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ODDS AND ENDS OF ODDS AND EVENS: AN INQUIRY INTO STUDENTS' UNDERSTANDING OF EVEN AND ODD NUMBERS

ABSTRACT. Differences in preservice elementary school teachers' perceptions between divisibility by two, or 'evenness', and divisibility by another number have been observed. This led to an inquiry into participants' understanding of the parity of the whole numbers. The results reveal that the equivalence of the number properties of being 'even' and being 'divisible by two' is not taken for granted. Rather, the parity is often perceived as a function of the last digit of the number. The extent of this perception is investigated. Some pedagogical approaches are considered.

The distinction between even and odd is as ancient as the study of Mathematics. Plato defined Arithmetica as 'the theory of the even and the odd' (van der Waerden, 1955). For Pythagoreans, even and odd were not only the fundamental concepts of arithmetic, but indeed the basic principles of all nature. Aristotle (Metaphysics A5) expressed this point of view listing the even–odd antithesis among the 10 basic principles of nature, together with bounded–unbounded, unity–plurality, male–female, etc. (ibid, p. 109). The Pythagoreans in their number mysticism looked upon even and odd as the roots of all things. The even numbers were called feminine, the odd ones masculine. Many abstract concepts were defined by numbers. Number 5 for example was identified with marriage, as the sum of the first masculine and the first feminine numbers (Burton, 1985).

A distinction between even and odd numbers is basic in our practice, mathematical practice and daily practice, and it has mathematical and cultural origins. Nevertheless, students' understanding of even and odd has received scan attention in mathematics education research. Students' work with odd and even was used as a powerful example in investigation of high school students' reasoning (Edwards, 1992). Students' ideas regarding even and odd were used by Ball (1993) in a discussion of pedagogical dilemmas in teaching 3rd grade students. However, the topic of even and odd probably was perceived as not rich enough to become the main content focus of any particular research. Similarly, in this study the topic of even and odd was not among the original goals of investigation. It emerged through the analysis of students' understanding of divisibility in a larger study on learning elementary concepts of number theory (Zazkis and Campbell, 1996a, 1996b).

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The general objective of this study was to describe preservice elementary school teachers' understanding of the parity of whole numbers. This narrow focus was chosen to allow in-depth analysis of relatively small but fundamental mathematical concepts. Specific questions included:

- How do preservice elementary school teachers use properties of parity in their reasoning about divisibility? Do they generalize parity to divisibility by another number, when a problem invites such a generalization?
- What strategies are used by preservice elementary school teachers to determine the parity of the given number? How does a given representation of a number (decimal, other than ten base, prime decomposition, product or sum) determine the preferred strategy?
- How can a pedagogical approach improve understanding of parity in the context of divisibility?

MOTIVATION

Students treat divisibility by two differently from divisibility by another number. What is so special about two? The following entry in a student's journal triggered a more formal inquiry into this phenomenon. Stephanie wrote:

Two is a very special prime number. It is even, and it is the only prime number that is even. All the other prime numbers are odd. Even though we don't know all the prime numbers, we can be sure that no other even prime will be found because it will have a factor of 2, and that makes it not prime.

I tried the following variation on Stephanie's observation: 'Seven is a very special prime number, because it the only prime number that is divisible by seven.' Somehow, it didn't sound convincing.

PARITY HEURISTICS

I will use the term 'parity heuristic', abbreviated PH, to determine a problem solving strategy in which properties of even and odd numbers are being utilized. The following claims are examples of PHs.

- The sum of two even numbers is an even number.
- The product of 23 and 17 is not divisible by 46 because 46 is even and (23×17) is odd.
- If $a = 2k$, where k is a whole number, then a is an even number.

A variation on PHs can be made, so that instead of parity a divisibility by another number is being considered.

- The sum of two divisible by 7 numbers is divisible by 7.
- The product of 23 and 17 is not divisible by 69 because 69 has a factor of 3 and (23 $\times$ 17) does not have a factor of 3.
- If $a = 7k$, where k is a whole number, then a is divisible by 7.

These heuristics can be generalized, such that instead of parity a divisibility by any possible prime or natural number is being considered. In this case each PH above is a particular example for the corresponding claim below.

For natural numbers a, b, n,

- If $n|a$ and $n|b$ then $n|(a + b)$.
- (23 $\times$ 17) is not divisible by (23 $\times$ n), for all $n > 1$, $n \neq 17$.
- If $a = nk$, where k is a whole number, then a is divisible by n.

Please note that even though in these examples the generalizations were made for any natural number, at times only a generalization for any prime number is possible (see further example of a proof for irrationality of $\sqrt{2}$).

How natural are these variations or generalizations for students who are successfully applying PHs? Do they see the similarities? Do they make the connections? These issues, among others, are addressed hereinafter.

DATA COLLECTION

The data in this study were drawn from two main sources: clinical interviews and written questionnaires. Fifty seven interviews were conducted during three years of the research study on preservice teacher's understanding of introductory number theory. These interviews were conducted with groups of volunteers from three different cohorts enrolled in a core course of ‘Foundations of Mathematics for Teachers’ over three consecutive years. The interviews were conducted after the topics pertaining to number theory were studied in the course. They dealt with a variety of issues related to introductory number theory, including divisibility and factorization, prime decomposition, prime and composite numbers, division algorithm and divisibility rules. For each cohort of interviewees a different semi-structured questionnaire was developed. By ‘semi-structured’ I mean that although the list of questions was prepared in advance, the interviewer had the freedom to diverge from this list and follow up with additional prompting or clarification questions when necessary. As a result at times not exactly the same questions were addressed by the interviewees from the same cohort. The issue of odd and even numbers was not the focus of any of these interviews or specific interview questions. This topic emerged and required a separate consideration during previous analyses, reported in Zazkis and Campbell (1996a, b) and Campbell (1996).

The questionnaire was designed and administered for the last cohort, and was answered by all 73 students enrolled in the course. The questionnaire presented below was designed with a specific focus on strategies identifying even and odd numbers.

> Several numbers are listed below. For each number decide whether it is odd or even and circle your decision. Explain your decision briefly.
>
> (1) 1234567 even/odd
>
> (2) 34_{five} even/odd
>
> (3) 121_{three} even/odd
>
> (4) 3^{100} even/odd
>
> (5) 3^{99} even/odd
>
> (6) $2^{100} + 3$ even/odd
>
> (7) 6^{71} even/odd
>
> (8) $7^{50} \cdot 3^{40}$ even/odd
>
> (9) $1234567 \cdot 2^{40}$ even/odd

Item 1 was chosen to identify the main argument used in determining the parity of a number, in fact, I was interested to see whether something alternative or additional to the ‘last digit’ strategy would be mentioned. Items 2 and 3 were chosen to determine students’ awareness of the fact that the ‘last digit’ strategy is specific for and dependent upon decimal representation of a number. Items 4–7 were chosen to observe students’ ability to discuss a number given in its prime decomposition without knowing its conventional decimal representation. It was chosen to determine to what degree students can reason about a number as being ‘odd’ or ‘even’ without knowing its last digit. Items 8 and 9 provided additional complexity in considering products of powers.

RESULTS AND ANALYSIS

Excerpts from the interviews

The following excerpts exemplify students' use of PHs (parity heuristics). They also show opportunities to make variations on PHs or generalizations of them and students' responses to such opportunities. The conversation below took place after Jennie found out that 391 was divisible by 23 and that in fact $391 = 23 \times 17$.

Interviewer:	Is 391 divisible by 46?
Jennie:	It's an odd number [391], and 46 is an even number, so I'd say no, even if 46 is a multiple of 23.
Interviewer:	And how about 69, do you think 391 is divisible by 69?
Jennie:	I can't be sure, both are odd, I'll have to look into it harder, I'll have to check.

This excerpt presents a rather typical reaction of the participants. A PH helps infer that 391 is not divisible by 46, that is, odd is not divisible by even. Using a more general variant of this heuristics one can conclude that a number divisible by 3 (here 69) cannot be a divisor of a number that is not divisible by 3 ($391 = 23 \times 17$). However, this variation was not made by Jennie and her conclusion was reached by means other than considering factors.

In the next excerpt a similar PH is implemented in a different context, which doesn't seem to invite a consideration of odd and even.

Interviewer:	Consider the number $K = 6 \times 147 + 1$. Is it divisible by 6?
Jennie:	Yes, because 6 is a part of prime factorization, well, I don't know if 147 is factored out, but because 6 is here, it is divisible by 6, 6 it's a factor.

[Interviewer suggests to try it out with a calculator. Jennie tries and explains.]

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Jennie:	By adding the 1, we changed the number from being an even number, to an odd number, because it's an odd number, and 6 is an even number, it's not divisible by 6.

Here the number K is represented as a sum of a multiple of 6 and one. Therefore it is obvious that the number is not divisible by 6 and further, leaves a remainder of 1 when divided by 6. However, this line of reasoning was not utilized by Jennie. After her initial confusion, probably caused by an incorrect parsing of the expression, Jennie implemented a PH: odd numbers are not divisible by six.

In the following excerpt Alice was asked to list all the factors of 117, after the number was represented in its prime decomposition and its prime factors were identified.

Interviewer:	Would you please now look at the number 117.
Alice:	Um hm.
Interviewer:	And I've done the calculation and $117 = 3^2 \times 13$.
Alice:	Okay. . .
Interviewer:	Okay. Will you please tell me, what are the prime factors of 117?
Alice:	Prime factors?
Interviewer:	Um hm. . .
Alice:	Would be 13 and 3.
Interviewer:	So 2 prime factors.
Alice:	Yeah.
Interviewer:	Okay. Now I'm asking you, what are the factors of 117? Can you list all the factors? Not only prime factors, all the factors of 117?
Alice:	All the factors. Okay, you could have 1,3,9, and 13.
Interviewer:	Anything else?
Alice:	(pause) Maybe, I don't know. So I would just go through on my calculator and divide them all. So I'd go, divided by 2 =, it wouldn't be anything, and then I'd go divided by (pause), I would just go through like that and find all its factors. . .
Interviewer:	You are dividing 117 by 2 and by 5. . .
Alice:	Yeah. . .
Interviewer:	To find out other factors?
Alice:	Yeah. Well I just could go through the whole system. . .
Interviewer:	Looking at this. Okay, so you've divided 117 here by 2 and by 5, by 7, how would you go on?
Alice:	I would just keep going up, well I know it wouldn't be even numbers because this is, this one's odd, so I'd just do, just go on up there (laugh), the number tree...

The observation that Alice didn't make a connection between factors and prime factors and didn't derive factors using the previously identified prime factors will be considered elsewhere. In order to find all the factors of 117 Alice decided to 'go through like that', that is, to try numbers on her calculator. Alice understood that she could "just go through the whole system", but she probably wanted to save work and effort. Therefore she acknowledged that the factors 'wouldn't be even numbers'. She understood that an even number could not be a factor of 117 because 117 was odd. However, in her trials to find the factors of 117 she didn't avoid numbers like 15 or 21. A PH was implemented by Alice, however a simple variation on this PH , such as '5 (or 7) is not a factor of 117, therefore a multiple of 5 (or 7) cannot be a factor of 117' has not been derived.

Many students believe that even numbers are more likely to have certain properties than odd numbers. In the conversation below Ann expresses her mistaken belief that the product of odd numbers can't be a perfect square, while the number $2 \times 12 \times 16$ is a perfect square because it is the product of even numbers.

Interviewer:	Do you think there is a whole number N, such that $N^2 = 3 \times 17 \times 19$? Do you think it is possible to find such an N?
Ann:	(pause) No, because when you times these two numbers, these three numbers here, you're going to get an odd number and odd numbers can't have a, can't be whole numbers when they're square rooted.
Interviewer:	Let's think about number 25. . .
Ann:	Oh yeah. . .
Interviewer:	Odd number. . .
Ann:	And 9, and 9. (laugh) Um, (pause) I don't think so, I don't think you would be able to find the whole number, because these are prime.
Interviewer:	Would you please elaborate on this a bit further?
Ann:	(pause)
Interviewer:	I agree with you. 3 and 17 and 19, they're three prime numbers. So what, can you find another whole number N that when you square it, it equals this product?
Ann:	(pause) I have to use my calculator, I don't know. I don't think there is one though.
Interviewer:	Okay. I'll ask you another one. $N^2 = 2 \times 12 \times 16$, do you think there is a whole number N, such that N^2 gives this product?
Ann:	Hmm, yeah. Because this is 24×16, they're both even numbers, and I think there's a good chance that you would get a whole number for a square root.

Similar beliefs that a perfect square must be even are demonstrated in a further discussion of the results from the written questionnaire. There are additional examples associated with beliefs about even and odd. At this point I would like to bring up one such example from my classroom practice. In the beginning of a discussion on decimal representation of rational numbers, it was pointed out that such a representation is sometimes finite and sometimes infinite and periodic. The cases of 1/2 and 1/3 served as generic examples for this phenomenon. Students were given a list of unit fractions (1/4, 1/5, 1/6, 1/7, 1/8, 1/12, 1/13, 1/15, 1/16, 1/24, 1/25) and were asked to predict, without calculation, whether the decimal representation of these fractions would be finite or periodic. A majority of students predicted that the decimal representation of 1/12 and 1/24 would be finite since those were ‘nice even numbers’.

In our final two excerpts the students are asked to give an example of a 5-digit number that leaves a remainder of 1 when divided by two. A request for a 5-digit number allows us to observe how the examples are generated.

Interviewer:	Can you think of a 5-digit number that leaves a remainder of 1 when divided by 2?
Mike:	7, it gives you 3 remainder 1.
Interviewer:	Yes, and what about a larger number?
Mike:	You can go 81, its 40 times 2 and then add one.
Interviewer:	OK. And if I asked you for five more examples of such numbers, what would you write? Also, I asked for a 5-digit number.
Mike:	It’s the same. You go 2 times a number, find that number and add 1, so this one will give a remainder of 1 in division.
Interviewer:	Another question: Can you give me a number that leaves a remainder of 3 when divided by 17?
Mike:	20.
Interviewer:	It is a good example. How did you find it?
Mike:	$17 \times 1 = 17$ and you just add 3 to it.
Interviewer:	And if I ask you to find a large number like this, 4 or 5-digit number.
Mike:	Can I use a calculator or work longhand?
Interviewer:	Whatever you choose, what is going to be your approach?
Mike	I’d do multiplication with 17 and then you add 3 to it.

Of course the correctness of Mike’s approach is obvious. What I find remarkable here is that a number that leaves a remainder of 1 when divided by 2 and a number that leaves a remainder of 3 when divided by 17 are found in a similar way. While in the second case the use of the form 17K+3 is efficient, in the first case any odd number satisfies the desired property. The fact that Mike didn’t pick any odd number, but described its construction by ‘go 2 times a number, find that number and add 1’, suggested that the connection between the property of being odd and the property of leaving a remainder of one in division by two was not well established. The next excerpt shows a connection between ‘odd’ and ‘leaving a remainder of 1 in division by two’, although in quite an unexpected fashion.

Interviewer:	Can you please think of a 5-digit number that leaves a remainder 1, when divided by 2?
Cindy:	(Pause) I’m thinking it would probably have to be an odd number, because all even numbers would be evenly divisible by 2. . .

Interviewer: OK. . .

Cindy: And, (pause), I'm trying to think of what number to put on the ends, but I'll have 1 (pause), I don't, actually maybe it's not possible, I don't know. . .

Interviewer: What is not possible?

Cindy: To have a remainder of 1, but. . .

Interviewer: You said a moment ago something about even and odds. . .

Cindy: It couldn't be an even number. . .

Interviewer: It cannot be an even number, so it must be an odd number. . .

Cindy: Um hm. . .

Interviewer: So when you know that it must be an odd number, what do you think about now?

Cindy: Well I think of the prime, actually not prime, but , (pause) I don't know, I'm probably stumped. Uh, (pause) I guess maybe just look at simpler cases, just look at 3 and 5 and 7 and. . .

Interviewer: 3, 5 and 7, okay there are simpler cases when you look at them. . .

Cindy: (pause) 2 is in the 3 once, remainder 1. . .

Interviewer: (pause) Okay, so you have written the number which is 10,003. You divided by 2, and this is your answer: 5551, remainder 1. Oh, it was hard, was it?

Cindy: (Laugh) (Pause)

Interviewer: Can you give me another number with 5 digits, that when divided by 2 has a remainder 1?

Cindy: I'll have to play around with those numbers . I'd keep 3 on the end. . .

Cindy understands that a number that leaves a remainder of 1 in division by 2 cannot be an even number. Interviewer extends this claim and suggests that it must be an odd number. However, this doesn't help Cindy to generate examples. She prefers to 'play around' with numbers and in each case to check her example by performing long division. It seems that Cindy does not think of odd numbers as a complementary set for even numbers. Her desire to check the examples is probably based on a belief that not all uneven numbers satisfy the desired property.

All the above excerpts show the power of PH as a tool in problem solving and decision making. The common theme in these excerpts is that heuristics regarding divisibility by 2 are successfully implemented by participants, however variations on the same heuristics requiring to consider divisibility by another number instead of the parity are not implemented. In what way is divisibility by 3 or 7 is so different from divisibility by 2? Results from the written questionnaire shed a light on this question.

Written questionnaire

Last digit as an indicator of parity. As a response to item 1 (asking to determine and explain the parity of 1234567) all the students claimed that the number was odd because it ended in seven. Two students added to the consideration of the last digit an explicit representation of the number 1234567 as $2k + 1$. It was evident that the parity of the number was perceived as a function of its last digit. This yet did not indicate that the perceptions of evenness as divisibility by 2 was missing, however, if it existed it was not a primary consideration in this group of participants.

In items 2 and 3 (asking to determine and explain the parity of 34_{five} and 121_{three}) the majority of students correctly converted the numbers and concluded their evenness or oddness based on the last digit in the decimal representation. Nine students made a mistake in conversion in at least one of the items. Ten students didn't attempt to make a conversion and made their (obviously wrong) decision based on the last digit of numbers represented in other-than-ten bases. For this minority of students the 'last digit' rule was applied without an understanding for what situation it was appropriate.

Arguments used by students in considering the questionnaire items 4–9 were in most cases consistent across the questions and repeated themselves. Since the interest of this research is in heuristics used and not the correctness of the answers, the analysis is presented considering students' decision making strategies.

Product/sum of even and odd numbers. Consideration of the parity of the elements in a product or a sum was the main strategy used by the participants. Claims such as '3^{99} is odd because this is a product of odd numbers' or 'we're multiplying by odd all the time therefore the result can't be even' were repeated frequently. At times these claims were complemented or substituted by an argument applying recently acquired terminology, such as 'odd numbers are closed under multiplication'. Seven students added to their arguments examples of simple exercises demonstrating a product of two even numbers, a product of two odd numbers or a product of even and odd. In three cases students added to these arguments proofs, considering a product of $2k + 1$ by $2n + 1$ or a product of $2k + 1$ by $2n$. Even though these arguments, examples or proofs were mostly accurate, they seemed at times unnecessary because the existence of the factor 2 was obvious in the representation of all the numbers.

Recognizing the factor 2. The argument based on the existence of the factor 2 in the given numbers could be most helpful in determining and explaining the parity of the number. However, the frequency of the use of this argument depended on the questionnaire items. While in item 6 the evenness of 2^{100} for most was taken for granted, the evenness of 6^{71} in item 7 was explained mostly as a product of the even numbers and the factor 2 was mentioned by only ten students. In item 9 (asking to determine and explain the parity of $1234567 \cdot 2^{40}$) the existence of the factor 2 was mentioned by 25 participants. Mostly it was mentioned by an operational formulation, such as ‘the number was multiplied by 2 therefore it is even’, rather than by a structural one, such as ‘the number is a multiple of 2’ or ‘the number has the factor 2’.

Only six students based their decision in items 4, 5 and 8 (considering the numbers 3^{100}, 3^{99}, and $7^{50} \cdot 3^{40}$, respectively), all presenting odd numbers, on the fact that the numbers were given in their prime decomposition and 2 didn’t appear as a factor. Students who have chosen this argument did so consistently across the items.

Last digit cycle pattern (LDCP). In some cases the participants attempted to determine the last digit of the number, using variations on the following strategy: $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = ...9$, etc. Therefore the last digits of powers of 3 have a cycle of 3, 9, 7, 1, repeating every four digits. From here some students determined that the last digit of 3^{100} was 1 and of 3^{99} was 7, and made their conclusion knowing the last digit of the number. Other students felt it was sufficient to claim that the last digit would be either 1 or 3 or 7 or 9, so the number would be odd in any case. Similarly, when the last digit of a power of 2 is considered, the pattern of last digits is the cycle of 2, 4, 8, 6. For the powers of 6 the last digit is always 6. In what follows I refer to this strategy as LDCP (Last digit cycle pattern). It was used by 24 students on items 4 and 5 (3^{99} and 3^{100}), by 13 students on item 7 (6^{71}) and by 4 students on item 6 ($2^{100}+3$). When the factor 2 appeared explicitly (item 6), for the majority of students there was no need to find out the last digit. When the factor 2 was implicit (item 7) or absent (items 4 and 5) , the need to know the last digit was increased. This finding is consistent with the recognition of the factor 2; its presence makes a statement much stronger than the lack of it.

Considering exponents and perfect squares. Fifteen students consistently repeated the mistake of confusing an even exponent with an even factor. For those, 3^{100} appeared to be even, because ‘100 is even’, while 3^{99} was odd, because ‘both 3 and 99 are odd’. Similarly, $7^{50} \cdot 3^{40}$ appeared to be even. The influence of the size of the exponent on this confusion has yet to be investigated.

Other than implicit identification of 3^{99} with 3×99, there could be a different reason for students' confusion. It seems possible that these students were influenced by exercises done in their class in which for natural numbers n and x, n^x was shown to be a perfect square for even x. Six students mentioned correctly that the number 3^{100} was a perfect square while the number 3^{99} was not. However the inference 'perfect square implies evenness' could point out students' dependence on the non-mathematical everyday meaning of 'perfect' and 'even': If 'even' is taken to mean 'without rough ends', then 'perfect' cannot be 'un-even'.

Synopsis. I didn't attempt to provide a comprehensive analysis of all students' answers. For example, the decision of five students to represent $7^{50} \cdot 3^{40}$ as 21^{90} in order to discuss its parity was left out of the scope of this discussion. Rather, I focused on arguments and strategies in participants' decision making.

There is strong evidence that in considering the parity of the number, participants focus on the last digit. When the last digit is not explicitly given in the number representation, there is a tendency to calculate it. It is interesting to note that only 4 of those applying LDCP for items 4 and 5 (3^{100} and 3^{99}) used LDCP for item 6 ($2^{100} + 3$). It is possible that the appearance of the factor 2 is much stronger than the lack of it. That is, if 2 is a factor, the evenness of a number is promised. If 2 is not a factor, other strategies must be considered.

Furthermore, multiplication by 2, multiplication by an even number or appearance of 2 as a prime factor was mentioned by 25 students on item 9 ($1234567 \cdot 2^{40}$). However, lack of 2 as a prime factor was mentioned only by six students in item 8 ($7^{50} \cdot 3^{40}$). This is a further confirmation for the assumption that for many students the factor 2 determines evenness, while the lack of it does not necessarily determine the lack of evenness. This finding is consistent with the findings of previous research, in which determining the divisibility of $M = 3^3 \cdot 5^2 \cdot 7$ by 7 appeared to be a much easier task than declining the divisibility of M by 2 or 11 (Zazkis and Campbell, 1996b). To infer divisibility or parity it is sufficient to consider the factors, however, to decline divisibility or parity considering the factors, one must understand the uniqueness of prime decomposition guaranteed by the Fundamental Theorem of Arithmetic.

CONCLUSION

Differences in preservice elementary school teachers' perceptions between divisibility by two, or 'evenness', and divisibility by another number have been observed. This led to an inquiry into participants' understanding of the parity of the whole numbers. Results of the questionnaire shed a light on students' thinking of even and odd numbers through the analysis of strategies students applied to determine the parity of the given numbers. Students' choice of strategies seem to provide an insight into students' lack of generalization of the Parity Heuristics. To summarize, evenness is not perceived by many students as equivalent to divisibility by 2. Evenness is often thought of as a property dependent on the last digit of a number. Also, while existence of 2 as a factor was used by students as a consideration determining the 'evenness' of a number, the lack of 2 as a factor was not identified with 'oddness'. Moreover, some students mis-connected the property of parity to other number properties, such as the property of being a perfect square.

The even/odd distinction is very dominant in a way students approach certain numerical situation. I suggest that the gap between evenness and divisibility by another number could be bridged by recognizing and emphasizing the equivalence of evenness and divisibility by 2. According to Skemp 'to understand something means to assimilate it into an appropriate schema' (Skemp, 1971, p. 46). A question I would like to entertain with respect to mathematics teacher education is how can one understand better what has been already understood, that is, assimilated. I wish to extend Skemp's claim by suggesting that to understand something better means to assimilate it in a richer or more abstract schema. I suggest that when mathematical concepts become in one's mind particular examples of more general mathematical concepts, a richer schema is constructed. This happens for example when familiar integer numbers become an example of a commutative additive group, or when a familiar square becomes a particular example of a parallelogram. With respect to the topic of investigation in this research, I suggest that a richer schema is constructed when 'evenness' is seen as a specific case of divisibility by a prime number.

In thinking of 'bridging the gap' I have in mind the population of adult students that contributed to this study. However, such a gap could be avoided if in students' exposure to even/odd terminology we postpone as much as possible the attention on the last digit property. In general, as a result of our previous study, it was suggested that a discussion of divisibility rules be delayed until students acquire some conceptual understanding of divisibility (Zazkis and Campbell, 1996a). In case of parity such a delay is crucial because of the early age at which children are learning these concepts. A definition developed by Debora Ball's third grade students claimed that the number was even 'if you can split it in half without having to use halves' (Ball, 1993). This informal definition of divisibility by two, that captures the essence of divisibility using terminology available for young students, is most appropriate. I don't recommend to avoid the last digit rule, just to postpone it, in order to achieve odd/even classification by substance and not by form.

Labeling numbers divisible and not divisible by two as even/odd assists in creating conceptual schemas, and makes divisibility by two exceptional in comparison to divisibility by other numbers. In many languages, such as Hebrew, French, Russian and Korean (representing 4 different language groups) the literal meaning of the term 'even' is 'pairable' or 'paired'. In English (as well in Chinese and Japanese) the words 'even' and 'odd' are commonly used outside of the mathematical context, bringing with them a variety of meanings and connotations. 'Even', outside of a mathematical context, means 'smooth', 'balanced', 'equal', 'exact' or 'precise'. 'Odd' means 'strange', 'exceptional', 'not regular, expected or planned'. This may be a possible source of difficulty for an English speaking student to label a number that is a *perfect* square as an *odd* number. Further research could examine the influence of a learners' native language on their understanding of even and odd.

FOR FURTHER CONSIDERATION

This research reflected upon experience with one specific group of preservice elementary school teachers. There is no attempt to generalize these findings or to claim how typical the observations reported here are of the population of preservice elementary school teachers. Can similar mathematical behaviour be observed within more 'mathematically sophisticated' populations? I would like to present for the reader's consideration the following two vignettes, that can be seen as my 'action research'.

Vignette 1

A famous proof of a theorem that the square root of 2 is not a rational number is based on a parity heuristic. My observations from discussing this proof with a class of preservice secondary mathematics teachers, most holding majors and minors in mathematics, are shared by colleagues.

To prove that the $\sqrt{2}$ is not a rational number we assume that it is rational and therefore can be represented as a/b, where a and b are whole numbers, $b \neq 0$, and the fraction is chosen in its reduced form , that is, $(a, b) = 1$.

$$\sqrt{2} = \frac{a}{b}. \tag{1}$$

From this assumption, by raising both sides of the equality to the second power we get that

$$2 = \frac{a^2}{b^2} \tag{2}$$

or

$$2b^2 = a^2. \tag{3}$$

At this point a contradiction can be claimed since the expression on the left of (3) has an odd number of factors in its prime decomposition, while the number of prime factors of the expression on the right side is even. This observation contradicts the Fundamental Theorem of Arithmetic, which guarantees the uniqueness of prime decomposition. However this route of a proof is perceived as somehow ‘tricky’ and a more popular conclusion for a proof is as follows: (3) means that a^2 is an even number and therefore a must be an even number. Representing it as $2k$, we get

$$2b^2 = (2k)^2 \tag{4}$$

or

$$b^2 = 2k^2. \tag{5}$$

Therefore b must be an even number. This claim contradicts our original assumption that the fraction a/b has been chosen in its reduced form and completes the proof.

A standard exercise after the proof for $\sqrt{2}$ is discussed in class (popular route), invites students to generate a similar proof for $\sqrt{3}$. In my experience, the majority of students get stuck at stage (3). ‘OK, I get that $3b^2 = a^2$, but I cannot go on from here since it gives me no clue of whether a is odd or even, so maybe you can prove that $\sqrt{3}$ is not a rational number, but not in the same way we did this for $\sqrt{2}$’ – this declaration of Allison summarized thoughts and feelings of many students. In fact, the proof can be imitated with exactly the same steps and arguments, when the arguments for evenness of the numbers a and b are substituted with arguments on their divisibility by 3. The fact that such a consideration was possible for students only after considerable prompting and encouragement, could mean that divisibility by three was not considered in an analogous way to evenness. Helping students interpret the idea of evenness as divisibility by 2, led not only to a successful proof regarding $\sqrt{3}$, but also to the generalization of it for $\sqrt{p}$, where p is any prime number. Although divisibility by 2 and evenness are mathematically equivalent, the latter is perceived differently in being more familiar and friendly. The knowledge of this equivalence is not applied naturally and spontaneously even among students with a relatively solid mathematical background.

Vignette 2

As a final vignette I would like to provide an example from a graduate level course, whose audience were secondary mathematics and computing science teachers, working towards their Master's degree in Education. They were asked to write a computer program that determined the parity of its input. The primitive commands *even* and *odd* that serve this purpose in the computer language ISETL used in this course were to be ignored for the purpose of this exercise. In most computer languages one can divide the number in question by two and then determine whether the quotient is an integer. In ISETL the task is even simpler: divisibility can be determined using a primitive function *mod*, that receives two inputs and outputs the remainder in division of one by another. For example *22 mod 3* returns the value 1. Using the *mod* function, the task can be completed in one line: 'return (number mod 2) = 0'. What follows is a solution offered by one of the students, David, a Computing Science major.

```
last.digit:= func(x);
   return x mod 10;
end;

is_even:= func(number);
   if not is_integer(number) then return "please input integers only"; end;
   return last.digit(number) in {2,4,6,8,0};
end;
```

The function *last.digit* returns the last digit of its input. The function *is_even* returns 'true' if this last digit is an element in the set {2,4,6,8,0}, that is, the number is even. (The function *in* in ISETL receives two inputs, an element and a set, and returns 'true' if the first input is an element of the second. For example, the expression *3 in {4,5,6}* returns 'false'). I would like to note the excellent programming habits exhibited here: informative choice of names for functions and variables, informative and friendly error message for inputs that are not integers, structured programming in defining a separate function *last.digit* and using it in the main function, and outputting the value of the expression without an additional condition statement. (Less sophisticated programmers would write here in the third line a statement like

```
if last.digit(number) in 2,4,6,8,0 then return “true”;
   else return "false"; end;)
```

The function *mod* was definitely familiar to David as it was used in writing the *last.digit* function. However, the program written by David indicates that he was thinking of parity in terms of the last digit of the number, and not in terms of divisibility. So did about one-third of his classmates. The relative simplicity of the solution that takes into account divisibility in comparison to David’s solution is an additional indication of how robust the ‘last digit pattern’ could be even among ‘mathematically sophisticated’ students.

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