emotions of certain past experiences and by his own preferences; he may find the visual way of understanding more perspicuous, or the symbolic way more fruitful.

We propose that learners can also develop 'thick' understandings of concepts, that is, understandings that include more contextual, qualitative triggers. In this article, we first focus on the notion of 'thick' understanding and describe what it could entail. We then describe a novel computer-based environment that was designed to provide learners with opportunities to develop 'thicker' understandings of some elementary number theory concepts. Finally, we present the results of a

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study carried out with a group of preservice teachers interacting with this environment and use the results to examine the nature of 'thick' understanding in mathematics more closely.

2. What is a thick understanding?

The adjective 'thick' [3] comes from anthropology where it applies to descriptions of events or social scenes that are layered, rich and contextual. This same understanding has been appropriated into educational research in a number of different contexts, the most recent of which is Shaffer and Resnick's [4] usage of the term 'thick authenticity' to describe the various kinds of authentic learning. 'Thick authenticity refers to activities that are personally meaningful, connected to important and interesting aspects of the world beyond the classroom, grounded in a systematic approach to thinking about problems and issues, and which provide for evaluation that is meaningfully related to the topics and methods being studied' (p. 203). Prior to Shaffer and Resnick's work, Ricki Goldman-Segall used the concept of 'thick' in utilizing multimedia to do ethnographic research of children's ways of thinking [5, 6].

For our purposes we appropriate this use of the adjective 'thick' to describe a learner's layered, rich, contextual, and often affective understanding of a mathematical concept. Likewise, we use the adjective 'thin' to describe a learner's understanding of a mathematical concept that lacks richness and connectivity. A thin understanding of a mathematical concept would include the memorization of a definition or a procedure with no clear awareness of how such a definition or procedure connects to other mathematical concepts or to one's own previous understanding.

A learner moves from a thin understanding to a thicker understanding – that is, they thicken their understanding – along two dimensions of extension: an increase in the number of conceptual layers related to a concept and an increase in the thickness of each conceptual layer. Of course, the demarcations between layers are not always well-defined. In what follows we provide two examples of mathematical concepts and the conceptual layers they might involve. Our first example – square numbers – is used to further exemplify the notion of thick understanding in general. In the second example we characterize thick understanding as it relates to one of the focus themes of this study – prime numbers – and the particular population of this study – pre-service elementary school teachers.

2.1. *Example of thick understanding*

As a learner's understanding of a concept thickens along the two dimensions the individual layers become intertwined and it is not always clear where one layer ends and another begins. For example, a thin understanding of square numbers might consist of the knowledge that a square number is of the form $n \times n$, such as 4, 25 and 100. Limited to this understanding a learner would, at first, only be able to generate a list of square numbers. With time, and an increase in experiences in generating such numbers, the learner would eventually be able to start recognizing specific numbers as being squares. His or her conceptual understanding of square numbers has thickened, if only slightly. However, it was done so within the same conceptual layer.

On the other hand, having the experience of representing square numbers visually as 'perfect' square arrangements as opposed to triangles or rectangles, or noticing that square numbers have an odd number of factors after successfully solving the *locker problem*†, would increase the number of conceptual layers related to square numbers, and as such, would also serve to thicken the learner's understanding of squares. A thicker understanding of square numbers might also involve observing how they are distributed – that they get further and further apart as one proceeds down the number line from one square number to the next. This experience can be further conceptualized as the sequence of differences being an arithmetic sequence of odd numbers.

These properties that we have mentioned as part of a 'thicker' understanding of square numbers are, of course, all implied by the initial 'thin' definition. However, they belong to certain contexts and experiences (drawing shapes, using number lines, looking at opaque properties, solving problems) that can be triggered in future thinking. They contribute to the cognitive (connectivity to other mathematical concepts) and emotional (connection to specific contexts and experiences) multidimensionality of the learner's understanding.

2.2. *Thick understanding of prime numbers*

A 'thin' understanding of prime numbers may consist only of a definition, often cited as 'primes are numbers that have only 1 and itself as factors', and several examples of small primes. Introducing a different definition can thicken this layer of definitions and numerical properties – primes are numbers that have exactly two factors. Examining the difference between the two definitions makes it clear that the latter is preferable as it explicitly excludes the number 1 from the list of primes. Awareness of existence of infinitely many prime numbers further contributes to creating a thicker layer.

Understanding of primes can be thickened also by the experience of arranging objects into rectangular arrays and realizing that prime number of objects can be arranged only in one-dimensional (or line) arrays. This is a different layer of understanding. A further understanding of primes may include a layer that consists of different algorithms and procedures for determining primality, such as factoring or the sieve of Eratosthenes. Again, each layer of this thicker understanding is implied by the definition, but they are contextualized in specific experiences that can serve as triggers in the future.

In this article, however, we are interested in an altogether different set of experiences than those presented above. We are interested in how students' experiences with a computer-based microworld can help to thicken their understanding of some elementary number theory concepts. In what follows we first introduce this microworld and discuss its potential ability to create a layered, rich, and contextual

† A school has 1000 lockers — one per student. On the first day of the new school year the first student to arrive runs through the school and opens every locker. The second student closes every second locker; the third student changes every third locker (opens it if it's closed, closes it if it's open); the fourth student changes every fourth locker — and so on until the last student either opens or closes the 1000th door. How many doors are left open, and which ones are they?

experience for students. We then present the results of a study designed to examine this microworld's actual effectiveness in providing such contextual experiences.

3. Thickening and a computer-based microworld

3.1. *Microworlds*

Many researchers have argued, along with Goldenberg [7], that well-designed computer-based learning environments can provide a scaffold for reasoning by fostering the development and use of visual and experimental reasoning styles, which greatly complement the traditionally taught symbolic deductive methods. However, for the learning of elementary number theory, few, if any, such computer-based environment exist.

Factor trees and number lines provide one type of visual representation frequently encountered in the domain of elementary number theory. Independently from these visual tools, calculators can support a certain range of numeric experimentation. However, each tool is often used in isolation: neither factor trees nor number lines can be represented on calculators. In contrast to these isolated tools, there exists a class of environments called 'microworlds' which, as Edwards [8] describes, 'embody' or 'instantiate' some sub-domain of mathematics. The *microworld* is intended to be a mini-domain of mathematics that essentially brings such tools together into a phenomenological whole. Thus, microworlds can be seen as specific forms of external representation of a subset of mathematical ideas. The challenge for mathematics educators is to design microworlds that can offer such new external systems of representation while at the same time making them more context-dependent, more descriptive, and less detached from the world in which learners live.

3.2. *Number worlds*

In this study we focus on one such computer-based microworld called *Number Worlds*†. Number Worlds, shown in figure 1, is a web-based applet that can best be described as an interactive number chart. The centre grid contains a two-dimensional array of clickable cells. The numbers shown in the cells depend on the 'world' that has been chosen. Although the basic objects of Number Worlds are the positive integers, or natural numbers forming the **Natural World**, the user can choose another set: **Whole World**, **Even World**, **Odd World**, and **Prime World**‡. Each world displays the corresponding set of numbers.

It is also possible to change the numbers shown in the cells. The user can increase or decrease the start number by 'one row', that is, by the value of the grid width, or simply by one 'cell'. The **Reset Grid** button resets the start number to the first number of the current world. The appearance of the grid can be affected by changing

† The 'Number Worlds' microworld was written in Java and is available on-line as an applet at http://hydra.educ.queensu.ca/;java/NumberWorlds/.
‡ This approach was inspired by Brown *et al.* [9], who uses similar ways of re-examining ideas in elementary number theory by shifting the focus from natural numbers to other domains.

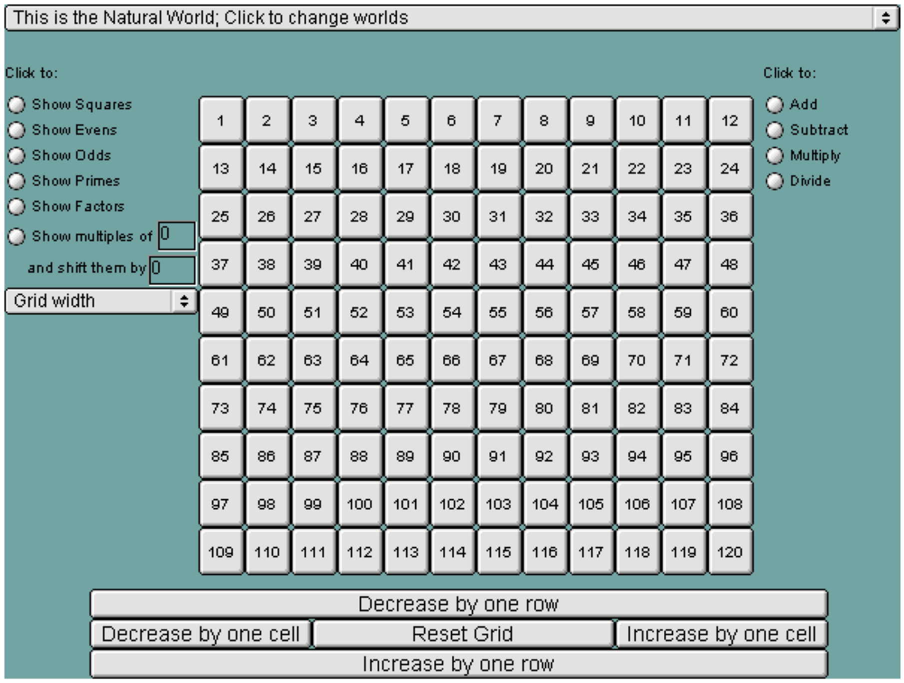


Figure 1. The Number Worlds microworld.

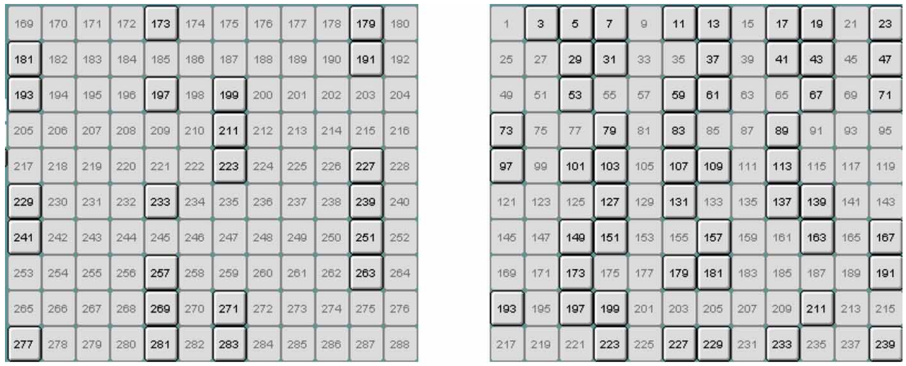
the value of the **Grid width** menu. By selecting values from one to twelve, the user can change the number of columns displayed, and thus the total number of cells. There are always exactly ten rows.

Within each world, the user can highlight certain types of numbers: **Squares**, **Evens**, **Odds**, **Primes**, **Factors**, and **Multiples**. Further, the multiples that are chosen can be shifted by an integer in order to create any arithmetic sequence of 'non-multiples'. Finally, the four basic arithmetic operations are available to the user. The result of the operation, in addition to the two inputs, is highlighted on the grid while a string representing the operation and the result appears above the grid.

3.3. *Number worlds and thickening the understanding of primes*

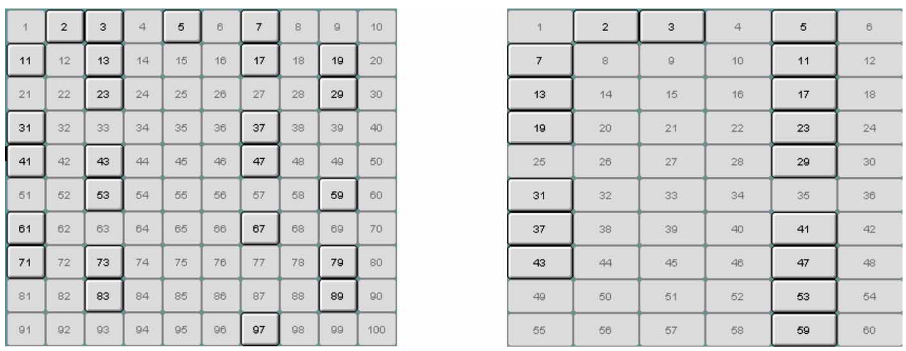
To illustrate how Number Worlds can thicken understanding we focus once again on the concept of prime numbers. We also refine our notion of conceptual layers by distinguishing three types: *numerical*, *visual*, or *procedural*. We do so for the purposes of discussion rather than for the purpose of imposing some kind of structure on the organization of the layers.

As a trivial extension, Number Worlds suggest an additional procedure of checking for primality – just 'show primes' in the microworld and scroll down to check whether a number in question is highlighted. A more substantial contribution of experience with Number Worlds comes from 'seeing' many primes 'shown' at once on a number grid. The knowledge that there are infinitely many primes could be reinforced and articulated by showing very big primes. Of course the display is not infinite, but the fact that prime numbers keep reappearing as the screen is scrolled down to 'large' numbers helps in creating the feeling of no-end (see figure 2(a)).



(a) Large primes in Natural World (b) Show primes in Odd World

Figure 2. Highlighting primes.



(a) Natural World, gw = 10 (b) Natural World, gw = 6

Figure 3. Highlighting primes (gw 10 and 6).

The experience of ‘seeing’ many primes in one display also creates understanding of the distribution of primes: it is often assumed by students that prime numbers are rather ‘rare’, while ‘seeing’ a lot of them on any screen helps adjust this expectation. The understanding of prime distribution could be further thickened by selecting Show Primes within the Odd World (see figure 2(b)), which highlights the cells with the familiar prime numbers (with the exception of number 2). Moreover, all twin primes (that is, primes that differ by 2) will appear in adjacent cells.

Furthermore, when working in Number Worlds and showing primes with different grid width and attending to their distribution, it becomes evident that primes appear only in certain columns (see figures 3(a) and 3(b)). For example, when a grid width is set to 6, starting with the second row, there are only two columns – the first and the fifth – in which primes appear. Consideration of this display can thicken the understanding of distribution of primes and also their structure in both visual and numerical layers – they cannot be a multiple of 2 or 3 and therefore cannot appear in columns 2, 3, 4 or 6. If this observation is further formalized to the fact that, with the exception of 2 and 3, all primes are one more or one less than a multiple of 6, then a procedural layer has also been thickened.

Having mentioned possible avenues for thickening students' understanding of primes, we wish to share how Number Worlds has thickened our own understanding of primes: We came up with the generalization of the twin-primes conjecture. Twin primes are easy to detect on the visual display – not only in Odd World, but also in Natural Worlds. So with the help of Number Worlds we notice not just 3 and 5 or 17 and 19, but also, for example, 191 and 193. In addition, showing primes at different even width of the grid, there are always appear to be two highlighted cells one underneath the other, that is, the difference between the numbers in these cells is the width of the grid. For example, with grid width set to 10, we notice pairs like 79 and 89 as well as 307 and 317; with grid width set to 12 we notice pairs like 101 and 113, 337 and 349, 431 and 443, etc. This observation of the number worlds display leads to the following generalization of the twin prime conjecture: given even number n, there are infinitely many pairs of primes of the form $(p, p+n)$. Then the twin primes conjecture becomes a special case where $n=2$. We encourage readers to verify this conjecture for their choice of n.

4. Research design

4.1. *Situating the study*

This article is part of a larger study that more generally examines the effects that a computer-based learning environment has on preservice elementary school teachers' understanding of elementary number theory concepts [10]. Prior research on the concepts in question – factors, multiples, division with remainder, prime factorization, and distribution of primes to name a few – has shown that these concepts are not well understood by this population [9, 11–16]. As such, one of the main goals of our studies was to create an experimental and visually based environment that would present these problematic concepts in a novel and contextually rich fashion.

4.2. *Methodology*

We asked 90 pre-service elementary school teachers enrolled in a course 'Principles of Mathematics for Teachers', to use Number Worlds. The course includes a chapter on elementary number theory that deals in part with concepts of divisibility, divisibility rules, prime and composite numbers, factors and multiples, greatest common factor and least common multiple. This chapter, as well as a textbook-based homework assignment on the topic, was completed shortly before the participants were introduced to Number Worlds.

The participants were provided with written instructions about web access to Number Worlds, a description of the microworld's commands and capabilities, and a list of suggestions for explorations to be carried out before turning to the main assignment. The list of tasks for exploration is found in Appendix A.

These tasks were intended to familiarize the participants with various aspects of Number Worlds and create an environment of experimentation and conjecturing. To support the participants, computer lab hours were scheduled at different times on different days. Participation in the lab was optional; nevertheless about one-half of the participants chose to work, at least for part of the assignment, in this environment.

Our data for the research project consists of three main sources: (1) observations of students' work during the lab time, (2) written assignments, and (3) clinical interviews.

1. During the allocated lab hours, the work of the students was observed and supported where necessary. We noted frequently asked questions, chosen routes for exploration, student's conjectures, as well as their approaches toward testing their conjectures. We also used these observations as a guideline for designing the interview questions.
2. Following the tasks for exploration, participants were provided with an additional list of tasks, to which submission of written response was requested. Participants had two weeks to complete the assignment. The list of tasks can be found in Appendix B. These tasks specifically requested that the participants make an observation and, whenever possible, generalize it.
3. After the completion of the written assignment, clinical interviews were conducted with seventeen volunteers from the group. The interviews lasted 30–50 minutes; the Number Worlds microworld was available to the participants at all times during the interview. These interviews had a semi-structured character. That is, the questions were designed in advance (see Appendix C), but the interviewer had the liberty to follow up with prompts, include additional questions, or omit questions due to time considerations. These interviews were then transcribed.

For the purposes of this article, however, we focus on only three questions from the interview and one question from the written responses to the tasks. These were chosen because the participants' responses to them best exemplify the diverse and contextually rich understandings that the Number Worlds helped our participants access. The question were:

Interview questions

- Here we are looking at the multiples of 7, with a shift of 2. The grid width is 7.

  What would happen if we change the shift to 5?
  What would happen if we change the multiples to 8
  (Returning to multiples 7, shift 2) How could we get 2 highlighted?

- Consider each picture. Is it a picture of factors, multiples, primes or squares? How do you decide?

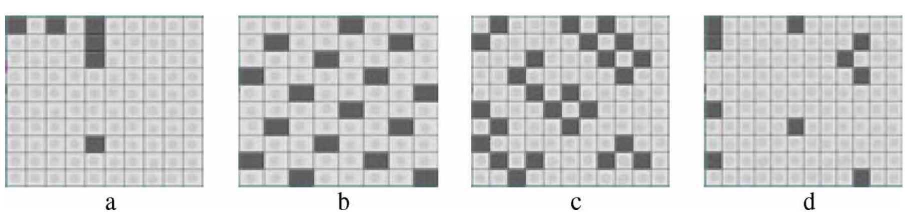


Figure 4. Various Number World Images.

- General invitation for reflection: What have you found surprising or helpful or interesting in your experience with Number Worlds?

Written response task

- Write a paragraph that describes the visual image you have of multiples and the visual image you have of factors.

5. Results and analysis

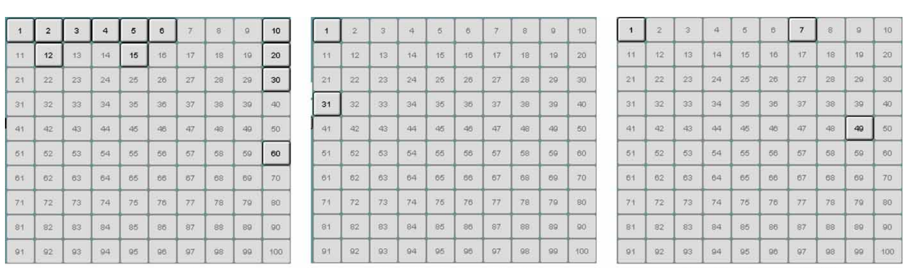
The data from all three sources were analysed and eventually organized according to the three themes that emerged most prominently from the study, namely, the understanding of factors, multiples, and primes. We first present the results in the form of representative excerpts from the interviews and from the written tasks. That is, we present how the participants' understanding of the concepts of factors, multiples, and primes were thickened from their experience with Number Worlds. We then discuss the affective overtones that these new understandings produced across the participants and across the tasks.

5.1. *Factors*

Because our participants had been exposed to the concepts of elementary number theory in their mathematics course shortly before their work with Number Worlds, they had already developed several layers of understanding of factors, regardless of whether those understandings were correct, fragmented, or incomplete. They were first introduced to the concept of factor in the context of 'divisibility'. That is, if *a divides b* then *a* is a factor of *b* (and *b* is a multiple of *a*). This definition was most often used within a numerical layer of understanding, that is, providing a method for numerical verification as to whether or not a given number is a factor of another. They had then used this concept of factor to further explore prime factorization (procedural layer) and factor trees (procedural and visual layer).

Number Worlds, first and foremost, provides a very different visual representation of factors from what had been seen in the course content. When '*show factors*' is selected and then an input is given all the factors of that input, including the number itself, are highlighted. A composite number like 60 will display many factors (see figure 5(a)), a prime number will result in two factors being displayed (figure 5(b)), and showing factors of a square number is the only way to produce an odd number of highlighted cells (figure 5(c)).

One of the powerful visual understandings that students are left with as a result of this representation of factors is that of 'clustering' of factors near the top of the



(a) Factors of 60 (b) Factors of 31 (c) Factors of 49

Figure 5. Showing factors.

number grid. Cathy makes this statement explicitly during the interview in referring to figure 4(a).

Cathy:	Okay, well I would say this is showing multiples of, I mean factors of a number.
Interviewer:	And why do you think that?
Cathy:	(pause) Because of the way that they're clustered at the top, like one separate down at the bottom, so these would be the smaller factors in a larger number... Um, and just the fact that it's, there's no pattern, or there's no continuing pattern.

Cathy also mentions the fact that there is 'no pattern'. This observation was very common among the participants. In the context of the written task where, in comparing factors and multiples, almost every participant made mention of this fact. In the interview setting, where the above comment was made, only three participants mention this lack of pattern. This is likely because in this setting not having a pattern narrowed the choices from four to three as the images of primes (figure 4(c)) and squares (figure 4(d)) also produced no easily discernable pattern. However, rather than simply classifying Cathy's claim as the recollection of her written response we propose that when she says 'no pattern, or there's no continuing pattern' the key word for her is 'continuing'. We believe that Cathy is referring to the finiteness of the highlighted cells, a common theme that will be discussed below.

Returning to the idea of 'clustering', Katherine also makes reference to it, but her claim is more implicit, as she opts instead to speak of the specific numbers that may be involved.

Katherine:	Um (pause) number 1 could be factors and (pause)...
Interviewer:	Okay, why is it?
Katherine:	I'm trying to visualize it, because, well I mean it's like one like this, you could probably do more of like (pause) because if we did like 5 and you have that one in 5, or if you did like 6, and if you did it 1,2,3,6. So you could see that going this way, hmm, 12, we did 24 show factors then 1,2,3,4,6,12,24. So you could see like the one row above...
Interviewer:	Okay, so this idea of having the top row highlighted...
Katherine:	Yeah.

This new visual understanding of factors proved to be a very powerful one. In fact, almost all of the participants made some claim to the effect that factors would be presented as being clustered at the top, in the first row, or around the smaller numbers. However, new this visual understanding of factors carries an old misconception. Prior research has shown that the idea that factors 'cluster' near small numbers is a robust misgeneralization [15] that ignores numbers composed of large primes such as $11021 = 103 \times 107$. Such numbers would not have any factors, other than 1, among the small numbers. However, the spatial relation of the four factors 1, 103, 107, and 11021 would still be representative of clustering near the 'top', if all the numbers up to 11021 could be displayed. So, while not necessarily displacing the robust misgeneralization of factors being small, the novel visual understanding of factors adds a new, and correct sense of spatial distribution of factors.

This new conceptual layer was itself thickened with another visually represented fact, the understanding that the number of factors is finite. This observation was most prevalent in the participants' responses to the final task of the written assignment where they were asked to compare the visual images of factors and multiples produced by Number Worlds. Comments such as 'factors are finite' and 'factors don't go on forever, they stop at the number you want to factor' were common among the responses to this question. However, not everyone's comments are as concise.

> Aimee: Um, something like this, I would think it has to do with factors, because um, just because it's like a closed member, it doesn't seem to continue as like if let's say that I said, um, show factors of like 64, you see how nothing goes higher than, and it's kind of bunched together . . .

When commenting on the first image Aimee makes the statement that it is like a '*closed member*'. In doing so she is likely referring to the finiteness of the set of factors. The use of 'closed' can be seen as an inappropriate attempt to use terminology recently introduced in the course. Aimee also makes references to the way that factors are 'bunched together'.

Andrew also comments on the finiteness of the highlighted cells during the interview. However, in doing so he adds an explanation as to why they are finite that is a variation on the idea of 'clustering'.

> Andrew: Well that's a factor [pointing to figure 4(a)].
> Interviewer: And how do you know that?
> Andrew: Um, well for one thing you've got a finite amount of highlighted squares.
> Interviewer: How do you know it's finite?
> Andrew: Okay, because there's, hmm, yeah because it could go on forever. Um, well when we see factors, you see a steadily decreasing number of spaces in between each highlighted square, and that's what we're seeing here . . .

We should point out that Andrew was the only participant who did not explicitly make reference to the way the highlighted cells clustered. Instead, he speaks of the spaces in between highlighted cells. This in itself would not be that significant, except for the fact that Andrew was also the only participant who attempted to explain why the number of spaces decreases as one moves up the grid. He does so in responding to the last interview question in which he is asked if he found anything surprising in working with the Number Worlds applet.

> Andrew: Well, when you're showing factors [. . .] the next greatest factor will always be only around half as much, so you, that steadily decreasing pattern is never something I had thought about.

Of course, this generalization is not true, as it works for only half the numbers, that is, for numbers that have a 2 in their prime factorisation (the even numbers). However, in making this statement Andrew demonstrates that he understands that the 'clustering' of numbers is the result of the nature of all factors rather than simply being the way it is. We feel that Andrew has demonstrated the existence of another conceptual layer of understanding of factors that, although visually in nature, is based on numeric and procedural interpretations. Furthermore, we feel that this layer can be thickened through more experience with Number Worlds examining factors with this frame of reference in mind. Such experience would most likely allow Andrew to rethink his generalization of 'half as much'.

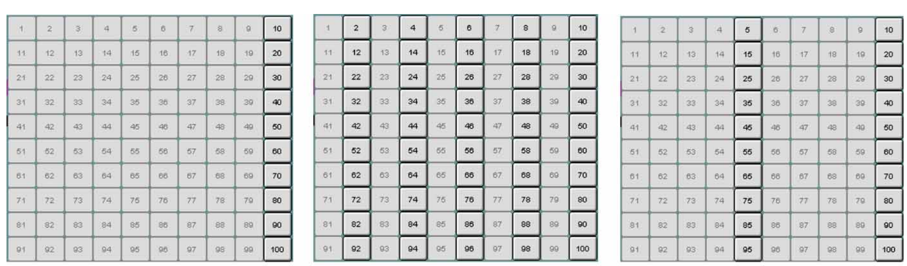
5.2. *Multiples*

The relationship between the input 'show multiples of' and the grid width determines the type of pattern. For example, in the Natural World, showing multiples of ten on a grid width of ten results in one highlighted column (see figure 6(a)) referred to as a 'stripe'. Showing multiples of two or five on the grid width of ten (see figure 6(b) and 6(c) respectively) results in five or two highlighted stripes, respectively. In general, a pattern of stripes emerges when the grid width is divisible by the input to 'show multiples of'.

However, this is not the only way to produce a pattern. Figure 7 shows multiples of 2, 4, 5, and 7 on a grid width of 9. We refer to the patterns in (b) and (c) as 'diagonals', and refer to the pattern in (d) as 'disconnected diagonals'. The pattern in (a) can be seen as diagonals as well, however, it is more naturally identified as a checkerboard pattern.

These images are examples of the following general relationship: 'diagonals' are displayed by multiples of m on a grid width of n, where $n \pm 1$ is divisible by m, and 'disconnected diagonals' are displayed by multiples of m on a grid width of n, where neither n nor $n \pm 1$ are divisible by m. 'Checkerboards' are a special case of 'diagonals' and they are displayed by multiples of two on any odd grid width. (This precludes the creation of a 'real' checkerboard pattern of width eight.)

Because of the extensive work the participants did in trying to generate and manipulate patterns of stripes, diagonals, and disconnected diagonals the most



(a) Multiples of 10, gw = 10 (b) Multiples of 2, gw = 10 (c) Multiples of 5, gw = 10

Figure 6. Showing multiples on a grid width of ten.

(a) m = 2, gw =9 (b) m = 4, gw = 9 (c) m = 5, gw = 9 (d) m = 7, gw = 9

Figure 7. Relationships between multiples and grid widths.

powerful impression that Number Worlds left was with respect to multiples. In responding to the interview question regarding the different images no one hesitated in stating that the second image represented multiples. This is most clearly demonstrated in Aimee's comments.

> Aimee: Okay, um, like when we've done any like the multiples, we've come across a lot of diagonal lines... So um, automatically I would think that this would be... would be a multiple, it would have to be with multiples.

Some of the participants did not stop here, however, and felt it important to figure out what multiples were being displayed. Cathy was one such person.

> Cathy: Okay, this would be multiples of something, 1,2,3,4,5,6,7,8,9,10... of 5 in some grid width, well like multiples of 5... because of the pattern... continuing pattern... consistent pattern.

In doing so, however, she also elaborates on the pattern in way that Aimee does not – 'because of the pattern... continuing pattern... consistent pattern'. The repeated use of the word *pattern* with increasingly specific modifiers speaks to the importance she places on this feature of multiples.

Andrew also calculates what the 'multiples of' is. We present his responses here, however, not because of this fact, but because we wanted to demonstrate how important the number spaces between highlighted cells is to him. To do this we present very truncated excerpts from his discussion on primes and squares as well.

> Andrew: So that's multiples [pointing at image 4b], because you have the same amount of space between each, I think it's multiples of 5 (pause)... Yeah, it almost looks like primes [pointing at image 4c]... there doesn't seem to be the steadily decreasing number of spaces which would indicate factors... I don't think it's multiples for sure [pointing at image 4d], because there's the variance between the highlighted squares is not even, it is always even with multiples...

Andrew uses phrases such as 'the same amount of space between each', 'the steadily decreasing number of spaces', and 'the variance between the highlighted squares is not even, it is always even with multiples'. We have already mentioned that Andrew's focus on the spaces as opposed to the highlighted cells is interesting to us because it seems to lead in the direction of more analytic work. Andrew already demonstrated this in the context of factors. For multiples, such a disposition would allow a learner to ignore the distraction of varying grid width. We also feel that attention to spaces, or differences, is the only way to satisfactorily resolve that the last image represents squares. As such, we see the focus on spaces as first and foremost thickening the visual layer of patterns by adding some numerical understanding to it. Eventually, however, we foresee that with more experiences the numeric layer, would be strengthened by coordination of spaces and/or differences.

We now turn our attention to multiples in the context of grid width. The added facet of grid width has introduced a mathematical relationship into Number Worlds that is not inherent in our usual understanding of multiples. This mathematical relationship between 'multiples of' and grid width is the major element of Number Worlds and allows learners to produce and manipulate the aforementioned patterns. As such we now present some of the discussions on multiples in the context of manipulating grid widths.

Even though the participants had by in large done well on Written Response Task 2 (see Appendix B) not one of those interviewed deferred to the generalizations they had produced in their written work when responding to the Interview Question 1 (see Appendix C), namely, that stripes occur when the 'show multiples of' entry is a multiple of the grid width, diagonals occur if the grid width is one more or one less than a multiple of the entry, and so on. Instead they all chose to, in one way or another, generate the cells that will be highlighted and from this determine the pattern. As such, this is indicative of the existence of a procedural layer of understanding that developed as a result of their exposure to Number Worlds. This is seen in Amber's response.

Interviewer:	What do you think is going to be highlighted?
Amber:	When you show multiples of 7 and shift them by 2?
Interviewer:	Yeah . . .
Amber:	Um, I guess 9, 16, 23, 30, 37 . . .
Interviewer:	And how do you make this decision?
Amber:	Because multiples of 7 include 7, 14, 21, so if you add 2 to each of those, then that's the number that you're going to get.
Interviewer:	Okay, and now if I keep the same width I change multiples of 7 to multiples of 8. What do you think will happen?
Amber:	Which is multiples of 8, that would be diagonals I think, because it would be 8, then it would be 16 and it will be shifted over, so it'll be (pause) I think it'll be 13, 21, and . . .

David and Kori also generate the cells, but in a slightly different way than Amber.

Interviewer:	What will happen now if we change the multiples to 8?
David:	Oh . . .
Interviewer:	Like show multiples of 8, rather than 7?
David:	Oh okay, we'll have an interesting line, we'll have it a diagonal going across, because of the grid width, so um let's try that, so the shift is 0 yes?
Interviewer:	So as you predicted, there's this diagonal going on and you said something, it relates somehow to the grid width, can you please elaborate on this?
David:	Oh uh, because the grid width is, is 7, um, it's going to push, well I don't know how to explain it, but . . . I know that in the first row we won't have any numbers highlighted. In the second row, we'll have the first, in the third row we'll have the second number, in the third row we'll have the third number, in the fourth row, 1,2,3, in the 5th row we'll have the fourth number, in the 5th row or the 6th, in the 6th row, and on until, well we're going to skip, that's interesting um, we skip this one because it's, yeah, because this is only you have 7 numbers, and the 8th number will be in the next column. So then this diagonal will start again.

Interviewer: Immediately you said diagonal and you immediately pointed to your diagonal going right, I wonder where did this image come from?

Kori: Well at first, it's because the grid width is at 7 so therefore is you have multiples of 8, it's always going to be um, it's hard to explain, it's one wider than the grid, so therefore it's, it goes a whole length and then 1, and the whole length plus 2, because it's 7 and you're going one more, it's hard to explain, but it always puts it, offsets it by 1 because you're always counting 7's but by adding multiples of 8 it's offsetting it by 1. And so that's why I got the diagonal pattern, and then you can just look from there and see where exactly it is and then the shifting . . .

David and Kori found a way of generating the cells in a spatial sense as opposed to the more numeric way the Amber chose. We see these as representing two further conceptual layers for mediating an understanding of Number Worlds. Amber's understanding is very numeric in nature and does not take into account the grid width at all. On the other hand, David and Kori have an understanding that is more visual, although not solely so, that creates a connection between 'multiples of' and grid width. However, neither of these understandings speaks to the definition of multiples, *per se*. Perhaps the powerful visual representations of multiples coupled with the introduced complexity of operating on grid width and 'multiples of' simultaneously created too big an obstacle for the participants to overcome. On the other hand, the temptation of a readily available procedural layer of how to predict where highlighted cells are may have negated the need to develop further layers.

5.3. *Primes*

We have already provided an in-depth discussion of how primes are featured in Number Worlds. As such, we will proceed directly with the presentation of some of the results.

Most of the participants were troubled by the apparent 'pattern' presented in image 4(c) as this upset their constructed understanding that there is 'no pattern' for prime numbers. This can be seen in Vanessa's response in which she is so struck by the apparent pattern that she is willing to consider it as an image of multiples.

Vanessa: The third one I'm not sure about, um, I can see that there's a pattern, somewhat, um, so I'm pretty sure it's not primes and I'm pretty sure it's not factors either, just um from, yeah I'm pretty sure it's not factors, so I would just from like deductive logic I would conclude that this is multiples . . .

Interviewer: Multiples, okay. And what are they multiples of?

Vanessa: Um (pause) I'm really not sure because it's really throwing me off that there's um like for right here, there's a number like right beside it, um, and that's not the case here, um, I'm just, I don't know, this one is really . . .

Interviewer: Yeah, okay . . .

Vanessa: Throwing me for a loop . . .

Interviewer: It seems like um you have a hard time accepting it's multiples and the only way . . .
Vanessa: Yeah . . .
Interviewer: That you are accepting it is because you don't think it's any of the other ones . . .
Vanessa: That's right.

Vanessa is demonstrating that she has a visual understanding of primes that, although not serving her well, is clearly a context dependent layer formed through her interaction with Number Worlds.

Andrew demonstrates how this conceptual layer can be thickened.

Andrew: And this guy here [pointing at image 4d] . . .
Interviewer: The last one?
Andrew: What have we got to show? (Pause) That might be part of a series of factors, but, or part of a very large factor, because it has some of the qualities that we've seen here, but just looking to see what else we can do. (Pause) Yeah, I don't think it's multiples for sure, because there's the variance between the highlighted squares is not even.
Interviewer: So it's not multiples, could it be primes?
Andrew: I don't think so, because primes is very steady pattern, well not pattern, but familiar look to it, not matter where you go, you're always going to get something like that . . .

Whether or not there exists a pattern for primes is open to the interpretation of what is meant by pattern. Certainly there is no repeating pattern of primes, but there is regularity in its distribution, both in placement and density. These ideas are not lost for Andrew.

Cathy demonstrates that she possesses a different layer of understanding that is more numeric, and perhaps a little procedural, in nature.

Cathy: This looks weird (laughter) [pointing at image 4c] that is, okay so we've got primes, multiples and factors. I've never seen anything like that before, so I either haven't done it or haven't seen it . . .
Interviewer: Remember it doesn't necessarily start at 1 . . .
Cathy: Right okay, I'm going to start at 1, 1, 2, 3, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13 (pause) I'm thinking it might be . . .
Interviewer: You've listed 1,7,11,13,17,23 and 29 . . .
Cathy: Which happen to be all prime numbers, and all odd numbers, but the space between them is 1, 6, 4, 2, 4, like skipping by, skipping by 6, skipping by 4, skipping by 2, skipping by 4, skipping by 6, skipping by 6, skipping by 2, skipping by 6, that's a prime, looks like a relationship between prime numbers . . .
Interviewer: What, why, how do you know that?
Cathy: Because I know that a prime number plus 2 plus 6, is a pattern of the primes . . .
Interviewer: Oh okay. So how come you think um, now that you know it's prime, why, how come you think at the beginning when you first looked at it, that you didn't know it, I mean it seemed like

something was, it seemed like you had a gut feeling that maybe it wasn't primes, or something else . . .

Cathy: Yeah, um (pause) well because I don't think I ever did show primes and saw the pattern, I think in my, because on the homework we were working with the pattern of diagonals and relationship between the patterns and we were asked questions about those, so this was just a pattern, I may have seen it, but I hadn't studied it or become familiar with it, so I had to just figure it out. I didn't recognize it right away, or I hadn't studied them.

In considering this image Cathy invokes a strategy of looking at the difference between highlighted cells. This is the first time she focuses on the spaces between cells as opposed to simply recalling a familiar pattern. She does this, as she says, because '*I hadn't studied it or become familiar with it*'. However, in doing so she recognizes a pattern of differences between cells as 1, 6, 4, 2, 4, . . . and claims that this is a pattern of prime numbers. This understanding was surprising to us and its origin seems to be rooted in the examination of the distribution of prime numbers on various grid widths in Written Response Task #3 (see Appendix B) of the written assignment, or in a similar exercise performed in the context of a textbook-based homework assignment. Regardless, this understanding seems to be contextual.

Finally, we look at Katherine who, like many other participants, invoked deductive reasoning to settle on the nature of image 4(c). In doing so she also demonstrates how powerful her image that prime distribution is not very dense.

Interviewer: Okay, so what about this one here, the third one?
Katherine: And then that one, oh we're left with primes . . .
Interviewer: Yeah, you don't like that option?
Katherine: No, I don't think I like it that much.
Interviewer: Why not?
Katherine: Because, this just seems too close. Prime numbers aren't that close. I don't know, it's just, it just doesn't seem like an accurate representation, because if you do like primes, like if these two cells are highlighted, it means there's like a difference between them, but with prime numbers, there's a difference between primes too though, not this much though. I don't know . . .

Katherine is uncomfortable with the fact that the highlighted cells are '*too close*' and makes the very firm statement that '*prime numbers aren't that close*'. We speculate that this resistance to the presented image of prime numbers is a hold over from a conceptual layer constructed from classroom experience where prime numbers are *special* numbers and, as such, less numerous.

5.4. *Square numbers*

The inclusion of square numbers in the interview was more of a result of wanting to provide a context in which to discuss factors, primes, and primes. As such, its inclusion was fruitful. In trying to determine what image 4(c) was many participants spoke about factors, multiples, and primes. As a result, as an instrument for discerning participants' conceptual understandings of square numbers it produced almost nothing of value, nor was it expected to. We therefore include this section only for the purpose of informing the reader that we have not forgotten to, or failed to, analyse the data with respect to square numbers.

5.5. *Affective overtones*

One characteristic aspect of our notion of thickening – which, for example, helps distinguish it from the notion of a schema – relates to the non-cognitive overtones attached to different layers of understanding. Thus far, we have focused primarily on the cognitive dimension of participants' understanding of various elementary number theory concepts. However, in our interviews, we were struck by the extent to which these understandings were embedded in larger experiential structures: their development was accompanied by feelings of novelty, attraction, surprise, and satisfaction, and supported through the participants' construction of rich, relevant analogies and metaphors.

For example, Amy discusses the affective dimension of her experience. Note how her emotional response to the patterns created by multiples is supportive of her understanding of multiples: 'The repeating, infinite pattern is calming. I can see with my eyes open the pattern on the grid right in front of me. They all seem to be linked together, no matter how far away from the next multiple they are, they are all connected to each other'. For Kori, the novelty of the representation provoked a new way of thinking about multiples and factors that was similarly supportive of her developing understanding: 'I didn't really realise the patterns of multiples until it was showing like visually in front of me, I never, like I knew they went on but I didn't, the difference between the multiples and the factors and the patterns that they created, I didn't really know that before. I never thought of it before'. Lisa explains how her explorations were motivated by a sense of enjoyment: 'It was fun, just playing around with different numbers, seeing the patterns you could make, like often times I wouldn't look at the numbers, I would look at rows and look at patterns, and how those corresponded to each other and put the numbers in afterwards, so experimenting around with that, I could figure out different ways of getting the things without necessarily having to use numbers'. The things she figured out through playing and experimenting were thus coloured by her sense of 'fun'.

We believe that it is only through these kinds of thick experiences, where cognitive, affective, and aesthetic modes of meaning making support each other, that learning becomes transformative, as it was for Vanessa, who describes her 'new' way of looking: 'I will definitely take what I've learned about manipulation of numbers and seeing patterns with me as I use numbers in other contexts of my life [...] I definitely learned a lot of different ways of looking at numbers, I don't just, like I'm looking at the number 64 right now, before I was just kind of taking advantage of, of it being 64, but now I kind of think how is that related to the other numbers that are in the set of numbers that I'm looking at'.

To a certain extent, we can assess the participants' emotional connection to their understandings through the types of understandings they constructed, particularly through the personal, and powerful metaphors they brought to bear. Diane seems to construct an analogy for multiples that is at least partly inspired by the microworld's visual representations: 'Multiples work together creating a pattern of some sort, such as a collection of columns, a checkerboard, a diagonal or collection of connected and disconnected diagonals. They spread far and wide, and keep on going forever, but are still somehow connected. Multiples make me think of emails which are written by one person, starting out at that one point and then keep getting forwarded and sent on and on...'

If we consider the view of learning as 'connecting to what you already know' then we see in Diane's analogy a powerful evidence that she has succeeded in building a solid layer of understanding about multiples, one that is connected to her everyday experiences. She goes on to say that, 'Factors make me think of handwritten letters which are sent to only one or a few people. The number for which you are finding the factors is like the sender who is far from home, and they send a letter home always (how is 1–always a factor), and then to certain people close to home (there are 2, 3, sometimes 4 or 5) and then the odd letter to a person in between. The number of letters sent, though, is limited, and they don't get passed on and on'. Her two analogies not only provide rich ways of thinking about two different concepts, but also reveal the ways in which the two concepts are related – their similarities and differences.

The participants constructed many different analogies for prime numbers, factors and multiples, some directly inspired by Number Worlds and others less so. We are intrigued by the possibility that the preponderance of such constructions is related to thickness of understanding. Such a possibility would certainly be supported by the arguments of scholars such as Lakoff and Nuñez [17], who have argued that all (mathematical) understanding is achieved through metaphorical thinking. Certainly, it seems plausible that analogies such as Diane's create the potential for greater triggering: the more connected an idea, the more accessible it is.

6. Summary and conclusions

In this study we focused on the nature of preservice elementary school teachers' understandings of several concepts in elementary number theory evoked in the Number Worlds microworld. In particular, we focused on the concepts of factor, multiple and prime number. We examined the notion of 'thickness' with respect to understanding mathematical concepts and alluded to various attributes of Number Worlds that may have been responsible for provoking different layers and depths of understanding, what we call 'thick understanding'. We have drawn on data from the interviews conducted with the participants and their written assignments. In the interviews, the participants were involved in problem-solving situations by being asked to solve a series of questions related to Number Worlds. The assignments, in addition to offering several problems, invited a more discursive situation where the participants could 'tell us what they know' about prime numbers, factors and multiples. What we find significant in our study is the way in which the properties, definitions, and descriptions that the participants offered in that discursive situation were actually triggered during the interviews.

Our goal was to identify ways in which different layers of understanding emerged in the mathematical activities of participants. The novel visual representations offer, by definition, a new layer of understanding, but they also provided the participants with an unfamiliar way of 'seeing' ideas they had already encountered – provoking a certain amount of cognitive dissonance and therefore focusing of attention in an emotional context that seems to have been positive. Participants' ability to mobilize different layers of their understandings as triggered in a problem-solving situation is the most compelling evidence of achieving a thicker understanding of concepts. We have shown that the participants' experience with Number Worlds was significant in this achievement. What we now have is a map of available possibilities – possibilities that can be used to inform the design of activities, assignments, and possibly new microworlds.

Appendix A. Tasks to explore in number worlds

1. Can you highlight the even numbers without using the Show Evens button? Can you highlight the odd numbers?
2. Set your grid width to 10. Show multiples of 10. What could you do to highlight all the numbers in the third column?
3. Can you find a number smaller than 45 that has more factors than the number 45?
4. Can you get the numbers 7, 14, 21, ... highlighted? Can you get the numbers 8, 15, 22, ... highlighted?
5. Change to the Odd World. Try adding two numbers. What do you notice? Try multiplying two numbers. What do you notice? Do the same thing in the Even and Prime worlds.
6. What could you do to highlight the number 169 (without clicking it directly!)? How about the number 320? Use the Increase by one row button to check your answer.
7. Can you get all the numbers in the first row of the grid highlighted? (Hint: you can change the grid width.)
8. Choose Show Primes in the Natural World. Now go to the Odd World and choose Show Primes. What do you observe about the location of the highlighted numbers?

Appendix B. Written response tasks

Task 1. Focus on factors

1. Can you give three examples of numbers that have exactly 2 factors?
   Can you give three examples of numbers that have exactly 3 factors?
   Can you give three examples of numbers that have exactly 4 factors?
   Can you give three examples of numbers that have exactly 5 factors? Check your examples with Number Worlds.
2. Can you find a number greater than 500 that has exactly 4 factors? Describe how you found this number and explain how you could find another such number.
   Can you find a number greater than 500 that has exactly 5 factors? Describe how you found this number and explain how you could find another such number.

Task 2. Stripes, checkerboards and diagonals

1. Describe at least three different ways for creating a grid of stripes, working with different grid widths. (Two examples of a 'grid of stripes' are shown below.) Describe a general procedure for creating a grid of stripes.

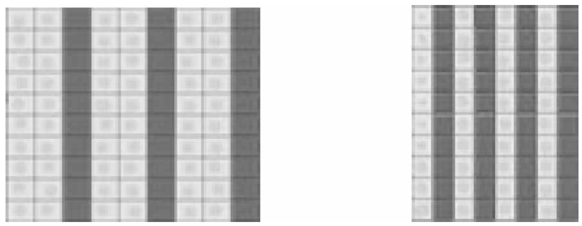


Figure 8. Grids of stripes.

2. Describe at least three different ways for creating a checkerboard grid. Describe a general procedure for creating a checkerboard grid. Is it possible to create a checkerboard for any grid width?
3. Describe at least three different ways for creating a grid of diagonals (either left to right or right to left). Describe a general procedure for creating a grid of diagonals.
4. Describe at least three different ways for creating a grid of 'disconnected diagonals', as shown in the figure below. Describe a general procedure for creating a grid of disconnected diagonals.

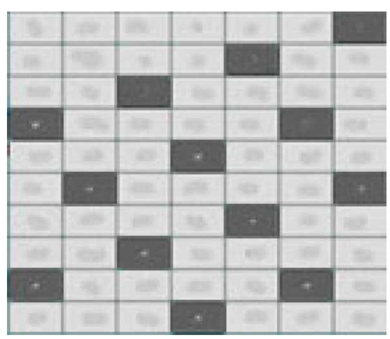


Figure 9. Disconnected diagonals.

5. What do all the visual images you created in a) through d) have in common?

Task 3. Prime distributions

1. In the Natural World, set your grid width to 4. Choose Show Primes. What do you notice about the highlighted numbers? Explain.
2. Now set your grid width to 6. Choose Show Primes again. What do you notice about the highlighted numbers? Explain.

Task 4. Multiples

1. Using Number Worlds with a grid width of 10, someone has chosen to show the multiples of k. If there are three cells highlighted, what are the possible values of k? Explain your answer.

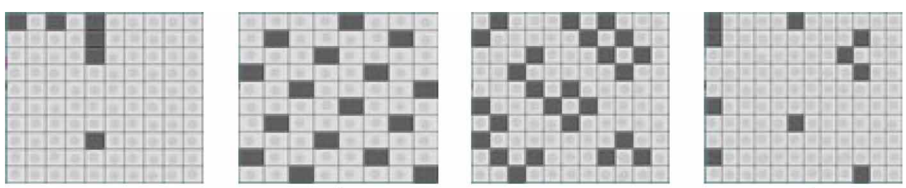
2. How would the answer in part (a) change if more than three cells were highlighted? Explain.
3. How would the answer in part (a) change if the grid width was different than 10?

Final task

Write a paragraph that describes the visual image you have of multiples and the visual image you have of factors.

Appendix C. Interview questions

1. Here we are looking at the multiples of 7, with a shift of 2. The grid width is 7. What would happen if we change the shift to 5?
   What would happen if we change the multiples to 8
   (Returning to multiples 7, shift 2) How could we get 2 highlighted?
2. The grid width is 10, with multiples of 4, shift 2 highlighted. Can you create a grid of stripes by changing only **one** thing (either grid width, multiples or shift)?
3. The grid width is 9, with multiples of 4, shift 2 highlighted. The first number in the grid is 190. How else could we highlight the same numbers?
4. Show how you could verify the following statements using Number Worlds: (a) the set of even numbers is closed under addition; (b) the set of odd numbers is closed under multiplication; (c) the set of prime numbers is closed under multiplication.
5. Is there a multiple of 7 in the 9th row of your grid? What about the 23rd row?
6. Consider each picture. Is it a picture of factors, multiples, primes or squares? How do you decide?



a b c d

Figure 10. Various Number World images.

7. General invitation for reflection: What have you found surprising or helpful or interesting in your experience with Number Worlds?

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