BOOK REVIEW

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Learning and Teaching Number Theory: Research in Cognition and Instruction by S.R. Campbell and R. Zazkis (eds.) Ablex Publishing, ISBN 156750 6534, November 2001, 245 pp

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When starting to write this book review, I asked myself – what would a potential reader of this review, a mathematics educator involved in teacher training or in teaching postgraduates, or a mathematics education researcher, expect to find in my review? When reading such reviews, what am I, as a teacher educator and a researcher, interested in? I believe that one answer is – I would like to get enough information to know whether or not to read the book under review.

I found the task of conveying this message quite a responsibility. On the one hand, this review should endow the reader with a good sense of what this monograph entails, and to enable him/her make their own independent decisions about it. On the other hand, the review should offer a warranted critique of the strengths and weaknesses of the monograph and possibly a concluding opinion. To answer this twofold demand, my review includes descriptive and critical summaries of both the general picture created by the monograph as a whole, and of each chapter as an independent unit.

Before going into details, I would like to state that my bottom line is that this monograph would be of great interest to both the teacher educator and to the mathematics education researcher as representatives of the mathematics education community, and to leading mathematics teachers in elementary, as well as secondary schools. This monograph offers a unique opportunity to address a rich variety of aspects dealing with learning and teaching of number theory. It is a well chosen and carefully designed collection of articles, revolving around the investigation of ways to promote prospective teachers' and college students' understanding of number theory, and around the examination of their related performance during and after participation in relevant courses. Number theory is presented both as an important mathematical subject area and a valuable, 'friendly' field for examining and promoting general mathematical skills like conjecturing, generalizing, proving and refuting mathematical statements. The editors

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Campbell and Zazkis, while highlighting the values of number theory, call attention to and protest its being a rather neglected area in mathematics education research. They thus present the collection of articles in the monograph as an initial attempt to fill the void, and it is this original, novel nature of their contribution, that grants this collection extra strength. Clearly, the editors achieved their explicitly stated goal "to identify and demonstrate some of the different kinds of problems and ways of thinking that can be investigated in a program of research into learning and teaching number theory and its implication for cognition and instruction" (p. 2). And even beyond that, as Selden and Selden explain, they have focused "on questions and new directions for investigation, some ranging well beyond number theory itself" (p. 214).

This review first relates to the outline of the monograph by means of an overview of the different types of the chapters included, and a brief description of the special contribution of each chapter to the global picture of learning and teaching of number theory. In conclusion, I call attention to several motives related to wide-ranging issues regarding cognition and instruction, which recur in the monograph.

An overview

The monograph consists of eleven chapters that can be classified in various ways, some of which are suggested by the editors in chapter 1 (p. 9). My categorization of the articles is done with reference to the monograph's title – that is, by focusing on aspects of cognition and instruction. While to some extent all articles in the monograph are concerned with cognition and / or instruction related to number theory, the varying weight given to these two aspects in the different articles sets up the spectrum of this monograph. The first and the last chapter frame the monograph by making explicit its aims and discussing the role that number theory can and should play in K-post secondary mathematics curricula. They also provide a general survey of the scope of research presented in the monograph, some theoretical underpinnings, as well as additional research questions and suggestions for possible educational implications.

The nine remaining chapters discuss issues related to the learning and teaching of number theory, and they can be grouped in the following manner: Four chapters are devoted mainly to students' conceptions, images, difficulties, and linguistic imprecision when dealing with elementary number theory issues (addressing prospective teachers in chapter 2, 3, 4, and undergraduate computer science students in chapter 5). Three chapters focus on didactical considerations regarding the design of instruction and teaching elementary and advanced number theory problems to prospective teachers (discussing general criteria for the choice of tasks to be presented in class in chapter 6, didactical advantages of using generic proofs in chapter 9, and analyzing a specific, recommended task in chapter 7). The two remaining chapters are mainly concerned with connections between the teaching and the learning of number theory, with reference to the role of conjectures and induction proofs in class (conjectures in chapter 8, and induction in chapter 10).

It should be noted that by covering various dimensions of learning and teaching number theory, the collection of articles in this monograph provides a significant amount of information regarding cognition and instruction of undergraduates and prospective teachers. Still, each of the nine chapters presents a specific, independent case that is interesting in itself. The following section will allow a glance at their contribution.

A brief summary of the distinct contribution of each chapter

The brief summaries of the different chapters will be sequenced according to their presentation in the monograph.

As mentioned before, in **Chapter 1**, "Toward Number Theory as a Conceptual Field", the editors of the monograph, Stephen Campbell and Rina Zazkis present their aims in editing this book. They, then, concisely discuss the main concerns of number theory as a mathematical subject area, characterize elementary number theory as distinguished from advanced number theory, and briefly review the role of elementary number theory in the K-12 and post-secondary curricula with reference to the explicit and implicit ways in which teaching and learning of number theory are addressed in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and in the new *Principles and Standards* (NCTM, 2000).

At this point, Campbell and Zazkis move on to shed some light on the articles included in the monograph. They start by discussing the studies' constructivist orientation and the theoretical frameworks (e.g., Vergnaud's theory of conceptual fields, and Dubinsky's Action-Process-Object-Scheme (APOS) theory) that are used to analyze the data. Then they define number theory as a conceptual field (NTCF) and suggest how the articles in the monograph contribute to this emerging area of research in mathematics education.

This opening chapter is not only a good introduction to the monograph; it is also valuable for the description of the possible contribution of number theory issues to learners' mathematical knowledge of whole numbers and operations, and their suggestions for further examination of this area.

In **Chapter 2**, “Coming to Terms with Division: Preservice Teachers’ Understanding”, Campbell discusses preservice teachers’ understanding of the differences between whole numbers division and rational number division, and the difficulties they encountered in coming to terms with these two mathematical situations. Campbell interviewed twenty-one volunteer preservice elementary teachers enrolled in a course that covered basic topics from elementary number theory, and in this chapter he analyzes their correct, as well as their incorrect, ideas when solving the different tasks.

The findings indicate that, while the participants’ correct responses are commonly based on calculations, and rarely on divisibility criteria, the basis for their erroneous solutions vary. For example, prospective teachers encountered difficulties with the demand that in whole number division the remainder and quotient are whole numbers (arriving at the solution 0.5 when asked about the remainder of 21 divided by 2). They also encountered difficulties with the demand that the remainder be smaller than the divisor (arriving at the solution the quotient is 9 and the remainder 3 for 21 divided by 2, because $2 \times 9 = 18$ and $18+3 = 21$) (p. 20–21).

The chapter suggests some reasons for the apparent mistakes, relating incorrect responses mainly to participants’ inability to distinguish between rational number division and whole number division with a remainder, to their tendency to inappropriately extend the applicability of familiar processes that are correctly used in certain activities to novel situations, and on their overdependence on interpreting formal referents of arithmetic division using informal language (p. 36). It concludes by raising some related questions for further research.

In **Chapter 3**, “Conceptions of Divisibility: Success and Understanding”, Anne Brown, Karen Thomas and Georgia Tolias report on a study that examined the understanding of ten prospective elementary teachers, enrolled in a course that addressed issues drawn from the number theoretic mathematical contexts regarding basic concepts of introductory topics in number-theory. The authors are interested in the individual’s ability to progress from action-oriented responses to explicit inferential reasoning that may reflect an understanding of mathematical operations and properties concerning the multiplicative structure of the set of natural numbers (a la Freudenthal).

The shifts in the participants’ reasoning are illustrated and described as going from stage 1, i.e., performing actions successfully with little awareness of general mechanisms (knowing by doing); to stage 2, where actions and conceptualization influence each other but the individual still cannot make inferences about the success or failure of actions without actually carrying them out; and finally, to stage 3, when actions are consciously guided by and reasoned about through applying one's conceptualization of the task – work in this stage includes the ability to make predictions of the success of future actions without direct experimentation.

The authors' analysis of the data is done by means of a theoretical framework that combines the APOS theory with a stage model adapted from Piaget's work in *Success and Understanding*, and the discussion includes a description of previous, related publications by Zazkis and Campbell (1996a, 1996b), where the development of the divisibility concept is analyzed. The authors mention some advantages of the stage model for analyzing an individual's grasp of divisibility, and suggest that an awareness of the roles played by the various aspects of multiplicative structure is an essential first step in developing the coherence necessary to have a schema for divisibility (p. 46–48).

The chapter concludes with a comprehensive discussion of the findings in light of the theoretical framework and of previous publications. Special emphasis is put on detailed pedagogical suggestions referring to a number of number-theory oriented didactical conclusions (e.g., to emphasize the central role of multiplication, or to make links across LCMs different representations), and some general conclusions regarding students' mathematical behavior. The authors' last statement raises the pressing need of more research to extend the related, existing body of knowledge.

In **Chapter 4**, "Language of Number Theory: Metaphor and Rigor" Zazkis addresses the increasing awareness of the mathematics education community to the importance of communication and discussion in mathematics classes, and calls attention to the crucial role that precision should play in students' language when expressing mathematical ideas. Zazkis examines the terminology used by prospective elementary teachers, enrolled in a course called "Principles of Mathematics for Teachers", when referring to the notion of divisibility during individual interviews.

Her findings indicate that most participants use a mixture of informal and formal mathematical terminology. The language used by the interviewers during their interviews was considerably different from the language used by the interviewees, even though the five mathematically equivalent statements that express the notion of divisibility were familiar to the participants from their textbook and from class discussions. Prospective teachers claimed, for instance, that a number 'cannot be divided' or 'cannot be divided evenly' by 2, meaning that it was not divisible by 2, and occasionally went even as far as to invent words like *timesing* saying, "You're timesing it by 6, it's a multiple of 6".

Zazkis also shows that the terminology used by the participants may reflect their grasp of division. That is, prospective teachers' saying that a number ‘can be divided evenly’ reflects images and processes consistent with a partitive view of division, and their saying that a number ‘goes into’, ‘fits into’, or ‘can be put into’ another number points to their quotitive or measurement view on division. Moreover, several prospective teachers mention that the result is ‘without remainder’, pointing to thoughts about whole number division, whereas other prospective teachers mention division ‘with no decimals’ indicating reference to rational number division.

Zazkis identifies possible reasons for the participants’ application of informal terminology. Some suggested reasons refer to the learner e.g., the prospective teachers’ need to confirm for themselves the meaning of the word intended by the interviewer. Other reasons refer to the structure of the word ‘divisible’ – suggesting that like other verbs using the suffix ‘-able’ or ‘-ible’, e.g., ‘edible’ means ‘can be eaten’, it is reasonable to interpret ‘divisible’ as ‘can be divided’.

All in all, the analysis of the participants’ responses and their descriptions of divisibility led to the identification of four different, though not necessarily disjoint, themes in application of informal vocabulary: (a) attempting to interpret the word *divisible*, (b) invoking images and processes, (c) seeking confirmation of meaning, and (d) overemphasizing.

In **Chapter 5** “Understanding Elementary Number Theory at the Undergraduate Level: A Semiotic Approach”, Pier Ferrari focuses difficulties experienced by Italian first-year university computer science students enrolled in an introductory modern algebra course when dealing with elementary number theory. His study was motivated by four main goals: (a) to discuss undergraduates’ performances when dealing with elementary number theory problems, particularly emphasizing impredicative problems; (b) to test the notion of semiotic control; (c) to test the APOS framework with a different population and to relate it to semiotic control; (d) to begin an analysis of the influence of language and format in the statement of problems on students’ performances.

The findings indicate students’ tendency to be “procedural rather than conceptual”, for instance, when linking between division and prime decomposition (p. 110), and the extreme difficulties they encounter in solving rather simple tasks that lack a well-known solving algorithm. The notion of ‘semiotic control’ and the author’s suggested criteria for evaluating students’ behaviors proved useful when interpreting the prospective teachers’ behaviors. Among the criteria mentioned are students’ ability to judge and consider the applicability of different strategies of solutions, and their ability to work with different representations of the same concept. It was found, for instance, that participants faced major difficulties in linking divisibility to factorization.

These data suggest that Dubinsky's APOS scheme and Zazkis and Campbell's analysis of the development of divisibility concepts also apply to undergraduate computer science students. In a sense, 'semiotic control' may enrich the APOS framework, as it allows one to analyze behaviors with respect to the interpretation of statements, providing tools to detect and analyze difficulties caused by poor mastery of language.

The author concludes by stating that elementary number theory has proved a subject suitable for analyzing undergraduates' semiotic control of their behavior, and then describing some pedagogical opportunities elementary number theory affords at the undergraduate level for the development of advanced mathematical thinking.

In **Chapter 6**, "Integrating Content and Process in Classroom Mathematics", Ann Teppo brings a fresh didactical breeze to the monograph that up until this point has focused on clinical studies, mainly examining participants' cognitive understanding of subject matter issues. This chapter provides a detailed description of an interesting example for how teaching number theory can be facilitated by a didactical approach of 'reflective discourse'. It describes "a classroom activity based on ideas of number theory that successfully integrates content and processes in the spirit of the new *Principles and Standards*" (p. 118).

By focusing on one 50 minutes lesson, analyzing the mathematical tasks presented, some teaching approaches (e.g., using the 'empty chart') and the classroom vignette, Teppo invites us to join her in leading her students on a journey towards the formulation of new sociomathematical norms (a la Yackel and Cobb, 1996) causing them to take part and "become involved in a new type of classroom mathematics" (p. 118).

Teppo shows how the prospective teachers who entered her course believing that mathematics is basically a procedurally oriented subject and that studying mathematics is about memorizing formulas and rules gradually became more and more engaged in a wide range of mathematics processes, including organizing information, looking for numerical patterns in order to make generalizations, raising and testing conjectures about these generalizations, and forming abstractions. These processes served for discussing and formulating number theory notions such as factorization, divisibility, and prime and composite numbers.

In designing and analyzing her didactical plans, Teppo pays much attention to the mathematical content as well as various aspects of communication, i.e., the importance of students' expressing ideas and critically listening to their peers, using precise terminology, making conjectures, defending ideas and justifying their choices. In conclusion, Teppo illustrates didactic tools and approaches which teachers, who aim to motivate and create an atmosphere of inquiry in their classes, can find useful.

In **Chapter 7**, "Patterns of Thoughts and Prime Factorization", Anne Brown presents a "thought-provoking problem" that, at least at first glance, seems quite difficult and very perplexing. In her phrasing: "though the problem is elementary, no one found it completely trivial or obvious, and its solution elicited a variety of strategies. The strategies as well as the stumbling blocks provide a few insights into the subtleties of the use of prime factorization as tools for reasoning about multiplicative reasoning" (p. 133). The chapter illustrates a way in which a specific representation of a problem, i.e., the prime decomposition of the elements of a given sequence, triggers a wide range of mathematical issues, ideas and difficulties to be clarified and discussed in class.

Brown opens the chapter by giving six entries in a sequence, all in prime-factored form, and then challenging the reader to pause, before going on with his / her reading, and solve the following task: (a) Write the next six entries in the sequence, all in prime-factored form, (b) Write the 200^{th} term in prime-factored form, and (c) Describe a method that will provide the prime factorization of the nth term of the sequence. Then, she describes the origins of the problem and explains that by means of this (and similar) problems it is possible to trigger the examination of prime factorization to identify, compare, and contrast the multiplicative properties of natural numbers.

She continues by presenting a number of strategies for related solutions, accompanied by enlightening explanations and comments regarding students' preferences and difficulties. She concludes by raising and discussing some related pedagogical issues, and 'the interested reader' is then offered some additional, similar problems for further thought.

In **Chapter 8**, "What Do Students Do with Conjectures? Preservice Teachers' Generalizations on a Number Theory Task", Edwards and Zazkis discuss some didactical considerations that underlie the design and the presentation of the *Diagonals in a Rectangle*, a generalization task. The task is special in the sense that it is embedded in a geometrical setting, but its solution requires some number theory considerations such as the idea of GCD.

The authors describe their (as teachers) follow up process on their students' performance on the task by means of the 'problem solving journal' in which the prospective teachers (as students) were asked to record all of their attempts to solve the problem. In addition to the wide instructional background, attention is paid in this chapter to the nature of the prospective teachers' solutions, their problem-solving strategies, and their specific conjectures about possible rules or formulas. The latter, the conjecturing-process, is described by the authors as follows: "it is this part of 'preproof' process that we were most interested in exploring in the research" (p. 141).

Indeed, the findings of this study contribute significantly to reported data on the range of prospective teachers' responses to disconfirming evidence when exploring conjectures. Here, information is gathered when relating to a non conventional number theory problem. The findings indicate, for instance, that most of the participants react appropriately the first time they encounter evidence that does not confirm their conjectures. Still, a few participants do not give up their conjectures due to the negating evidence; they either choose to ignore the evidence or focus on the part of their solution that is consistent with their initial claims.

In **Chapter 9**, "Generic Proof in Number Theory", Rowland opens by stating his view that "the potential of the generic example as a didactic tool is virtually unrecognized and unexploited in the teaching of number theory". He is, thus, "urging a change in this state of affairs" (p. 157). Accordingly, this chapter is dedicated to highlighting and discussing the pedagogical advantages of 'generic proofs' and to the place the author believes they should have in number theory courses.

Rowland first discusses the 'conviction, explanation and illumination' purposes of proofs in mathematics classes, and, while referring to published views (e.g., Reuben Hersh and Gila Hanna), states that "In the teaching context, the primary purpose of proof is to *explain*, to illuminate why something is the case rather than to be assured that it is the case" (p. 159). The author, then, illustrates and discusses the teacher's role in promoting students' *inductive* inference as well as their *deductive* reasoning, explaining that, naturally, the teacher's decisions regarding what proofs might be acceptable in a given context, is to a great extent dependent on his / her purposes in 'proving'.

Then, the author presents and analyzes a very rich and stimulating collection of number theoretic examples, some from the literature and others from his own experience as a teacher, thereby illustrating how generic examples might point to general arguments. This collection includes reference to 'the summation of consecutive odd numbers', 'Gauss and the sum 1+2+3+...+100', 'Euler's Φ-function', and to 'Wilson's theorem'. Rowland, then, relates to the need for a list of principles underpinning the construction and presentation of generic proofs in number theory, and while he understands that it is premature to offer a definitive list, he makes a first step in this direction by suggesting five guiding principles. The author, then, discusses his related work with undergraduate students, describing their obstacles when addressing number theory statements.

In the last section of this article ‘pedagogical suggestions and proposals for further research’, Rowland challenges his initial, extreme position of questioning the necessity of formal proofs in general symbolic notation. He takes a ‘more moderate stance’, making ‘three modest and conservative suggestions’, based on his five principles, providing support for students bridging the gap between generic understanding and general exposition (writing ‘proper’ proofs) (p. 180).

In **Chapter 10**, “The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction”, Harel addresses relationships between the learning and teaching of mathematical induction – a significant proof technique in discrete mathematics and in number theory that can provide a context to enhance students’ conceptions of proof. The author explores students’ conceptions and difficulties with mathematical induction in a standard instructional treatment, and in an alternative instructional system. He points to major deficiencies of the standard treatment of mathematics induction: It is handed to students as a prescription to follow; problems can be solved by means of mathematical induction with little understanding of it; and its presentation consists of sequenced problems, from easy to difficult in the view of the author / teacher, rather than in accordance to students’ conceptual development.

The alternative treatment of mathematics induction took into account these deficiencies. That is to say, the novel treatment considered the possible causes for failure of the standard approach by including phases corresponding to the levels of conceptual development. It was implemented in a teaching elementary number theory experiment that was carried out with twenty-five junior prospective secondary teachers. The DNR system of pedagogical principles – the duality principle, necessity principle, and repeated reasoning principle for designing, developing and implementing mathematics curricula – is the conceptual basis for this instructional treatment.

Harel specifies that the most significant result in this study is “that in this alternative treatment students changed their current ways of thinking, primarily from mere empirical reasoning… into transformational reasoning” (p. 206). That is to say, the new pedagogical approach helped students in developing their proof schemes. This chapter describes a most structured study about the pedagogical aspects of designing and carrying out the teaching of number theoretic topics, while emphasizing issues of mathematical induction.

Finally, in **Chapter 11**, "Reflection on Mathematics Education Research Questions in Elementary Number Theory", Annie Selden and John Selden provide an inclusive overview of various general issues that are highlighted in the monograph. They identify the following common threads running through the various articles: "the potential of number theory for teaching and learning of problem solving, reasoning and proof; questions regarding the language and images of divisibility; philosophical stances taken; theoretical frameworks used; and implications for teaching" (p. 214).

The authors interweave their discussions with some enlightening comments to which they add illustrations from their own classes, research and readings, and refer to points that need further clarification by means of additional research. In their own words, "in the spirit of the monograph and drawing on it, we focus primarily on questions and new directions for investigation, some ranging well beyond number theory itself" (p. 214).

To illustrate, let us look at their discussion of the first topic, i.e., the role elementary number theory could play in promoting students' mathematical reasoning, generalization, abstraction and proof. The authors first refer to the related statements made in the NCTM *Principles and Standards* (2000) and to related chapters in this monograph (chapter 6, 9 and 10), describing the chapters' contribution of data related to the issues under consideration and posing a number of questions for further research (e.g., regarding the effectiveness of generic proof for teaching).

The authors go on by describing their own teaching experience of abstract algebra to prospective secondary teachers. They indicate that while proofs are an integral part of abstract algebra courses, in these courses students often find themselves struggling with both, issues related to abstraction and with the constructions of proof. "In contrast, in elementary number theory, students deal with objects (integers) and operations (ordinary multiplication and addition) that are familiar to them. Hence, they can concentrate on discovering and constructing proofs without being distracted by simultaneously having to extend their conceptions of the operations and objects they are studying" (p. 215).

In this spirit, the authors suggest that, in order to promote students' performance with mathematical proofs, elementary number theory statements not involving excessive abstraction be presented. The authors illustrate such a statement, and describe how they have successfully tried it with their university students in a 'bridge' course. They conclude by suggesting three major reasons why number theory is ideal for introducing students to reasoning and proof.

Similar comprehensive discussions are presented with reference to each of the listed topics.

This chapter is a special contribution to the monograph, demonstrating via an examination of the other papers and an extended discussion, how number theory lends itself to promoting teaching and learning of general mathematical issues.

Closing Circles and Opening New Horizons

I have two reasons for labeling this section: *Closing Circles and Opening New Horizons*. First, in this section I intend to close circles I have opened in the introduction to this review by offering teacher educators and mathematics education researchers some warranted recommendations regarding this monograph, and by suggesting new horizons of work (teaching and researching) that may evolve from their reading. The other reason for this title is the nature of closing circles and opening horizons I have found in the monograph itself. While addressing various issues related to cognition and instruction of number theory, topics that are the focus of one chapter, time and again, are backed up by the findings reported in other chapters. Moreover, the circles of cognition and instruction are firmly linked and the role each of these plays in the different chapters creates a puzzle worth solving. On the other hand, in the words of Selden and Selden, this monograph "intentionally raises more questions than it answers", aiming to convince the reader that "many interesting questions in the teaching and learning of number theory await their attention" (p. 213) and thus, the reading of this monograph, may open new horizons of interests and deeds, in teaching and investigating students' performance with number theory using various types of instruction.

I opened this review by asking – what message does this review carry to the teacher educator, and to the mathematics education researcher? Do I believe that they will find any interest in reading this monograph? If the answer is, yes, in what respect? and Why? Before going into details, the answer to both is: YES. I do see many varied benefits for each of them in reading this monograph, and I have also provided them with the flavor of each chapter via a concise summary, so that they will be able to get their own ideas regarding the different issues dealt with here.

Now, I would like to close the circle by separately addressing teacher educators, and mathematics education researchers, who are assumed to have read this review up to this point, and to have acquired a sense of moderate familiarity with the various chapters. I also suppose that both are convinced by now that "number theory offers many rich opportunities for explorations that are interesting, enjoyable and useful. These explorations have payoffs in problem solving, in understanding… other mathematical concepts, [and] in illustrating the beauty of mathematics" (NCTM, 1989, p. 91).

First, the teacher educator for elementary (and perhaps even secondary) mathematics teachers, is going to find many ideas on how to teach number theory as well as how to use elementary number theoretic tasks as a rich setting for promoting his students' mathematical reasoning, problem solving and mathematical communication. As specified in the *Overview* section, the articles in the monograph are concerned with either cognition or instruction both with regard to number theory. If one is eager to start by getting ideas regarding ways to present number theory to students, (s)he would start by thoroughly reading chapters 6, 7, and 9, which are instruction oriented. One may then continue by reading chapters 8, and 10 that give balanced attention to cognition and instruction, in order to become familiar with both ideas for teaching and a notion of how students may react to such instructional steps. Still, since teaching should consider what students find easy, what they find difficult and why, a teacher educator is likely to find interest in chapters 2, 3, 4, and 5 that focus on students' conceptions, reasoning, and difficulties. On the other hand, perhaps teacher educator should begin this adventure by reading the editors' introductory chapter (1), and to conclude by getting some more ideas, and general perspectives, by reading Selden and Selden's chapter (11).

All in all, this monograph explicitly (as the main focus of certain chapters), and implicitly (in the background of some chapters), offers a spectrum of instructional ideas, approaches and tasks. The tasks are presented either as teaching or as research tools – appearing in the introduction to the chapters, or in their description of the study, or as suggestions for additional related problems. They address various mathematical issues, including:

(a) **elementary number theoretic notions** such as, divisibility (e.g., in chapter 2, 3, 4, 5, 7), prime decomposition (e.g., in chapter 2, 5, 6, 11), LCM – least common multiple (e.g., in chapter 3, 7), GCD – greatest common divisor (e.g., chapter 5, 8), and connections between equivalent expressions for the same notion as well as distinction between non-equivalent notions (e.g., in chapter 3, 4).

(b) **Advanced number theoretic issues**, such as the division algorithm/ theorem (e.g., in chapter 2), and Wilson's theorem (in chapter 9).

(c) **Additional mathematical settings**, like sequences (e.g., in chapter, 7, 10) and rectangular figures (in chapter 8). The reader is usually provided with satisfactory mathematical highlights and didactical comments that point out the special offering of the different tasks, their solutions, and occasionally, students' prevailing ideas and difficulties when solving the tasks.

The problems are presented in various ways, including numeric, parametric, (e.g., in chapter 5) and verbal (chapter 2, 3) representations of 'solve' tasks (with or without calculators); 'reversed' tasks – providing the 'solution' while asking about the given (e.g., in chapter 3, 5); 'generalize' tasks (e.g., in chapter 6) and 'prove' tasks (e.g., in chapter 9, 10, 11). The numeric representations included tasks with whole numbers (e.g., chapters 2, 3), prime factorizations of whole numbers (e.g., 33·52·7 in chapter 2, 3), expressions that include multiplication and addition (e.g., 6·147+2 in chapter 2), sequences and others.

The sequencing of these tasks during interviews and in teaching sessions is occasionally discussed and elucidated by the authors. As mentioned in the previous section, additional didactical issues are presented, for instance, in the following chapters: Rowland (chapter 9), convincingly argues for the use of generic proofs for advanced number theory theorems, and suggests five principles for selecting particular cases. Selden and Selden (chapter 11) further explain why and how elementary number theory should serve to promote students' ability to cope with formal proofs. Harel (chapter 10) shows that the sequencing of tasks should be consistent with students' conceptual development, and that in the case of mathematical induction, starting with challenging tasks is more beneficial than going 'from easy to difficult'. Teppo (chapter 6) describes in detail her instructional steps, addressing individuals, small groups, and then, the whole class, using various approaches and tools (e.g., the divisor table), to trigger inquiry, conjectures, justifications, and genuine communication.

It is noticeable that I decided not to bring examples of the tasks, and I should emphasize that in no way do I claim to have exhausted the richness of tasks and didactical offerings of this monograph. I left for the reader much to reveal by himself / herself when reading the monograph.

Clearly, I believe that the mathematics teacher educators may enjoy and can benefit from reading the monograph. Now, what about the mathematics education researchers? What interest will they find in this monograph?

A mathematics education researcher is going to learn a lot about the initial research steps made in "working *towards* a systematic definition of number theory as a conceptual field" (p. 8). For example, the data reported here, regarding university students' difficulties when solving elementary number theory tasks, and their use of vague terminology (e.g., in chapter 5, 6), their confusing whole number with rational number division (e.g., in chapter 2), their problems in raising conjectures validating statements and proving (e.g., in chapter 8, 9, 10), and possible reasons for these difficulties, should be studied carefully when designing continued studies. Also, the didactical approaches reported, occasionally with no accompanying research (e.g., in chapter 7), and the additional, pedagogical suggestions made (e.g., in chapter 6, 7) should be systematically examined.

The researcher is also invited to accept the challenge made by the authors, since in most chapters and mainly in the framing two chapters (1 and 11) the authors themselves raise questions for further research. For example, in chapter 11, Selden and Selden state when addressing Harel's chapter (10), "are there ways to help students become better at it [generalizing]? ... further studies could help provide details that might help teachers engender such reasoning in their students" (p. 216). Furthermore, when addressing Rowland's chapter (9) they say, "one question to ask from a pedagogical point of view is whether a specific generic proof is likely to be illuminating, even if one can easily find a suitable particular case" (p. 216). They then relate to a number of chapters that discuss students' conceptions and difficulties, saying that, "it would be an interesting research question to see what sorts of mental images of the division algorithm, or of prime factorization, that university students bring with them and to what degree they are aware of using inner vision or inner speech" (p. 218). Also, "issues regarding divisibility have been raised by various chapter authors, for example, the necessity to distinguish the indivisible units of integer division from the infinitely divisible units of rational division, as well as the importance of negotiating successfully between the modular ($2 \cdot 10 + 1$), fractional ($10^1/_2$), and decimal (10.5) representation of '21 divided by 2' .... All of these are ripe for further investigation" (p. 220).

Indeed, this monograph can be regarded as a necessary and valuable 'first step' in the investigation of number theory as a conceptual field, and as a promising field for promoting students' mathematical reasoning. Still much more interesting research work in this area is needed.

In conclusion, this monograph is recommended to mathematics educators and to mathematics education researchers. But even more so, heads or coordinators of mathematics education departments should encourage their teams to read and discuss the various issues highlighted and those merely insinuated in this monograph during department meetings. They may reflect, for instance, on the impressive integration of theory, research and didactical implications found in the monograph. For example, the repeated use of Dubinsky's APOS theory for the analysis of the participants' solutions (e.g., in chapter 3), the use of research findings to validate and enlarge the APOS theoretical framework (e.g., in chapter 5), attempts to integrate this theory with another theory to create a new, more suitable theoretical framework (e.g., in chapter 3, 5), or to use the APOS theory in designing and ranking the difficulty of number theory tasks (e.g., in chapter 5).

In order to enrich these discussions, the department members are invited to add to their reading two very significant extras: Fischbein's (1993) analysis of students' formal, algorithmic and intuitive knowledge and Zazkis's (1999) analysis of students' reactions to number theory tasks by means of the intuitive rules theory. These may contribute considerably to the interpretation of the data reported about students' errors and difficulties, about possible reasons that underlie students' performance, and, perhaps, to novel instructional trends.

REFERENCES

Fischbein, E.: 1993, 'The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity', in R. Biehler, R.W. Scholz, R. Straesser and B. Winkelmann (eds.), *Didactics of Mathematics as a Scientific Discipline*, Kluwer Academic Publishers, Dordrecht, Netherlands, pp. 231–245.

NCTM [National Council of Teachers of Mathematics]: 1989, *Curriculum and Evaluation Standards for School Mathematics*, Reston, Virginia.

NCTM [National Council of Teachers of Mathematics]: 2000, *Principles and Standards for School Mathematics*, Reston, Virginia.

Zazkis, R.: 1999, 'Intuitive rules in number theory: Example of "The more of A, the more of B" rule implementation', *Educational Studies in Mathematics* 40(2), 197–209.

Zazkis, R. and Campbell, S.: 1996a, 'Divisibility and multiplicative structure of natural numbers: Prospective teachers' understanding', *Journal for Research in Mathematics Education* 27(5), 540–563.

Zazkis, R. and Campbell, S.: 1996b, 'Prime decomposition: Understanding uniqueness', *Journal of Mathematical Behavior* 15(2), 207–218.

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