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# DIVISIBILITY AND MULTIPLICATIVE STRUCTURE OF NATURAL NUMBERS: PRESERVICE TEACHERS' UNDERSTANDING

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This study contributes to a growing body of research on teachers' content knowledge in mathematics. The domain under investigation was elementary number theory. Our main focus concerned the concept of divisibility and its relation to division, multiplication, prime and composite numbers, factorization, divisibility rules, and prime decomposition. We used a constructivist-oriented theoretical framework for analyzing and interpreting data acquired in clinical interviews with preservice teachers. Participants' responses to questions and tasks indicated pervasive dispositions toward procedural attachments, even when some degree of conceptual understanding was evident. The results of this study provide a preliminary overview of cognitive structures in elementary number theory.

This study is a contribution to the growing body of research on teachers' content knowledge in mathematics. The specific content under investigation is elementary number theory. Elementary concepts of number theory, despite their importance to the field of mathematics, have received scant attention in mathematics education research. Previous studies have used concepts of elementary number theory as a mathematical context for investigating different issues; for example, Martin and Harel (1989) used notions of divisibility in research on preservice teachers' understanding of mathematical proof. Leron (1985) adapted a theorem on the infinity of prime numbers to illustrate a more constructive approach to indirect proofs. Lester and Mau (1993) used prime factors in research on problem solving in a course for prospective elementary school teachers. Movshovitz-Hadar and Hadass (1990) applied proofs for irrationality of square roots of prime numbers for investigating the pedagogical role of paradox and conflict resolution in the education of prospective mathematics teachers. In our research, number-theory concepts themselves are the primary focus of investigation, rather than a means to another end.

The main emphasis of this study is concepts involving divisibility and the multiplicative structure of natural numbers. Previous investigations of multiplicative structures have focused mainly on contextual situations, that is, a story (word) problem that can be solved by applying either multiplication or division (e.g., Ball, 1990; Graeber, Tirosh, & Glover, 1989; Vergnaud, 1988). In accordance with this line of investigation, Schwartz (1988) considered arithmetical concepts and operations as

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a modeling activity in terms of practical applications and Greer (1992) provided an extensive summary of multiplication and division as models of situations. These studies and many others have revealed numerous complexities in understanding elementary arithmetic in various pedagogical contexts. We attempt to complement these studies with an investigation of multiplicative structures of natural numbers *per se*. By *multiplicative structure* we do not mean the multiplicative structure of a problem situation but rather the more abstract multiplicative structure of natural numbers independent of situation or context (Freudenthal, 1983, p. 112). That is to say, we consider multiplicative structure in terms of conceptual attributes and relations pertaining to and implied by the decomposition of natural numbers as unique products of prime factors, as defined by the fundamental theorem of arithmetic.

The objectives motivating this study were threefold—(a) to explore preservice teachers' understanding of elementary concepts in number theory, with emphasis given to concepts involving divisibility and the multiplicative structure of natural numbers; (b) to analyze and describe cognitive strategies used in solving unfamiliar problems involving and combining these concepts within this context; (c) to adapt a constructivist-oriented theoretical framework for the analysis and interpretation of these strategies and to model the cognitive structures supporting them.

The purpose and utility of this investigation is to design and aid the implementation of pedagogical methods that meet contemporary professional-development standards for teachers in the conceptual understanding of mathematics. We agree with Steffe (1990) and many others that improvement of mathematics education starts with improvement of the mathematical knowledge of teachers. Improvement of the mathematical knowledge of teachers starts with a deeper understanding of their existing knowledge and its construction. This study provides details of knowledge, or, to use Schoenfeld's terminology, serves to provide a finer granularity (Schoenfeld, Smith, & Arcavi, 1992) in the domain of divisibility and factorization.

## METHODOLOGY

### *Participants*

Twenty-one preservice elementary school teachers participated in the study. They were volunteers from the group of students involved in a professional development course called "Foundations of Mathematics for Teachers." The mathematical background and experience of the participants varied considerably. Concepts or topics of their curriculum included factorization, least common multiple, greatest common divisor, prime and composite numbers, prime decomposition and the fundamental theorem of arithmetic, and divisibility and alternative "divisibility rules" for the numbers 2, 3, 5, and 9.

### *Data Collection Procedures*

Within a 2-week period, approximately a week after the number-theory-related topics were covered in the course, individual clinical interviews with preservice elementary teachers were conducted using an instrument that allowed for flexibility in probing and clarifying participants' understanding of number theory concepts. The instrument was designed to reveal our participants' ability to address problems by recall or construction of connections within their existing content knowledge. Our objective was not to determine statistical occurrences of particular understandings in any detail but rather to probe for and determine distinctive qualitative features of cognitive structure commonly exhibited in this domain. The questions covered a spectrum ranging from elementary number concepts (e.g., *What does it mean to you that a number is an even number?*) to more subtle and sophisticated problems requiring deeper insight into the elementary theoretical properties and relationships of numbers (e.g., *What is the smallest positive integer divisible by every integer, 1 through 10?*). The specific subset of questions used in this report is described in the next section.

During the interviews, which lasted about 1 hour each, the participants were probed for understanding that may not have been apparent from their initial responses. A consequence of this methodological strategy was that not all, nor necessarily the same, questions were addressed by each participant. In circumstances in which participants experienced difficulties with a particular question, they were encouraged to reflect on and articulate the nature of those difficulties. In cases in which such activity proved inadequate for leading the participant to a realization of a solution or a resolution of his or her difficulties, the interviewer would progressively allude to or provide additional information. Calculators were available to participants upon request.

### *Interview Questions*

The questions were designed to clarify our participants' understanding of procedures and concepts relating to divisibility and to investigate their ability to make connections and inferences from them. All questions presented specific examples of numbers in order to minimize any complexities added by algebraic abstraction. The interview question sets analyzed for this report and the rationale for their formulation are as follows.

#### *Question Set 1*

Consider the number $M = 3^3 \times 5^2 \times 7$.

Is $M$ divisible by 7? Explain.

Is $M$ divisible by 5, 2, 9, 63, 11, 15? Explain.

We designed these questions to investigate our participants' understanding of the connection between the concepts of divisibility and prime decomposition. We were interested in determining whether our participants would take advantage of the fact that $M$ is given as a product of its prime factors, or whether they would actually calculate the value of the product and then divide. In our choice of numbers as divisors we included prime factors of $M$, prime nonfactors of $M$, and composite factors of $M$ in order to investigate the extent to which the nature of the divisor would educe differences in our participants' approach to inferring divisibility.

#### *Question Set 2*

(a) Is 391 divisible by 23?

(b) Is 391 divisible by 46?

We designed these questions to investigate our participants' understanding of the connection between the operation of division and the concept of divisibility by providing cases in which division is warranted (Part a) and in which division is subsequently unnecessary (Part b). We chose the numbers for Part a carefully to eliminate any obvious applications of divisibility properties for specific numbers such as 2, 3, 5, and so forth.

*Question Set 3*

Consider the numbers 12 358 and 12 368. Is there a number between these two numbers that is divisible by 7? By 12?

These questions served to assess our participants' ability to minimize or forego calculation and argue for the existence or nonexistence of a number with a particular property. We were interested in participants' understanding of the modular distribution of numbers sharing the same divisibility property. We were particularly interested in determining specific procedural and conceptual strategies used in addressing these questions.

*Question Set 4*

(a) The number 15 has exactly four divisors. Can you list them all? Can you think of several other numbers that have exactly four divisors?

(b) The number 45 has exactly six divisors. Can you list them all? Can you think of several other numbers that have exactly six divisors?

The participants were presented with either Part a or Part b of this question set, depending on their success with previous questions. We designed this question set to determine the extent to which the participants would systematically construct numbers with the desired properties. We also used it as an indicator of thematization of multiplicative structure in general.

*Data Reduction*

We recorded, transcribed and then categorized the interviews with respect to question sets, their difficulty, and identifiable cognitive patterns of various degrees of sophistication exhibited by the participants. We analyzed and coded responses to the interview questions in accordance with their contribution to the purposes for which the interview questions were originally designed. In particular, we were concerned with (a) applying and evaluating the adequacy of our theoretical framework (see below) for interpreting the development of divisibility concepts and (b) exploring procedural and conceptual relationships between divisibility and division. As our investigation progressed, a third area of analysis, with important implications regarding the first two, emerged, (c) the use of divisibility rules.

## THEORY

### *Action-Process-Object (APO) Framework*

The particular interpretation of constructivism used in this study is based on Dubinsky's action-process-object (APO) developmental framework (Dubinsky, 1991). Dubinsky developed this framework as an adaptation of some of Piaget's ideas that are central to the study of *advanced* mathematical thinking. This framework has been used previously in studies of undergraduate mathematics topics such as functions and groups (Ayres, Davis, Dubinsky, & Lewin, 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky, Leron, Dautermann, & Zazkis, 1994). We hoped that by applying this framework to a domain involving less advanced mathematical understanding, we would contribute to determining the extent to which APO can be useful for investigating the development of mathematical understanding in general.

A central tenet of Piaget's theory is that an individual, disequilibrated by a perceived problem situation in a particular context, will attempt to reequilibrate by assimilating the situation to existing schemas or, if necessary, reconstruct particular schemas to accommodate the situation. Dubinsky (1991) stated that the constructions involved are mainly of three kinds—actions, processes, and objects. An *action* is any repeatable physical or mental manipulation that transforms objects in some way. When the total action can take place entirely in the mind of an individual or just be imagined as taking place without the individual's necessarily running through all of the specific steps, the action has been *interiorized* to become a *process*. New processes can also be constructed by *inverting* or *coordinating* existing processes. When it becomes possible for a process itself to be transformed by some action, then we say that it has been *encapsulated* to become an *object*. We express the construction of connections that relate disparate actions, processes, or objects to a particular object as the *thematization* of the *schema* associated with that object. In this way, we take each object to be a *kernel* of a schema. These notions will be illustrated below as we explore the extent to which this theoretical framework contributes to an understanding of number-theory-knowledge construction and development.

The cognitive events of interiorizing an activity into a process, encapsulating a process into an object, or thematizing a schema are accounted for in this framework in terms of *equilibration*. For instance, some form of disequilibration within a particular context, such as a need to perform an action on an existing process, is taken to serve as a precondition of or motivation for encapsulation, with encapsulation subsequently restoring equilibration. Equilibration serves as an explanatory principle in what otherwise would be a purely descriptive framework.

### *Divisibility: An Abbreviated Illustration of Genetic Decomposition*

We used the action-process-object framework to guide the analysis and interpretation of the interviews for the manners in which the participants appeared to think about the specific topics and problems presented to them. We illustrate the application of the framework with an abbreviated phenomenological analysis or *genetic decomposition* of the concept of divisibility. This is a hypothetical analysis of the way in which the concept of divisibility could be constructed by a learner, viewed in terms of the APO theoretical framework.

A construction of divisibility as a conceptual object starts with specific examples of divisors. In early examples the divisors are usually small numbers such as 2, 3, 4, and 5. Initially, divisibility by 3, for example, is an *action*: A learner has to actively perform division and obtain a quotient of a whole number (no remainder) in order to conclude a posteriori that a number is indeed divisible by 3. Later, the activity of division may be *interiorized* as a *process*, in which the action is intended but not actually performed. In this case, the student has understood the notion that it is the division procedure itself that determines whether or not a whole number satisfies the "rule" or criterion for divisibility. In this way, the action/process distinction can be used to distinguish between what we refer to as *procedural activity* and *procedural understanding.*

Processes of divisibility with particular numbers may be *coordinated* to create new processes of divisibility, that is, processes for new numbers. For example, coordination may be demonstrated when divisibility by 2 and 3 is used to infer divisibility by 6. Furthermore, some processes can be *inverted.* For instance, knowing that the sum of a whole number's digits is divisible by 3 implies that the number itself is also divisible by 3 and can be inverted and used to construct numbers divisible by 3.

*Encapsulation* of divisibility as an *object* could result in an understanding of the concept of divisibility as an essential property of whole numbers independent of the procedural aspects of division. At this level the concept of divisibility is seen in terms of a bivalent, "yes or no" property of whole numbers. Eventually, it becomes possible to conceive that, for natural numbers $a$ and $d$, $a$ is, a priori, either divisible by $d$ or not divisible by $d$. An object of divisibility can be evidenced in making inferences such as, "If $a$ is divisible by $b$, and $b$ is divisible by $c$, then $a$ is divisible by $c$." When divisibility is related to other cognitive structures such as those involving factorization and prime decomposition, we would say that divisibility comes to be *thematized* to form a higher order object, or *schema.* The process/object distinction will be used to distinguish between what we refer to as *procedural understanding* and *conceptual understanding.* Conceptual understanding is indicated by, but not restricted to, encapsulation of processes as objects as it pertains to thematization of objects as well.

In accordance with our constructivist-oriented framework, we view mathematical formalisms as concise expressions of processes and objects that have been properly interiorized, encapsulated, and thematized. The "properness" of an object's interiorization, encapsulation, and thematization involves philosophical issues that would lead beyond the scope of this article. Nevertheless, we take "proper" here to mean "consistent with mathematical convention." Our pedagogical perspective in this regard is that conceptual understanding should converge toward an understanding of the mathematical formalism.

## DATA ANALYSIS AND INTERPRETATION

We begin with a spectrum of developmental levels of our participants' understanding of divisibility, analyzed and interpreted in terms of the APO framework. We consider the APO framework as both a model of cognitive structure and a lens for interpreting the data that allows the data to be seen meaningfully in many cases. However, as will become evident below, there are important areas where the data become blurred. Naturally, we have been selective in the presentation of our data in order to bring out what we feel are the general features of this cognitive domain. Some of these features may be further resolved with the presentation and acquisition of more data, whereas others may require refinements in the APO framework itself. Furthermore, we acknowledge the possibilities of alternative interpretations of the data, especially in the blurry areas, from within the APO framework and from alternative theoretical perspectives. We view these possibilities as offering the potential for further research in this area.

*Development of Divisibility Concepts*

*Actions.* A minority (6 out of 21) of the participants in this study group were able to consistently discuss and demonstrate an understanding of divisibility as a property of, or a relation between, natural numbers. However, the majority (15 out of 21) were unable to do so without at some point performing divisions that such understanding would render unnecessary. Of these, about half (8 out of 15) relied exclusively on division procedures. The latter tendency is an indication that their construction of divisibility had not developed beyond actions. Such a reliance on procedural activity justifies and supports adherence to and dependence on specific examples that, in turn, reinforce a strictly empirical attitude toward mathematics—as this excerpt from the interview with Nicole (arising from Question Set 1) exemplifies:

*Interviewer:* Suppose I have an even number which is divisible by 7. Say I've now divided it by 7. Would I still end up with an even number?

*Nicole:* You'd have to try. You'd have to try to see if it works.

The claim "you'd have to try to see if it works" or "you cannot be sure that the result is a whole number if you don't know what the result is" seemed to be typical in this group of preservice teachers. Even participants who attempted to provide explanations in terms of multiples and divisors often made statements expressing their tendency "to work it out to make sure." This level of understanding requires the carrying out of an action, not only to obtain but also to assure one's confidence in the answer. Thinking of divisibility as an action is further exemplified in the following excerpt (from Question Set 3).

*Interviewer:* Do you think there is a number between 12 358 and 12 368 that is divisible by 7?

*Nicole:* I'll have to try them all—to divide them all—to make sure. Can I use my calculator?

*Interviewer:* Yes, you may, but in a minute. Before you do the divisions, what is your guess, what is your bet?

*Nicole:* I really don't know. If it were 3 or 9 I could sum up the digits. But for 7 we didn't have anything like that. So I will have to divide them all.

[Indeed, Nicole performed several divisions to find the number that gives a whole number quotient when divided by 7 and only then answered the original question positively.]

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*Nicole:* Yes, there is one. Twelve thousand three hundred and sixty-two divided by 7 is 1766 exactly. No decimal part. So this is the number.

*Interviewer:* Do you think there is another number in this interval that is divisible by 7?

*Nicole:* I'll just keep checking, 'cause I can't see a pattern happening, I don't know an easier way that you do it to find—in a glance.

For Nicole the approach to decide whether there is a number between 12 358 and 12 368 that is divisible by 7 was to divide by 7 all the numbers in the given interval. "Keep checking" is the main strategy and may be the only strategy Nicole was aware of, because she claimed she didn't "know an easier way that you do it." Another action is evident in the following excerpt:

*Interviewer:* I'm asking you to look at the number, which is $3^3 \times 5^2 \times 7$. Do you think this number is divisible by 7?

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*Armin:* Okay, first I'm just multiplying $27 \times 25 \times 7$ and I get 4725, and now I need to divide them all by 7.

*Interviewer:* Okay.

*Armin:* So [I] get 675, so you have it divisible.

*Interviewer:* So this number is divisible by 7. Could you know this without using the calculator and without finding out the product of all the numbers?

*Armin:* Could I know it? Um, well, I know we discussed something in class about if, if one number is divisible by 7, then another number is divisible—or what was it—this number is divided by 7, and this number is divided by 7, then the sum of those numbers should divide by 7.

*Interviewer:* If I asked you whether this number was divisible by 5, what would you do?

*Armin:* I'd do the same thing.

Armin, responding here in Question Set 1, preferred to calculate the number $M$ and decide about its divisibility by 7 by performing division. Unlike Nicole, Armin seems to be aware that there are ways other than dividing to infer divisibility. She recalled a theorem related to divisibility, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$, which is not applicable in this case. When subsequently asked whether $M$ was divisible by 5, Armin still claimed, "I'd do the same thing," which is multiply and then divide by 5.

*Interiorization: from action to process.* Interiorization can be characterized as a shift from a procedural activity to an ability to reenact that activity in the imagination with some understanding of the influence of initial conditions regarding outcome. Such a shift is what we use to distinguish an action from a process. Thinking of divisibility as a process is exemplified in the following excerpt (from Question Set 3) in the interview with Jane. She starts with explicit division, but then refers to and draws her conclusion on the basis of *intended* division. This illustrates a procedural understanding, rather than just the procedural activity, of long division.

*Interviewer:* Do you think there is a number between 12 358 and 12 368 that is divisible by 7?

*Jane:* Let's see. [Performs long division] 12 359 divided by 7 gives remainder 4. So, 60, 61, 62,... 12 362 will be divisible by 7.

*Interviewer:* It's interesting. How did you know? I haven't seen you doing division.

*Jane:* If this one (12 359) gave remainder 4, the next one will give remainder 5, and the next one, 6; and the next one, 7; which means zero, or no remainder. So if you divide 12 362 by 7, there will be no remainder—it will be divisible.

Jane demonstrates a procedural understanding that an increment of one in the dividend will result in an increment of 1 in the remainder, and that the latter is taken modulo the divisor 7 (presumably with a concomitant increment of 1 in the quotient). Having calculated that 12 358 leaves a remainder of 4 when divided by 7, she counts up three numbers to reach the number divisible by 7. Still, Jane does not claim the existence of such a number before she actually finds one. Also note that in the last sentence quoted above, Jane provides a clear statement of divisibility in terms of division.

As the following excerpt indicates, Andy is in a transitional stage toward the interiorization of divisibility.

*Interviewer:* Would you please look at the number that is $3^3 \times 5^2 \times 7$. I would call this number $M$.

*Andy:* Okay.

*Interviewer:* Is $M$ divisible by 7?

*Andy:* (Pause) Um, okay, I know that this is 27 and this is 25,... and you're asking divisible by 7?

*Interviewer:* Mm hm.

*Andy:* Oh, um, I'd say "no."

*Interviewer:* And why do you think so?

*Andy:* Um, I guessed "no" because 25 isn't divisible by 7 and 27—oh no, maybe not—I wouldn't be able to guess, I'd have to multiply it out.

*Interviewer:* Do you think there is another way?
(Pause)

*Andy:* Or could I do—oh no—I could do "$3 \times 5$ is 15" and then add those two. (Pause) Could I do that?

*Interviewer:* So you have written "$15^6 \times 7$."

*Andy:* Mm hm, and then I would say that it would be divisible by 7 because you're multiplying it by 7, because 7 is a, a factor.

*Interviewer:* Factor. Please help me understand something. You looked at this expression, "$27 \times 25 \times 7$," and you couldn't draw your conclusion from here, and then you looked at this expression, which is "$15^6 \times 7$," and this helped you to draw your conclusion. Why was it easier for you to look at this [$15^6 \times 7$] than to look at this [$27 \times 25 \times 7$ ]?

*Andy:* Um, (pause) because this [$15^6$ ] is one number.

Early on, in the course of Question Set 1, Andy attempts to carry out the action, "to multiply it out," in order to solve the problem. Following the interviewer's suggestion to think of "another way," Andy "chunks" $3^3 \times 5^2$ to $15^6$ in order to infer divisibility. The interviewer ignores her error in an attempt to detect in what way the new expression can help her to draw a conclusion. It seems evident from her answer that she has failed to associate the product $27 \times 25$ with a single numerical entity. On the other hand, when 7 has one multiplicand, Andy claims it "would be divisible by 7 because you're multiplying it by 7." She is able to recognize 7 as a factor when it is one of two factors, but apparently cannot see it as a factor in the list of more than two factors. Minutes later in the interview Andy claims divisibility by 5 and denies divisibility by 2:

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| *Andy:* | Um, I guess, yeah I could. You probably could say that 5 would be a factor, but that 2 wouldn't be a factor. |
| *Interviewer:* | And why? |
| *Andy:* | Because, because 5 is, 5 is a factor of this number. |
| *Interviewer:* | How do you know? |
| *Andy:* | Because you've multiplied it to get the answer, to get the sum or total, whatever. |

We find Andy, despite referential difficulties between product and sum, to be in the transition stage from thinking of long division as an action to thinking of it as a process. Perhaps the interview stimulated this transition. Unfortunately, it was difficult to ascertain whether Andy actually constructed new knowledge in the course of the interview or whether she belatedly recalled it. She begins with a reference to explicit action, but later the action becomes intended. Andy seems to have been thinking of (possibly prime) factors of $M$ as something one would "multiply by" and this is a step toward procedural understanding.

Also note that, despite her ambiguity of reference, Andy appears to have been thinking of divisibility in terms of multiplication: "It ($M$) would be divisible by 7 because you're multiplying it ($M \div 7$ or actually $3^3 \times 5^2$ or $15^6$) by 7, because 7 is a, a factor." This appears to contrast with Jane's thinking of divisibility in terms of division: "So if you divide 12 362 by 7 there will be no remainder—it will be divisible." It is interesting to note Jane's and Andy's use of the future tense in the discussion of divisibility. Their statements—"it would be divisible" or "if you divide, it will be divisible"—may be interpreted as a transitional dependence on the procedural activity of division and/or multiplication. Their responses build on two different conceptual views of divisibility (see discussion), which Lena brings together in the excerpt below. Lena (also in an excerpt from Question Set 1) notes a seemingly trivial and yet profoundly important relationship between divisibility and multiplication:

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| *Lena:* | Yeah, well, I was thinking that, um (pause), I don't know what I was thinking I guess—well no, I was thinking that if the number was multiplied by 7, then is it divisible by 7, but I don't know if that really means anything. |

The assimilation of these perspectives in terms of object encapsulation and schema thematization eventually leads to a more comprehensive understanding of divisibility that can accommodate connections to other related concepts. First, however, we consider the development of new processes from existing ones.

*New processes: coordinating and inverting.* One of the tenets of Dubinsky's theoretical perspective is that new processes may be obtained from existing processes by coordinating existing processes or by inverting existing processes. The tasks presented to participants in the interviews made it possible for us to observe their constructions and their struggles with coordination, as well as with inverting.

According to the APO framework, the process of divisibility by 15 can also be taken as coordination of divisibility by 5 and by 3. In terms of coordination and inversion of processes, a number is divisible by 15 if and only if it is divisible by 5 and by 3. In other words, if both 3 and 5 are divisors of $M$, then 15 must be a divisor as well. In the following excerpts, Linda and Anita attempt to make such coordinations.

*Interviewer:* And what about 63?

*Linda:* If I'm multiplying right here, $3 \times 3$ is 9, $9 \times 7$ is 63, therefore I would suggest that it would be able to, to go in like that.

*Interviewer:* And why would you make that suggestion?

*Linda:* Um, because it's similarly the way I multiply the 3 and the 3, I know through a principle that I can multiply that as well with the 7, because it's all multiplication. I can't remember the name for it.

*Interviewer:* Mm hm.

*Linda:* So you multiply the 3 by the 3 and then multiply it by the 7. I would need $3^1$ and $5^2$ besides 63 to get $M$.

*Interviewer:* Okay. And 9?

*Anita:* Yes, because um, because you have 3 to the power of 3 and 9 is $3^2$, you can make 9 from $3^2$.

*Interviewer:* Okay. And how about 63?

*Anita:* Yes, because 63 is $9 \times 7$, and you have the 7 and you can make 9 by $3^2$.

Although Anita simply notices that $7 \times 3^2$ is 63, and that explains for her why $M$ is divisible by 63, for Linda pointing out how prime factors of $M$ form 63 doesn't seem to be sufficient. Linda implicitly represents $M$ as a multiple of 63. The "principle" for which Linda "can't remember the name" may reflect coordination of commutativity and associativity that led Linda to represent $M$ as $63 \times (3^1 \times 5^2)$. We've observed that participants' insufficient proficiency with these kinds of basic arithmetic calculations were one of the obstacles in coordination of processes (Campbell & Zazkis, 1994).

Tasks involving coordination appeared difficult for many of our participants. In fact, 13 out of 21 participants were able to infer divisibility of the number $M$ in the first question by 7, 3, and 5 by identifying these as factors in $M$'s prime decomposition. But only 6 of these 13 were able to coordinate these processes of divisibility to infer divisibility by 15 or by 63. Both excerpts from the interview with Bob in the section on encapsulation (below) illustrate some interpretive challenges that can arise from this phenomenon.

Another major difficulty was encountered with inverting processes. Checking whether or not an object has a certain property appears to be easier than constructing an object that has such a property. In a pilot study carried out in the previous year with similar participants and in similar circumstances, 16 participants who were able to check successfully whether a number was divisible by 15 using simple divisibility rules for 3 and 5 had significant difficulties providing examples of six-digit numbers divisible by 15 that are quite readily constructed using those very same rules. Most examples provided seemed to relate to structures associated with long division. These included numbers such as 150 000, 300 000, 151 515, or 153 045 (concatenating 15-30-45). However, when disequilibrated by a further request to determine the largest six-digit number divisible by 15, or give an example of a number without repeating digits—most of the participants resorted to trial and error. That is to say, they would guess a particular six-digit number and then divide it to see if it was divisible by 15. These attempts may not have been as random as they appeared to us at the time of the interviews: A possible interpretation here is that some of these participants may have, consciously or not, returned to long division with an eye open to discovering the subtleties of a much more demanding inversion of the algorithm. Be this as it may, even the participants who were successful in coordinating the divisibility rules for 3 and 5 were unable to follow through with inverting this procedure—which would have allowed them to obtain more easily a solution to the problem presented.

Further evidence of inverting a process was found in the students' responses for Question Set 4: When asked to give an example of a number that had exactly four (or six) divisors, all but three participants preferred to choose a number and then "check it" by listing and counting its divisors. If the "guess" was successful, participants were disequilibrated with a request to generate 10 more examples. We had anticipated that some of our participants would interiorize or encapsulate whatever particular procedure they may have been using. Armin had "guessed" that the number 6 had four divisors, and after noting that "it's any two prime numbers" that give the desired solution, she could easily generate additional examples such as 21, 35, and 55. Tara, after identifying the example of $45 = 3^2 \times 5$ as a number with six divisors, claimed that $3^2 \times 7 = 63$ and $3^2 \times 2 = 18$ also had exactly six divisors. Unfortunately, it was not clear from the interview whether Tara would be able to generalize "3" to mean "any prime" when asked to provide more examples. Linda, for instance, was very close to Tara in recognizing a pattern of this kind. Considering 45 and 18, she concluded "it must be $3^2$ times something," but her attempt to try to list the divisors of $3^2 \times 4$ placed her conjecture in doubt.

*Encapsulation: from process to object.* Encapsulation of divisibility as an object is indicated when a learner begins to distinguish the concept of divisibility from procedures of division and/or multiplication. The following is an excerpt from the interview with Bob, who explains divisibility of $M$ by 7 and 5 in terms of factors in the prime decomposition of $M$.

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| *Interviewer:* | Bob, I'm going to ask you to write down a number, please. And that number is $3^3 \times 5^2 \times 7$, and we're going to call this number $M$. Now, my first question is, is $M$ divisible by 7? |
| *Bob:* | Yes, it is. |
| *Interviewer:* | And would you explain why? |
| *Bob:* | Well if 7 (pause), let's see (laugh), $M$ is, or let's see, so 7 is a factor of $M$, therefore, it's divisible by $M$, pardon me, by 7. |
| *Interviewer:* | And how about 5? |

*Bob:* Five is also a factor of $M$.

*Interviewer:* Okay, and would $M$ be divisible by 2?

*Bob:* No, it would not, since 2 is, um (pause), since 2 is not seen here, it's not a factor of $M$.

*Interviewer:* Hmm, okay, and why do you feel that that's the case?

*Bob:* Um, explain this clearly (pause), since 2 is not one of the numbers that's being multiplied, the product, therefore, can't be divided by 2.

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*Bob:* Okay. And that since, obviously, 2 is a prime number, the prime number of 2 is not in this solution; therefore, whatever the product $M$ turns out to be, 2 cannot divide into that.

This excerpt illustrates that Bob has made some important connections regarding relationships between factors of multiplication with divisors, and that he may have encapsulated divisibility as an object. In particular, Bob's phrase "whatever the product $M$ turns out to be" expresses *the* crucial step required in crossing over from procedural to conceptual understanding. However, although some encapsulation may be indicated in determining divisibility by 7 and in refuting divisibility by 2, encapsulation of divisibility may as yet be incomplete. For some, the ability to infer divisibility on the basis of a connection with prime decomposition or multiplicative factors is not necessarily achieved simultaneously with the ability to infer indivisibility. Patty, for example, notes that both 7 and 5, as factors, were divisors of $M$, but then regresses to procedure when asked about divisibility by 2 and 11.

*Interviewer:* Okay. And will it be divisible by 2?

*Patty:* I would multiply each one and find out what the total number is. So $3 \times 3$ is $9 \times 3$ is 27, and this 25 is times 7. (Pause) It's not, 2 doesn't go into it evenly.

*Interviewer:* So you computed the number and you got 4725, and now you are sure that it is not divisible by 2.

*Patty:* Right.

*Interviewer:* But you were able to conclude about divisibility by 7 with, before you knew what was the number.

*Patty:* Mm hm.

*Interviewer:* So how is it?

*Patty:* Because 7 is a factor of it, so it's, what is it, the commutative law or associate law, 7 is a factor of it.

*Interviewer:* And what about divisibility of $M$ by 11?

*Patty:* I would divide 4725 by 11 to find out.

Like Andy in a previous section, Patty may be appealing to associativity in order to "chunk" $M$ as $(3^3 \times 5^2) \times 7$. Aside from this important question as to what constitutes a factor, it appears that Patty may not have made the connection that only factors of $M$ are divisors of $M$. In other words, she may have been thinking that 2 and 11 could possibly be divisors of $M$ even if they are not actually factors of $M$.

Unlike Patty, Bob discusses both divisibility and indivisibility in terms of $M$'s divisors and nondivisors. It is indeed tempting to conclude from his responses that

Bob has completely encapsulated divisibility. The next excerpt illustrates that this is not quite the case.

*Interviewer:* Would you think that 81 would divide *M*?

*Bob:* I'd want to find out what *M* would be, I guess that's the, the best thing, that's what I'd prefer.

*Interviewer:* Mm hm.

*Bob:* I guess knowing what *M* would equal, and then from there working backwards, finding which numbers can go into that.

*Interviewer:* Mm hm.

*Bob:* Um, right now I can't see whether or not 81 can go in there.

*Interviewer:* Okay. Uh, how about 63?

*Bob:* (Pause) Once again, um, we have 7. Now, 7 can go into 63, well [??] 3 can, as well (pause). Once again I'd have to solve for *M*, in order to find out whether 63 can divide *M*.

*Interviewer:* Okay, so when you say "solve for *M*," you mean, like, multiply it out and then divide by 63?

*Bob:* Yeah, exactly, exactly.

*Interviewer:* Okay. How about if you wish to divide *M* by 15?

*Bob:* (Pause) Um, well, since there's 5, $5^2$ in this problem, we know that the, that the units digit will be 5. Now 15 obviously has a 5 in it as well, therefore, quite possibly, 15 will go into *M*, and once again I'd have to solve for that.

Bob provides logical arguments when discussing divisibility of *M* by 7 and 5 and indivisibility of *M* by 2. However, when asked about 81, 63, and 15, Bob describes his strategy as "to solve for *M*," that is, to find out the value of the number and then perform division. From the previous excerpt it seems fairly evident that Bob recognizes that a factor of *M* will also be a divisor of *M*. It is remotely possible that Bob's logic only reflects a linguistic procedure; however, it is more likely that, like Andy, and perhaps Patty as well, his subsequent difficulties reflect deficiencies in fully understanding what constitutes a factor. This would explain his compromised ability to fully connect prime decomposition with divisibility. Has Bob not fully encapsulated divisibility or has he yet to thematize the relations between divisibility, prime decomposition, and multiplicative structure? This is a difficult call. Bob has at least encapsulated divisibility as an object to the extent that he recognizes a priori that some prime numbers will either divide or not divide *M*. However, it is apparent that his construction of divisibility as an object is not solid enough to accommodate composite numbers. That Bob has yet to make certain connections with other objects, such as factors and composite numbers, may illustrate that he has yet to fully thematize divisibility as a schema. On the other hand, perhaps he has yet to adequately coordinate specific cases in order to encapsulate "divisibility by *n*" as an object for all *n*—that is to say, for composite, as well as prime, numbers.

These interpretive ambiguities warrant further empirical investigation and may require further refinements in our theoretical framework, as well. Be this as it may, it appears that encapsulation of divisibility as an object does not necessarily occur as a singular event or realization. Patty's and Bob's interviews provide evidence that encapsulating divisibility involves coordination and inversion of specific examples of divisibility by specific numbers.

*Thematization: from object to schema.* Our interpretation of the Dubinsky theoretical framework is that thematizing divisibility as a schema involves constructing specific relationships between divisibility and other objects from number theory such as primes, factors, and so on. The following excerpts illustrate that Jane was able to coordinate divisibility smoothly by different primes to draw conclusions about composite cases.

| | |
|---|---|
| *Interviewer:* | And what about something like 63? |
| *Jane:* | Sixty-three, $7 \times 9$ is 63, so yeah, it would be. |
| *Interviewer:* | So, what have you done here? Would you please explain? |
| *Jane:* | I took some of the kind of factors, I took $3 \times 3$—so $3^2$, which is 9—and I multiplied that by 7, which gives me 63. And because $3^2$ and 7 are both part of the prime factorization, then the entire number is divisible by 63. |

Jane demonstrates her understanding of *M*'s divisibility not only by its prime factors, but also by composite products of its prime factors. Making this connection between factors and prime decomposition with divisibility appears to greatly contribute to the thematization of divisibility. Dana clearly illustrates the power of thematization of divisibility as a schema by applying the connection with prime decomposition to refute divisibility of 391 by 46.

| | |
|---|---|
| *Interviewer:* | Okay, um, would you say 391 is divisible by 46? |
| *Dana:* | (Pause) No, because 23 and 17 are both prime numbers. There is no 2 involved in there, it's just 23 times 17. |

Our concluding example in this section indicates in one sense how deeply thematization of a schema can reach and in another how deeply entrenched the obstacles to encapsulation can be. The following excerpt shows that Pam not only infers divisibility by 7 as an a priori property but can also explain how often this property is found within a contiguous set of whole numbers.

| | |
|---|---|
| *Interviewer:* | Do you think there is a number between 12 358 and 12 368 that is divisible by 7? |
| *Pam:* | I think there is. |
| *Interviewer:* | Do you know which number it is? |
| *Pam:* | Not yet, but I can find it if you want me to. |
| *Interviewer:* | No, you don't have to find it. But if you don't know what it is, how do you know it is there? |
| *Pam:* | Here we have 9 numbers. And I know that if I take any 7 numbers there will be one divisible by 7. And here I have 9, which is more than 7. |
| *Interviewer:* | Are you saying that if I pick any 7 numbers I wish, there will be one divisible by 7? |
| *Pam:* | I didn't mean that. What I mean is if you take these numbers one after another, there will be one of them divisible by 7. |

The understanding of this modular distribution of numbers that share a certain divisibility property is an indication of the depth of Pam's schema for divisibility, in that a strong connection may be made between it and multiplicative structure. Pam's idea that "every seventh number is divisible by 7" was part of the repertoire of only 7 participants in this group of 21 preservice teachers. Nicole, for instance, after finding the number divisible by 7 in the given interval, was asked whether she could have predicted the existence of such a number without calculating it. Her answer was negative, followed with the explanation: "The further you go, the more they grow apart. By the time you get up into numbers that are this high, the difference between the two numbers is only 10. There would be a larger difference between the two numbers (divisible by 7)." This description was accompanied by a hand-waving that indicated progressively increasing intervals. It is possible that Nicole was confusing multiplication with exponentiation here. Nevertheless, this and other instances led us to suspect that, for many of our participants, the process of repeated addition had not been properly encapsulated in terms of multiplication as an object. Deficiencies in relating addition to multiplication, and thus multiplication to divisibility, appear partially responsible for Nicole's reliance on procedure and would present obstacles to both encapsulation and further thematization of divisibility.

*Divisibility and Division*

Is the number 41 418 divisible by 177? Unless one's answer comes from divine inspiration or one has memorized the multiplication table for 177, there is but one obvious way to find out—divide. It was mentioned above that encapsulation of divisibility as an object must begin by discerning between divisibility as a property and division as a procedure. But even with a clear understanding of divisibility, there is no evident alternative strategy for answering the above question without performing division. Indeed, a majority of our participants applied this strategy when addressing the first part of Question Set 2: "Is 391 divisible by 23?" These numbers were carefully chosen to make it difficult to guess or to determine an answer using divisibility rules, yet easy enough to perform division, even without the help of a calculator. However, 6 out of the 21 participants applied strategies other than carrying out division.

Lena and Joan infer divisibility by performing multiplication. Here, Lena looks for a number that gives 391 when multiplied by 23.

*Interviewer:* I'm going to ask you, is 391 divisible by 23?

*Lena:* Hmm, (pause) I'm not sure if it's divisible evenly or not.

*Interviewer:* Mm hm. How would you go about answering that question for yourself?

*Lena:* Okay, this is what I would do. Twenty-three, um, if I was given this question, I would honestly just, you know, plug in a few numbers and multiply them by 23 to see how close I get to 391. So, I'm going to try that....

*Interviewer:* It was, "Bingo!"

*Lena:* Right, okay, well, the only reason I chose 17 is because I know that $7 \times 3$ is 21, and I know that it's not 7, because $7 \times 23$ is too small of a number, so I put a 1 in front of it.

Lena's approach is creatively based on estimation and considering the number patterns for the last digit. Joan uses a rough estimation to establish a starting point, and then converges toward the product via a tedious set of incremental multiplications.

*Interviewer:* Is 391 divisible by 23?

*Joan:* (Pause) Do you want me to figure it out, or just...

*Interviewer:* Sure, go ahead, take your time.

*Joan:* (Pause) Okay, yeah, it's 17 times.

*Interviewer:* Okay. Okay, could you explain to me how you've approached this problem, Joan?

*Joan:* Um, well, I just took as many, I kept sort of going up, well, from 23 × 10 it would be 230, so I went up from there. I multiplied by 13—it might be a bit more than that—and then ultimately I kept going up and up until I multiplied it by 17 and found that 23 × 17 is 391, so the 391 divided by 23 would equal 17.

Joan's approach consists of multiplying (with paper and pencil) 23 by 13, 14, 15, 16, and 17. To the question about her choice of strategy, Joan replies:

*Joan:* I always feel that it's easier just to keep multiplying, because I have to multiply anyways to figure out what this is and what this is.

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*Joan:* Um, I suppose I could have (done) long division, but it, um, whenever I get one like that, I always multiply to divide, because it's just, I find it easier.

Joan has some understanding of the relationship between multiplication and division, and she demonstrates a clear preference for the former. To emphasize, she prefers to perform five long multiplications instead of one long division. It may be the case that both Lena and Joan carry with them from elementary school some discomfort with long division. But during the rest of the interview, Joan and Lena perform long division several times with ease and without seeking the help of the calculator that was placed in front of them on the desk. Therefore, we suggest that, for these participants, the concept of divisibility was related to multiplication by the definition "$b$ is divisible by $a$, or $a$ divides $b$, if there exists a natural number $d$ such that $ad = b$." Both Lena's and Joan's activity demonstrate a search for such a $d$.

Karen, Anabelle, and Tara claim that 391 is not divisible by 23 because 391 is prime. The "primeness" of 391 was inferred in different ways. Karen claims that 391 is a prime number because the sum of its digits is a prime number.

*Interviewer:* Let's take the number 391. Is 391 divisible by 23?

*Karen:* Ugh, (pause) um, I don't think so, because when, when I add 3 + 9 is 12 and 1 is 13, and 13 is not really divisible, like the sum of the digits in 391 aren't really divisible. Thirteen is not really divisible by anything—it's sort of a prime, like the prime number. Um, so basically I don't think 391 is divisible by anything, because, because the sum of the digits is 13. I don't think it's divisible by anything, except for, um, 1 and itself.

Anabelle and Tara reach the conclusion about the primeness of 391 after they are unable to find a small prime number by which it is divisible.

*Interviewer:* Let's take the number 391. Would 391 be divisible by 23?

*Anabelle:* Twenty-three (pause), I don't know. I don't think so.

*Interviewer:* Hmm, and why do you think not?

*Anabelle:* Three hundred ninety-one, I think, is a prime.

*Interviewer:* And why do you think that 391 is a prime?

*Anabelle:* Because I don't think it had been divided by 2 or 3, or 5, or 7 (laugh).

In the next step Anabelle divided 391 by 11, and the fractional result on the calculator confirmed her conclusion about the primeness of 391. Tara doesn't stop at 11, but proceeds a little further.

*Interviewer:* Okay, alright, um, so if you were to determine for yourself one way or another whether or not 391 was divisible by 23, how would you go about it?

*Tara:* Um, I would find the prime factorization of 391....

*Tara:* I have a feeling this is probably a prime number....

*Interviewer:* And can you tell me how you're going about this?

*Tara:* Oh, yeah, um, I'm trying to find a prime that divides 391 evenly, and I tried 11 for some reason, but I thought maybe it would work. It didn't, I tried 13, it didn't work, so, so I have to come to the conclusion that maybe it's a prime number, uh (pause).

After about ten minutes of prompting in search of divisors of 391, attempts to establish its primeness, and facing the evident with the help of a calculator, Tara was asked why she didn't simply divide 391 by 23 from the beginning. Her answer was as follows.

*Tara:* I don't know. I guess, like, I, um, like I was saying with, I know there's a way to do it, prime factorization, and I know that 23 is a prime number, but I guess, um, I was assuming, for some reason, that as long as 391 was not a prime number, it would have a factor smaller than 23, a prime factor smaller than 23.

We believe in the context of the rest of Tara's interview that her reference to "prime factor smaller than 23" was not based on estimating the square root of 391 and looking for a factor smaller than a square root. We conclude that Tara simply wanted to see "small numbers" in the prime decomposition. In our research we have found additional evidence supporting some students' belief that "prime decomposition" means "decomposition into *small* primes" and that this belief coexists with their awareness of existence of "very big" primes. A detailed discussion of this issue can be found in Zazkis and Campbell (1994).

Stanley, similarly to Karen, tries to generalize a rule for divisibility by considering the sum of the digits. In his opinion, 391 is not divisible by 23 because the sum of the digits was not divisible by 23.

*Interviewer:* Okay, is 391 divisible by 23?

*Stanley:* By 23? Um, let me think. (Pause) I don't think so.

*Interviewer:* Okay, can you tell me, uh, a bit about what you've done here in terms of how you've thought about it, and how that corresponds to what you've written?

*Stanley:* Okay. Um, basically I'm just going on assumption, that I just learned in my last class there, in that, uh, if I add the digits of 391, it'll go 13, and if 13 is divisible by 23, then the number itself should be divisible by 23, but, uh, seeing as it's not, then the number isn't divisible by 23.

*Interviewer:* Okay, and before you used that, uh, or learned that rule, um, how would you have gone about answering the question?

*Stanley:* Uh, I probably would have sat and counted up 23 enough times until it gets close enough to 391, it'll either be under it or over it, or dead on, um, and depending what the result was, I would decide....

The "rule" that Stanley uses here was most likely confused with divisibility rules for 3 and 9. But even when specifically asked not to apply his rule, Stanley did not opt to use division, nor did he choose multiplication. He would rather have "counted up" by 23. Apparently, Stanley was more at ease with addition in determining divisibility than with either division or multiplication (this phenomenon is further investigated in Campbell & Zazkis, 1994).

As a matter of fact, four of these six students gave reasonable and mostly correct answers on Question 1, arguing divisibility of the number $M$ in terms of its divisors and nondivisors. It may be the case that they tried to avoid division in a search for more powerful strategies, accompanied by a belief that such strategies do exist for most, if not all, cases.

*Divisibility Rules*

Divisibility can always be inferred by performing division; however, carrying out the division algorithm can be tedious and time-consuming. If quotients or remainders are not required, divisibility rules allow for the possibility of inferring divisibility without performing division. Such "permission" may help the learner to separate performing division from considering divisibility as an intrinsic property of a number. Therefore, understanding and application of divisibility rules may help students move toward encapsulation of divisibility by a specific number as a conceptual object. On the other hand, there is a danger that divisibility as a property of a number may be procedurally reduced to seeking patterns of digits. For example, when Andy was asked, "Do you think there is a number between 12 358 and 12 368 that is divisible by 7," she attempted to find a number in this interval with the sum of the digits divisible by 7. A more common answer (repeated three times in this group of 21 students) was "Twelve thousand three hundred and sixty-three is divisible by 7 since 63 is divisible by 7." These responses do well to summarize the misapplication of the divisibility rules learned or reviewed by this group of students. As mentioned above, the participants were familiar with the divisibility rules for 3 and 9 (the sum-of-the-digits rule) and with the divisibility rules for 2, 5, and 10 (the last-digit rules). Eight out of 21 participants managed to inappropriately generalize or apply at least one of the rules in response to at least one of the questions. In the previous section we observed Karen's conclusion that 391 was prime because the sum of its digits, 13, was prime, and Stanley's conclusion that 391 was not divisible by 23 because the sum of its digits was not divisible by 23. Here is another example from Question Set 1.

*Jennifer:* I went $225 \times 7$ and got 1575 and figured 1575 isn't divisible by 7.
*Interviewer:* Okay, and the, the basis of your conclusion there?
*Jennifer:* Because I looked at the last digit....
*Interviewer:* Okay. And 9?
*Jennifer:* Um, 9 as well, is not divisible, or 1575 isn't divisible by 9 because I'm looking at the last two numbers, 75, and know that 75 isn't divisible by 9.

Jennifer denied divisibility of $M$ by 7 and by 9 by considering the last digit and the last two digits of the product 1575. When Anabelle was asked the third question ("Is there a number between 12 358 and 12 368 divisible by 7?"), she claimed, "I don't know how to figure out how a number is divisible by 7," and this, just after she had tested divisibility by 23 by long division. Anabelle's answer may indicate more than an unawareness of a divisibility rule for 7; she may be assuming the existence of such rules for most, if not all, numbers.

Karen, in order to find the prime decomposition of 391 considers the prime decomposition of 91. Her conclusion that 91 is prime confirms her previous idea that 391 may be prime.

*Karen:* Well, I, I think what I'd do is, I'd take 391 and try to think about, um, breaking it down into the prime factorization, if I could.

*Interviewer:* Mm hm.

*Karen:* But see, um, I sort of realized that if this were 390, it would automatically be divisible by 10 and 39, I think,... 39. But because it's 391, um, and also, like, if you look at the, even the 91 here, um, we know that 91—we don't have anything in the times tables that actually equals 91 too—we know that that's prime.

A point to note from this excerpt, aside from our main discussion here, is the claim that "we don't have anything in the times tables that actually equals 91." Indeed, in times tables for one-digit numbers typically memorized in third grade, nothing equals 91. Nevertheless, it is a fact that $91 = 13 \times 7$. Is it just a computational omission or is there a procedural dependence on times tables?

Tara, in the excerpt below, repeats Karen's assumption that "391 would be prime if 91 was prime," and later tests her conjecture that the prime decomposition of 391 can be found as a sum, whatever the "sum" could mean here, of prime decomposition of 300, 80, and 11.

*Tara:* Yeah, I'm trying to prime, find the prime factorization of 391, so that I can determine whether or not 23 divides into 391. Umm, (pause) hmm, and 391 is not prime? Can you tell me that?

*Interviewer:* No comment.

*Tara:* (Laugh) I have to determine that, right? Well, I guess, I mean, 391 would be prime if 91 was prime.

*Interviewer:* And how so?

*Tara:* Hmm, I don't know. I'm not sure of that statement either, but, hmm, for some reason like it seems to make sense to me and I don't know why (laugh). Hmm, (pause) actually, I have a question for you and I know you're not going to answer. I was going to say, "Now if I write it in this form and I prime factorize this, wouldn't that be the same thing as to find the prime factorization of this number?"

*Interviewer:* Oh, I see, um, you've written 391 as 300 plus 80 plus 11, and you'd like to know whether or not it would be equivalent to do the prime factorization of the three numbers in that sum as an equivalent procedure to finding the prime factorization of 391.

*Tara:* That's right.

*Interviewer:* Okay. Um, well, I would suggest that you experiment with that.

[Tara experiments]

*Tara:* Well, that wouldn't be the same. I couldn't do the prime factorization of 300 plus 80 plus 11 if I wanted to get the prime factorization of 391. (Pause) Hmm, well, uh...

Tara's conjecture, even though she rejects it after experimenting, may be an additional example of what is described by Matz (1982) as a "misapplication of linearity" or an "overgeneralization of distributivity." The erroneous claim that $\sin(a + b) = \sin(a) + \sin(b)$ is one of the "classical" examples of such overgeneralization. In Tara's case, a similar overgeneralization is apparent: prime decomposition$(a + b)$ = prime decomposition$(a)$ + prime decomposition$(b)$.

Many of our participants overgeneralized and misapplied divisibility rules when specific rules were not available. According to Matz (1982), these errors may be explained as students' reasonable, although unsuccessful, attempts to adapt previously acquired knowledge to a new situation. We also note, from these pseudodivisibility rules, not only students' propensities to grasp at procedures in the absence of conceptual understanding, but also a sense of disequilibration in the absence of a rule to use or follow and a subsequent sense of reequilibration from the creation of such pseudorules.

## DISCUSSION

The action-process-object framework has proved useful for describing the construction of mathematical knowledge and has provided a reasonable, although not totally unambiguous, vocabulary to describe students' difficulties in these constructions. The data indicated complex interpenetrating layers of cognitive structure underlying the concepts of number theory that extend deep into the elementary operations and concepts of arithmetic. The data also indicated some theoretical ambiguities with respect to coordination of processes and thematization of schemata. When analyzed and interpreted in terms of this framework, responses to questions and tasks such as considered herein worked particularly well in revealing the pervasiveness of procedural attachments, even when some degree of conceptual understanding is in evidence. We agree with Sfard (1991) that procedural and conceptual dispositions are not incompatible but rather are complementary. The manifestation of procedural understanding cannot be taken strictly as the absence of conceptual understanding. However, the amount of time and effort that a learner requires to resolve or complete a particular problem or task offers some indication of the degree of conceptual understanding involved. We have attempted to design questions that require significant time and effort to be solved procedurally and yet are readily resolvable with the appropriate conceptual understanding.

Some of the interview questions revealed a tension in the subjects between the desire to apply recently acquired mathematical knowledge on one hand and the desire to feel certain with the answer on the other hand. Calculating the number $M$ (in Question Set 1) or dividing all the numbers in the given interval by 7 (in Question Set 3) left the participants certain of their conclusions, although many of them made statements indicating their dissatisfaction with the chosen approach, for instance, "There should be a better way" and "I did it the long way—I couldn't think of any shortcut." But can one be certain of one's claim of existence of a number divisible by 7 between 12 358 and 12 368 without actually determining such a number? Can one be certain that the result of division would turn out to be a whole number without knowing what this number would be? Achieving such certainty is a step toward mathematical maturity as well as toward encapsulation of divisibility.

We have argued that encapsulation of divisibility as an object requires a movement beyond the outcome of actual or intended procedures *with* numbers toward a conceptual understanding of the intrinsic multiplicative structure *of* numbers. Many of our participants who demonstrated process or even object constructions of divisibility claimed they would have preferred to carry out the action in order to feel sure about their conclusions. This appears consistent with the findings of Martin and Harel (1989) and Fischbein and Kedem (1982), who observed that many students who were convinced by deductive arguments wanted further empirical verification.

However, procedural dependencies without some form of conceptual guidance often result in tedious and time-consuming "hit and miss" strategies (such as Joan's and Stanley's, above) potentially leading toward total disenchantment with the subject. As long as procedures lead to some form of conceptual understanding, "hit and miss" strategies can be quite fruitful. For some students (such as Jane above) the strategy of "hit and miss" progresses toward more guided and constrained strategies of "guesses and checks" and is essential as a midpoint between experimentation and structural insight. In some cases just a handful of successful guesses is all that is needed to uncover the general structure beneath such guesses and achieve a greater sense of certainty. We believe that a useful pedagogical approach for preservice elementary school teachers would be to accompany any abstract reasoning with particular examples and calculations, using calculators for calculations with larger numbers. Our hope is that at some level the particular calculations instantiating the abstraction would reinforce the abstraction and then gradually give way to it.

A major interpretive difficulty was encountered with respect to the encapsulation of divisibility as an object. This may simply reflect the fact that divisibility is a very complex cognitive structure. Understanding "divisibility by *n*" as a generalized object in some, and quite possibly all, cases is preceded by the encapsulation of separate processes of divisibility for specific numbers. These encapsulations need not occur simultaneously for all numbers. For example, in reviewing Bob's interview we suggest that he is thinking of "evenness," or divisibility by 5, as an object and of divisibility by 15 as an action or process. At some point, perhaps, a "critical mass" is obtained with the accumulating encapsulation of divisibility for specific numbers serving as a catalyst for the encapsulation of "divisibility by *n*."

This raises an important issue regarding the pedagogical role of divisibility rules. For individuals who are procedurally oriented, divisibility rules are likely to be considered procedurally, as well. This added complexity and heterogeneity regarding various division procedures may confuse and compromise encapsulation of "divisibility by *n*." For those who are well versed in division and its relationship with divisibility, the divisibility rules may serve to emancipate the concept from the procedure. Or, perhaps divisibility rules should be avoided altogether until after divisibility has been fully encapsulated. However, these conjectures can only be substantiated with further study.

Furthermore, encapsulation of divisibility appears to be intimately bound up with a more general thematization of divisibility as a schema—that is, with the construction of connections with other actions-processes-objects involving multiplication, division, factoring, prime decomposition, and so on. We have noted that as the framework is applied to more elementary concepts, the relationship between encapsulation and thematization blurs into a host of "cognitive substructures." To some extent this effect appears to be related to the fact that objects may be used in processes and within schemas without being adequately encapsulated. In the course of this investigation we have noted many cases that suggest that our participants appear not to have encapsulated basic objects such as factors, multiplication, and distributivity. These issues have been pursued in more detail in Campbell and Zazkis (1994).

Aside from these "finer-grained" interpretive difficulties, there is strong evidence for the central role of conceiving divisibility in terms of both multiplication and division. That is to say, in terms of division, $a \mid b$ ($a$ divides $b$) if and only if $b \div a = d$ where $d$ is a natural number; and in terms of multiplication, $a \mid b$ if and only if there exists a natural number $d$ such that $ad = b$. It seems that insufficient connections between these two definitions were a source of cognitive discord for many of the participants in this study. If this were true, it would be quite ironic in the sense that these differences are complementary and could be used pedagogically to highlight the important inverse relationship between multiplication and division. That is, in terms of natural numbers $a$, $b$, and $d$, $ad = b$ if and only if $b \div a = d$. In particular, a progressively sophisticated pedagogical approach with regard to this inverse relationship would seem to be most viable. Such an approach would emphasize the equivalence of factors and divisors and subsequently their representation in terms of prime decomposition.

We have provided examples demonstrating how both of these forms of understanding can be manifested as actions or processes, either by performing division or finding the "missing multiplier." We believe a proper encapsulation of "divisibility by $n$" requires a firm understanding of the inverse relationship between the operations of multiplication and division. This relationship may be perceived as trivial when multiplication involves two elements only; however, when there are more than two elements in the product (see Question Set 1), the relation of multiplication to division, as our data demonstrates, is often not implied. Making this connection between divisibility and prime decomposition appears to greatly contribute to the understanding of divisibility as a thematized schema.

Overall, our results suggest that in the schooling of the participants involved in this study, insufficient pedagogical emphasis has been placed on developing an understanding of the most basic and elementary concepts of arithmetic. At some point, if one is to *meaningfully* continue in mathematics, the basic concepts of arithmetic must be grasped. If this is not happening in the middle grades, then it should come as no surprise that many students fail to make a successful transition to algebra. We believe that developing a conceptual understanding of divisibility and factorization is essential in the development of conceptual understanding of the multiplicative structure of numbers in general. In fact, we suggest that the successful development of conceptual understanding in algebra requires a firm grounding in the conceptual understanding of arithmetic and elementary number theory.

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