Arithmetic Sequence as a Bridge between Conceptual Fields

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Abstract: Arithmetic sequence is used in this study as a means to explore pre-service elementary school teachers' connections between additive and multiplicative structures as well as several concepts related to introductory number theory. Vergnaud's theory of conceptual fields is used and refined to analyze students' attempts to test membership of given numbers and to generate elements that are members of a given infinite arithmetic sequence. Our results indicate that participants made a strong distinction between two types of arithmetic sequences: sequences of multiples (e.g., 7, 14, 21, 28, …) and sequences of 'non-multiples,' (e.g., 8, 15, 22, 29, …). Students were more successful in recognizing the underlying structure of elements in sequences of multiples, whereas for sequences of non-multiples students often preferred algebraic computations and were mostly unaware of the invariant structure linking the two types. We examine the development of students' schemes as they identify differences and similarities in situations presented to them.

Sommaire exécutif : Une séquence arithmétique est une séquence de nombres caractérisée par une différence commune entre des paires adjacentes. Dans le cadre de cette étude, nous voulons en savoir plus sur la façon dont les futurs enseignants et enseignantes de l'élémentaire analysent la structure sous-jacente des séquences arithmétiques. La séquence arithmétique est utilisée comme moyen d'analyser, d'une part, les liens que font les futurs enseignants et enseignantes de l'élémentaire entre les structures additives et multiplicatives, et d'autre part, plusieurs concepts liés à l'introduction de la théorie des nombres. On leur a posé les questions suivantes : Quelles stratégies cognitives les élèves utilisent-ils lorsqu'ils sont confrontés à des problèmes nouveaux liés aux séquences arithmétiques ? Quels sont les patterns qui sont manifestes aux yeux des élèves dans la structure mathématique des séquences arithmétiques ? Quels sont ceux qui ne sont pas reconnus ? Comment les élèves appliquent-ils les patterns reconnus à des situations données de résolution de problèmes ? Quels sont les outils et les notions mathématiques qu'ils utilisent ? Dans quels cas recourent-ils à une approche fondée sur les patterns et dans quels cas préfèrent-ils une approche computationelle ? Par quoi leurs choix sont-ils influencés ?

Cette étude s'inscrit dans le cadre théorique des champs conceptuels de Vergnaud. Un champ conceptuel, dans notre interprétation de Vergnaud (1994, 1996), est un ensemble de concepts, d'opérations et de théorèmes reliés entre eux. Selon la définition de Vergnaud (1996, p. 238), un concept est un triplet formé de trois composantes C = (S, I, ***S***), où S est l'ensemble des situations dans lesquelles ce concept prend une signification, I est l'ensemble des invariants opérationnels qui apparaissent dans les schèmes développés pour pouvoir faire face à ces situations, et ***S*** est l'ensemble des représentations symboliques (langage naturel, diagrammes, représentations graphiques, …) qui peuvent être utilisées pour représenter les relations impliquées dans ces situations, communiquer à leur propos et aider à les maîtriser. Vergnaud (1994, p. 58) définit les schèmes comme des systèmes invariants de comportements qui s'appliquent à des classes de problèmes bien définies.

Dans cette étude, nous avons réalisé des entrevues cliniques avec de futurs enseignants et enseignantes de l'élémentaire à qui nous avons présenté deux types de situations : (a) la vérification de l'appartenance d'un élément donné à une séquence donnée, par exemple : « 360 est-il un élément de la séquence 2, 5, 8 … ? » ; (b) la génération d'exemples : « Donnez un exemple de nombre élevé qui soit un élément de la séquence 2, 5, 8 … ». Nous avons analysé les schèmes dynamiques auxquels font appel les étudiants et étudiantes dans leurs tentatives de résoudre les situations, en particulier pour

1. identifier et décrire les stratégies (ou règles d'action) utilisées lorsque les participants se trouvent devant des situations problématiques liées aux séquences arithmétiques ;
2. analyser les stratégies des étudiants et découvrir les « théorèmes-en-action » sous-jacents ;
3. suggérer une piste pour le développement des schèmes individuels dans le contexte des situations présentées ;
4. analyser le développement des schèmes individuels sous l'angle des relations entre les champs conceptuels des structures multiplicatives, des structures additives et de l'algèbre élémentaire.

Nos résultats indiquent que les participants distinguaient nettement deux types de séquences arithmétiques, les séquences de multiples (par exemple 7, 14, 21, 28 …) et les séquences « non multiples » (par exemple 8, 15, 22, 29 …). Il était plus facile pour les étudiants et étudiantes de reconnaître la structure sous-jacente dans les séquences de multiples, tandis que pour les séquences non multiples ils préféraient souvent les computations algébriques et ne percevaient guère la structure invariante qui reliait les deux types. Dans les séquences de multiples, les étudiants et étudiantes reconnaissaient aussi bien les structures additives (différence commune) que multiplicatives (où chaque élément est le multiple d'une différence commune). Dans les séquences non multiples, la majorité n'ont reconnu que la structure additive. Plusieurs incitations ont même été nécessaires pour que certains étudiants et étudiantes perçoivent les séquences non multiples comme des « multiples modifiés » et mettent à profit cet aspect multiplicatif pour exécuter les tâches. Nous analysons le développement des schèmes des élèves à mesure qu'ils percevaient les différences et les similarités dans les situations qui leur étaient présentées.

En conclusion, le traitement traditionnel des séquences arithmétiques dans l'enseignement néglige à notre avis un aspect important : la structure commune des éléments qu'il y a dans toute séquence. Pour les apprenants et apprenantes en mathématiques, il est essentiel d'accorder une plus grande place à la reconnaissance des patterns et des structures. De plus, une telle attention pourrait s'avérer particulièrement profitable aux futurs enseignants et enseignantes de l'élémentaire, qui, au cours de leur carrière, seront plus probablement appelés à enseigner la reconnaissance des patterns que les manipulations algébriques.

Introduction

An arithmetic sequence is a sequence of numbers with a common difference between adjacent pairs. The topic of arithmetic sequence, along with other sequences, is usually introduced in high school, and the standard approach utilizes algebraic representation and manipulation. Despite being a part of a high school rather than elementary school curriculum, the topic of arithmetic sequence is frequently approached in mathematics courses for pre-service elementary school teachers. This is mainly because in many classical activities, such as those using figurative numbers, students can make use of the ideas of arithmetic sequences as tools for problem solving.

Little research that is not limited to a counting sequence has been done on students' understanding of arithmetic sequences. However, arithmetic sequences surface in the discussions of pattern recognition and of understanding relations, generalization, and problem posing techniques (Brown &Walter, 1990; Principles and Standards for School Mathematics, 2000). Suggestions for creative teaching of the topic have also been made, advocating more visualization in developing the formulas (Hurwitz, 1993). Furthermore, arithmetic sequences appear implicitly in the research on student recognition of linear patterns (Orton & Orton, 1999; Stacey, 1989). We believe that a deeper understanding of the additive and multiplicative structure of arithmetic sequences will enable elementary school teachers to provide a richer experience for their students in exploring patterns and in grasping the relationship among arithmetic operations.

In this study, we are interested in exploring pre-service elementary school teachers' understanding of the structure underlying arithmetic sequences. The following questions are addressed: What cognitive strategies are used by students when facing unfamiliar problems related to arithmetic sequences? What patterns are apparent to students in the mathematical structure of arithmetic sequences? What patterns remain unrecognized? How do students apply the recognized patterns in a problem solving situation? What mathematical tools and concepts are being utilized? In which cases do students chose a pattern-guided approach and in which cases do they prefer a computational approach, and what aspects influence their choice?

By exploring these questions, we contribute to a large body of research on additive and multiplicative structures (Fuson, 1992; Greer, 1992) and the connection between them. Furthermore, concepts of factors, multiples, and divisibility are inherent in the structure of elements of an arithmetic sequence and are employed by students as they approach arithmetic sequence–related problems. This study, therefore, contributes to the body of prior research on pre-service elementary school teachers' understanding of elementary number theory concepts (Campbell & Zazkis, in press; Zazkis & Campbell, 1996).

Theoretical framework

Vergnaud's theory of conceptual fields

In several publications over the past two decades, Vergnaud developed, proposed, and elaborated on his theory of conceptual fields (Vergnaud, 1988, 1994, 1996, 1997). According to Vergnaud (1996), the theory of conceptual fields aims to provide 'a fruitful and comprehensive framework for studying complex cognitive competencies and activities and their development through experience and learning' (p. 219).

The theory of conceptual fields is based on the understanding that a single concept may refer to several different situations, and a single situation may be analyzed using several interrelated concepts. The development of the theory of conceptual fields was motivated by the need to establish connections among explicit mathematical concepts, relations, and theorems, and between students' (at times implicit) dynamic conceptions and competencies related to these mathematical concepts, relations, and theorems. The following terms of reference have been defined and used by Vergnaud:

- A *conceptual field* (1996, p. 225) is a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts with different properties, the meaning of which is drawn from this variety of situation.
- A *concept* (1996, p. 238) is a tuple of three sets (S, I and $\mathcal{S}$), where S is the set of situations that make the concept meaningful, I is the set of operational invariants contained in the schemes developed to deal with this set of situations, and $\mathcal{S}$ is the set of symbolic representations.
- A *scheme* (1996, p. 222; 1997, p. 12) is the invariant organization of behaviour for a certain class of situations.
- A *theorem-in-action* (1996, p. 225) is a proposition that is held to be true by the individual subject for a certain range of the situation variables.
- *Concepts-in-action* (1996, p. 225) are categories that enable the subject to cut the real world into distinct elements and aspects, and to pick up the most adequate selection of information according to the situation and scheme involved.

Vergnaud (1997) analysed and described several conceptual fields, including additive structures and multiplicative structures, 'the two main conceptual fields of ordinary arithmetic' (p. 15), as well as the conceptual field of elementary algebra and the conceptual field of number and space. His analysis included the classification and hierarchical organization of mathematical problems and tasks (referred to as 'situations') for students to engage in, and the identification of appropriate schemes for students to develop in order to deal with these problems and tasks. Furthermore, possible theorems-in-action and concepts-in-action that students develop while struggling with situations within a specific conceptual field were identified (see Vergnaud, 1988, 1994, 1996, and 1997 for a comprehensive description of specific conceptual fields). Vergnaud's theory of conceptual fields aims to provide a theory 'that would make conceptualization the keystone of cognition' (1996, p. 224).

Further theoretical considerations

The theory of conceptual fields is comprehensive and integrative, two features that Vergnaud considers to be particularly useful. These features allow researchers to study a broad range of phenomena at different levels. We would like to emphasize two additional implied features of Vergnaud's theory of conceptual fields that make this theory a particularly useful frame of reference for our investigation. First, it is content-specific but not content-limited. That is, it is not restricted to the learning of specific topics, but rather can be extended to investigate an individual's learning in any domain. Though Vergnaud's examples of conceptual fields focus on elementary mathematics, they contain clear pointers that conceptual fields can be studied outside of elementary mathematics and even outside of mathematics in general.

The second and more crucial feature is that the theory of conceptual fields explicitly acknowledges the existence of established mathematical knowledge. This knowledge, when restricted to a particular domain or content, is described as a conceptual field, in which concepts and relationships are inherent. It also explicitly acknowledges a particular learner's knowledge as his or her scheme. An individual's scheme includes goals, rules of action, inference possibilities, and operational invariants, which are theorems-in-action and concepts-in-action. An individual's scheme is dynamic and functional. It is being developed and changed over time and it is aimed at achieving a goal.

Situations

Though Vergnaud is very careful to define rigorously the terminology used to present the theory, one term is left undefined and is therefore subject to interpretations. This term is 'situation,' which is a key feature in defining a scheme, a concept, and a conceptual field. Vergnaud's position makes clear that situations can be both routine and non-routine problems. In the mathematics education community, situations are often interpreted as contextualized story- or word-problems (Greer, 1992). Most of the examples of situations that Vernaud provides are of this kind. Consequently, in considering multiples and divisors, Vergnaud (1988) suggests that many of the considerations discussed with respect to the conceptual field of multiplicative structures are not meaningful for this domain of mathematics because 'it is accepted that the concepts of multiple and divisor concern pure numbers' (p. 159).

We suggest that Vergnaud's theory of conceptual fields is applicable to a broader interpretation of situations. We classify as situations not only contextualized problems, but also mathematical problems and questions that are 'abstract,' that concern 'pure numbers,' or that are 'decontextualized'—that is, not rooted in 'real world' context. For example, asking students to find two numbers that have a sum of 24 is a situation that belongs to the conceptual field of additive structures.

Cognitive development in a conceptual field

Vergnaud (1996) claims that the theory of conceptual fields is ‘a theory of representation and cognitive development’ (p. 220). The idea of a ‘situation’ plays an important role in Vergnaud’s theory of conceptual fields. In a way, situations serve as triggers in generating and promoting cognitive development. When students are faced with a new situation, ‘they use the knowledge which has been shaped by their experience with simpler and more familiar situations and try to adapt it to this new situation’ (Vergnaud, 1988, p. 141). This description is similar to Piagetian accommodation and assimilation. We would like to elaborate further and describe in detail the mechanism of learning, using the terminology of Vergnaud’s theory of conceptual fields. If taken over a long period of time, learning can be seen as an individual’s cognitive development. If taken over a short period of time, learning can be seen as a development of a particular scheme.

Theorems-in-action are identifiers of students’ knowledge, as they describe mathematical relationships, either correct or incorrect, that are taken into account by students when they choose a path to solve a problem. Vergnaud suggests that ‘theorems-in-action have the potential to be the links among situations in the conceptual fields’ (1988, p. 145). We add that theorems-in-action may also serve as separators rather than links, and that recognizing either could serve as a stepping stone to learning. Thus, learning within a conceptual field may occur in two ways: (1) A student may recognize differences in two seemingly similar classes of situations. As a result, different theorems-in-action will be invoked and different routes will be taken by the student in dealing with each of the two classes of situations. It may be the case that once the difference is recognized, a hierarchy is established—that is, one situation will appear easier for the learner than the other. (2) A student may recognize a common structure between two classes of situations that were formerly perceived as ‘different.’ This may lead to an adaptation of two previously used theorems-in-action into one more general theorem-in-action that is applicable for both classes of situations. Once the situations are perceived by a student as belonging to the same ‘unified class,’ the same scheme will be invoked. Furthermore, identifying the invariant structure in situations may serve as a bridge that takes a student from one conceptual field to another.

Methodology

Vergnaud (1996, p. 225) asserts that cognitive development should be analyzed both from the perspective of the mathematical situations in which students’ activities take place and from the perspective of the concepts involved in the analysis of the situations. He mentions two advantages in taking such a perspective on analysis:

1. It gives a way to study the situations, to identify similarities and differences between the situations as well as the repertoire of schemes that is progressively developed to deal with the situations.
2. It provides the tools to describe students’ at times implicit knowledge underlying their schemes in terms of operational invariants—that is, theorems-in-action and concepts-in action.

We attend to the above mentioned advantages as a guideline for analysis. First, we examine the situations from the mathematical perspective, as ‘mathematics is an indispensable tool for this analysis’ (Vergnaud, 1988, p. 142.). Then we explore students’ dynamic schemes that are invoked in their attempts to deal with the situations, specifically aiming to

1. identify and describe strategies (rules of action) used as participants encounter problem situations related to arithmetic sequences,
2. analyze students' strategies and uncover underlying theorems-in-action,
3. suggest a path for a development of individuals' schemes within the context of presented situations,
4. analyze the development of individuals' schemes in terms of bridges among conceptual fields of multiplicative structures, additive structures, and elementary algebra, and
5. test empirically the (above) refinement of the theory pertaining to scheme development.

Participants

Participants in this study were pre-service elementary school teachers enrolled in a course entitled Foundations of Mathematics for Teachers, which is a core course in the elementary teacher education program. In the early part of the course, students 'covered' the topic of arithmetic sequence. They were fluent in recognizing and labelling some sequences as 'arithmetic' and also in generating sequences given the first element and the difference. They developed and implemented formulas for calculating the *n*th element as well as the sum of the first *n* elements of the sequence. They were also reasonably proficient in modelling phenomena, such as the constant growth of a plant or the constant daily charges of a bank account, as arithmetic sequences and in solving related word-problems.

Interviews were conducted by both authors in the later part of the course, shortly after the ideas of elementary number theory—including divisibility, factors and multiples, and the division algorithm—were discussed in class. Twenty out of the 64 students enrolled in the course volunteered to participate in clinical interviews, which are the main source of our data.

Situations

The following interview questions represent the core of the situations that were presented to students.

1. *Describing and exemplifying.* Please give several examples of arithmetic sequences. What makes these sequences 'arithmetic'? Can you think of an example that is different from others?

2. *Testing membership.*

Consider the following sequences of numbers.

(a) 2, 5, 8, 11, 14, …

Do you know what the next element is? Is it arithmetic? What is the twentieth element? Is the number 360 an element in this sequence (assuming it is infinite)? Why? Is there another way to verify this? And how about 440? Is it an element in this sequence? Why? Is there another way to verify this?

(b) The same questions with respect to sequence 3, 6, 9, 12, … and numbers 360 and 440.

(c) The same questions with respect to sequence 17, 34, 51, … and number 204.

(d) The same questions with respect to sequence 8, 15, 22, 29, … and number 704.

3. *Generating examples of members.* Can you think of a 'large' number that is an element in the sequence 2, 5, 8, 11, 14, … ? (If necessary, 'large' was described as a three- or four-digit number). Can you think of a large number that is definitely not an element in this sequence?

The same questions were posed with respect to the sequences listed in 2(b), (c), and (d), above.

The interviewer could vary the sequences and the numbers in question, or suggest additional sequences and numbers, as they deemed appropriate. The interviewer also asked meta-questions, such as, ‘in what way is the task 2(a) different from the task 2(b)?’

The request to describe and exemplify an arithmetic sequence was considered as a warm-up question, the intent being to start with something easy in order to establish a supportive atmosphere. The request to present a different approach to the problem was used to get beyond the formula-based responses that could be invoked by a recently learned mathematical content. The situation of testing membership or generating examples of members developed for the interview can be seen as ‘twist’ or ‘inverse’ questions (Groetsch, 1999; Zazkis & Hazzan, 1998). A standard exercise related to an arithmetic sequence is to find the Nth element. In our interview ‘what is given’ and ‘what is sought’ in standard questions have been ‘twisted’ or ‘reversed.’ This type of questions invites students to examine the situation rather than automatically follow an established algorithm, and gives researchers an opportunity to gain a better understanding of students’ schemes.

Results and analysis

Mathematical analysis of situations and the interpretive analysis of students’ schemes are often intertwined. Mathematical analysis serves as a lens through which students’ schemes can be described. On the other hand, analysis of students’ schemes helps in identifying subtleties in mathematics that could otherwise be overlooked or ‘taken for granted’ by the researcher.

It became apparent from participants’ responses to situation 1 (the request to provide examples of several arithmetic sequences) that a particular class of arithmetic sequences is preferred by students. Each participant provided between four and eight examples of arithmetic sequences. Most of these examples were sequences of multiples of a small natural number, such as 3, 6, 9, 12, … or 5, 10, 15, 20, …, with a possible exception of the sequence of odd numbers. When the interviewer explicitly asked for ‘something different,’ the usual reaction was to provide sequences of multiples of ‘large’ numbers, such as 50, 100, 150, 200, … or 100, 200, 300, … etc., or to list multiples in a descending order. All the participants mentioned ‘common difference’ as a salient feature of arithmetic sequences and readily accepted other sequences, such as 2, 5, 8, 11, …, as ‘arithmetic,’ as they were in accordance with the ‘common difference’ definition. However, sequences that did not list multiples of a natural number were not a part of their immediate repertoire of examples. Attending to sequences of multiples versus sequences of non-multiples adds a dimension to our mathematical analysis of situations.

Mathematical analysis of ‘situations’

Arithmetic sequence is defined by its first element, denoted a or a_1, a difference denoted as d, and a recursive relationship of $a_n = a_{n-1} + d$ for $n > 1$. Informally, it is described as a sequence in which a ‘common difference’ exists between each pair of consecutive elements. An arithmetic sequence has the form $a, a + d, a + 2d, a + 3d, a + 4d, \ldots, a + (n - 1)d, \ldots$

We limit our considerations here to infinite arithmetic sequences of whole numbers. The idea of common difference is embedded in the definition of arithmetic sequence. Attending to this additive structure gives a possibility of generating lists of elements by successive addition. Another identifying feature of arithmetic sequences is that all elements of a given sequence have the same remainder in division by the common difference. This multiplicative invariant is easily observed by attending to the common form $a + kd$, where k is a whole number of each element. More formally we can state that for all a_i in any arithmetic sequence of whole numbers, $a_i \equiv c \bmod d$, where c is the constant remainder and d is the difference.

An interesting subset of arithmetic sequences is sequences in which the common remainder *c* in division by the common difference *d* is zero. They can be described as sequences of multiples of *d*. In a special case where $a=d$, these sequences are of the form

$a, a + a, , a + a + a, a + a + a + a, \dots$, or

$a, 2a, 3a, 4a, \dots$

or, in a more general case,

$md, (m + 1)d, (m + 2)d, \dots$.

Attending to the fact that all elements of these sequences are multiples of *d* gives a way to identify whether a given number is an element of a given arithmetic sequence, and also gives a way to generate elements belonging to the sequence in a non-sequential way. In specific, considering the sequence 17, 34, 51, 68, …, one can claim that 300 is not an element in this sequence because it is not divisible by 17, while 17,000 is an element because it is divisible by 17.

Extending this argument to a situation of a general arithmetic sequence should include consideration of remainder. Considering, for example, the sequence 8, 15, 22, 29, …, we note the following multiplicative structure: All the elements give a remainder of 1 in division by 7. A different way to describe this relationship is to say that the sequence 8, 15, 22, 29, … is obtained from a sequence of multiples of 7 by adding 1 to each element. Therefore, 704 is not an element in this sequence, while 701 is. Realizing that 'no remainder' or 'being a multiple' implies a remainder of zero enables the use the same approach for both cases.

While a reference to multiples of *d* identifies one sequence (or identical sequences from a certain place on), there are $d-1$ arithmetic sequences of non-multiples of *d*. Therefore, for $d>2$, a property of a number being 'non-multiple' of *d* does not determine its membership in any given sequence of non-multiples. A remainder in division by *d* identifies the specific sequence and sets up a partition of integers. The relevance of this observation becomes apparent further on, in the section 'Considering non-multiples.'

Students' schemes

A 'scheme' is defined by Vergnaud (1996, p. 222; 1997, p. 12) as the invariant organization of behaviour for a certain class of situations. The components of students' schemes are goals and expectations, rules to generate action and pick up information, operational invariants, and inference possibilities. Operational invariants are concepts-in-action that guide students to grasp and select relevant information, and theorems-in-action that guide students in treating this information.

Theorems-in-action describe properties and relationships that a student believes are true for a certain kind of situation. They influence rules for action that can be observed and described as a student's strategies in approaching the situations. Theorems-in-action can be stated explicitly in a student's explanation or can remain implicit. However, taking the chosen strategy as an indication of a student's awareness of the relations involved gives an opportunity to make inferences about an individual's theorems-in-action.

In what follows, we describe students' strategies in approaching the situations presented in the interview. Through these strategies, we analyze students' explicit and implicit concepts-in-action and theorems-in-action.

Listing the elements

Listing the elements in a given arithmetic sequence by adding the common difference will eventually generate 'large' elements and determine whether a given number is an element in the sequence. For participants in this study, listing the elements was not the preferred choice; nevertheless, this strategy was mentioned either as a verifying strategy, or as a default for not being able to generate a better strategy.

In the excerpt below, Chris is considering the sequence 3, 6, 9, … and the number 360. She realizes that the elements of the sequence are multiples of 3, but this doesn't give her confidence in validating her conjecture.

> Interviewer: You said it is a multiple of 3 and therefore you believe it will be in the sequence and then you said, 'I don't know,' so …
>
> Chris: Oh, uh, that would be my guess, yes it is, and quite honestly usually when I do these problems, if I were to solve, you asked me in this case to figure this out, I would go back and I would check, like I would sit there and I would write out the entire thing until I came to 360. That would be how I checked, unless I had someone to confirm that with.

Chris would have preferred to have an external confirmation for her conjecture. However, since 'someone to confirm with' is unavailable, 'write out the entire thing' serves the purpose of internal convincing.

Sue mentions listing the elements when asked whether it is possible to approach the question of whether 360 is an element in the sequence 2, 5, 8, … without relying on formulas.

> Interviewer: Okay. Do you think it is possible to figure this out for somebody who doesn't know this formula?
>
> Sue: Um, You could actually use the trial-and-error method and just keep on going until you get past 360, but that's going to take a long time.

'Keep on going' is how Sue describes her rule of action. Similarly, for Lily in the excerpt below, 'adding 3 each time' is the only strategy she can suggest in order to give an example of an element.

> Interviewer: Okay, let's try another one. You don't know whether 360 is an element in this sequence or not, but if I ask you, can you find some number which *is* an element in this sequence, can you find such a number?
>
> Lily: I could find a number that's an element just by adding 3 each time …
>
> Interviewer: Okay, but how about a big number? If I ask you, please give me an example of a three-digit number which is an element in this sequence. Yes, you can go on and add 3, but can you think of some other strategy?
>
> Lily: (pause) Hmm, not really, no.

The strategy of listing the elements or 'adding on,' is evidence of students' theorem-in-action, which indicates the additive structure of a common difference between pairs of consecutive elements. Eight students mentioned explicit calculation of all the elements up to a certain place as an alternative strategy or as a way of checking one's answer, though only two participants suggested listing the elements as their primary choice of strategy.

Applying the formula

The formula $a_n = a_1 + (n-1)d$ is applied in routine questions to find the Nth element when the first element and the common difference are known. Furthermore, it can be used to calculate any one of the four variables when the other three are known.

Using the formula was a popular choice of strategy in order to approach situations of determining membership. In the next excerpt, Eve explains her way of deciding whether 360 is an element in 2, 5, 8,…

Eve: Okay, I um, I used this formula here, I put the 360 equal to 2 + 3 x (N - 1) and I tried to solve for the N …

Interviewer: Okay …

Eve: Now if at the N I reached a conclusion where I cannot find a whole number for N, then that means that um 360 cannot be in this sequence of numbers. Because in order to have 360 to be in here, the N must be a perfect, uh no, a whole number …

Interviewer: And why does N have to be a whole number?

Eve: Oh, well because N represents Nth place in the sequence, right, if we don't have a whole number, then it's not in the sequence.

If 360 were an element in the sequence, solving for N would have determined the place of this element. Eve realizes that such a solution must be a whole number. However, several of Eve's classmates who have chosen to use the formula couldn't explain what N or X represented. Leah, approaching the same question, set as her equation 360=2+359d, explaining that 359 is 'the term before.' Jill, in the example below, sets an appropriate equation, but she exhibits an obvious confusion between the element in an arithmetic sequence and its place.

Interviewer: OK, what I see here, you set an equation 360=2 + 3x, what is your X?

Jill: X is the element, any element, it's um the number that you multiply the difference by, that's just, I don't know, I have to figure that …

Interviewer: What is this number you multiply the difference by?

Jill: One number less than the element I'm looking for.

Participants also found formulas helpful in generating 'large' elements. In the following excerpt, Larry chooses 701 as an example of a 'large' element in 8, 15, 22, … . This number appears as a generic example if one is aware of the form 7k+1 of the elements. However, 701 is generated by substituting into the formula 100 as a choice for N, without attending to the form.

Interviewer: Could you please give me an example of a large number which is an element of this sequence.

Larry: Uhh, okay, say (pause) 701 …

Interviewer: And how did you find 701?

Larry: By using the equation, the formula.

Interviewer: So here it is written A_{100} = 8 + 99 x 7, so you know that 701 is what?

Larry: Is the hundredth, the hundredth number in the sequence.

Application of formulas was suggested as the exclusive strategy by only one of the participants. However, 17 participants used formulas for situations similar to 2(a) and 2(d), but applied considerations of form and pattern for sequences similar to 2(b) and 2(c) (situations 2(a)–(d) are described in the Situations subsection in the Methodology section, above). This fact, taken together with a significant amount of prompting and invitation to think of 'another way' during the interview, suggests that participants preferred formulas when the pattern in the sequence was not obvious to them—that is, when they weren't aware of the multiplicative invariants in the structure of the elements. Every element of an arithmetic sequence satisfies the given formula—appears to be a general theorem-in-action that guides student's approaches in cases where more specific relationships among elements are not recognized. In such situations, the participants invoked a scheme previously established to deal with arithmetic sequence–related questions: the scheme of plugging numbers into the formula.

Attending to last digit

A strategy applied by two students consistently involved consideration of the last digit in a number. As shown below, such a consideration gave Mike clear guidance in some cases and left him on unstable ground in some other cases.

Interviewer: The sequence is 1, 7, 13, 19, 25, and so on. I'll pick a number, a big number, let's say 360, and my question is, if I continue this sequence, will the number 360 be one of the elements?

[…]

Mike: Well no, it wouldn't be because, (pause) well I was just looking at these numbers, 1, 7, the final digit is 3 and the final digit is 9, 5, and, oh wait a minute, yeah, no that would be 31 and there's the, and 31 and then 37, so the pattern, you're seeing a period of the pattern there, 1, 7, 3, 9, 5, 1, 7, 43, uh 49, and uh 55, and so on, so 360 wouldn't be in this sequence.

Interviewer: Hmm, that is interesting. Let me give you another number, how about 343. (pause) What do you think, is this number in this sequence, is this number not in this sequence, how can you work it out?

Mike: Well my first impulse would be to say, well yes it is, because we have a 43, but another strategy that I actually do employ when given problems of this nature, is lack of trust, so like by seeing that number there, I automatically say oh, there must be some catch to it and I'll have to figure out a formula to find it. Uh (pause) but I really don't know how to approach it right now.

[…]

Interviewer: Can you think of a number that you are sure is here or you are sure isn't here?

Mike: Okay. 61 would be the next number in this sequence and then um 92 wouldn't be in the sequence because the final digits of those numbers don't follow the pattern …

Interviewer: Um hm …

Mike: Um, (pause) now 343, as I say, my instinct is telling me that it would be part of the sequence because the final digit is, not only is the final digit 3, but it, the second digit is the same as well …

The sequence presented to Mike is 1, 7, 13, 19, 25, … . He recognizes the repeating pattern of last digits and claims with confidence that 360 is not the element because it doesn't fit this pattern. He also gives the example of 92 as something he is 'sure isn't there.' His theorem-in-action here is rather explicit: The last digit of a number must fit the pattern of last digits of the known elements in the sequence. However, this theorem-in-action provides no clear direction when considering numbers with a last digit that does fit the pattern of the sequence. With respect to 343, Mike's decision is less confident; he reports 'lack of trust' and a desire 'to figure out a formula to find it.' He further considers not only the last digit, 3, but also the last two digits, 43, which is in this case an inappropriate extension of a previously used scheme. As clarified by Vergnaud, theorems-in-action can be either true or false. Mike exemplified how the same strategy, attending to last digit, can result in identifying true relationships for some situations and in generating false arguments for others.

Considering multiples

Attending to multiples guided students to generate several theorems-in-action. In a sequence of multiples of a given number, in our examples of numbers 3 and 17, divisibility of a number by 3 or 17 determines whether or not it is an element in the given sequence and provides an immediate means to generate large elements.

Interviewer: Okay. How about a number like 94, do you think it is an element of this sequence?

Chris: (pause) No.

Interviewer: And why do you believe it is not?

Chris: Because I would say that 90 would be, because it would be a multiple of 3, and so the next one after that would be 93 and 94, no, because it would be 93 and 96.

Chris concludes that 94 is not an element in the sequence 3, 6, 9, … . However, her argument doesn't consider divisibility of 94. Her strategy is to generate elements that are close to 94, in this case 93 and 96, that are multiples of 3.

Interviewer: Okay. I would like you to look at a different sequence, and it is 17, 34, 51, 68, and so on. And I would like to ask you about the number 204. Is it an element of this sequence?

Dave: If it's a multiple of 17, it is.

Interviewer: And if it is not a multiple of 17?

Dave: Then it shouldn't be.

Interviewer: So this will guide your decision.

Dave: Um hm.

Interviewer: So 204 is indeed 17 x 12 …

Dave: Then it's in.

Interviewer: It's in. Can you please give me an example of a big number which is in this sequence?

Dave: 17,000.

Interviewer: Another one.

Dave: 17,051.

Interviewer: Okay. What makes you believe that 17,051 is an element of this sequence?

Dave: (pause) 17,000 is 1,000 x 17, and that's a multiple of 17 …

Interviewer: Um hm …

Dave: I also know that 51 is a multiple of 17, and so it's the 3 x 17, so I add 1,003 17's, I've still got a multiple of 17, it's still going to be in there.

Dave exemplifies his belief that any multiple of 17 is an element in the sequence 17, 34, 51, … . He immediately mentions 17,000 as a generic example for a 'large' number divisible by 17 and also builds on his previous knowledge of multiples of 17 in order to produce a new one. Any multiple of 17 is in the sequence, any non-multiple of 17 is not in the sequence—this is Dave's theorem-in-action. Both parts of this theorem-in-action are important and can be used as separate arguments. It appears that Chris's scheme included the first part of the theorem only. We observe that though Chris's concept-in-action, multiple, was similar to Dave's, the inference rules in her scheme were rather limited to infer the 'if and only if' relationship between a number being a multiple of 3 and its membership in a sequence 3, 6, 9, … .

In the next example, Larry describes his scheme as a general decision-making strategy.

Interviewer: And how about 440, is it in the sequence?

Larry: (pause) No, it's not.

Interviewer: Why not?

Larry: Because 3 doesn't divide 440.

Interviewer: Okay, so can you describe your general strategy and decision making here please?

Larry: Um, I'm just looking at the constant difference and I found that the constant difference is 3, therefore any multiple of 3 will be in this sequence, but then if you have a number that doesn't, that is not divisible by 3, then it will not be in this sequence.

At first glance, Larry's description appears accurate and comprehensive. However, a more detailed examination of his argument suggests that numbers divisible by 3 are elements in the sequence because 3 is a common difference. This is the case only if the first element is also a multiple of 3. For Larry, this could be an incomplete communication of an idea, as he later acknowledged inappropriateness of applying the same strategy for the sequence 8, 15, 22, … . However, for several other participants, the idea that numbers in an arithmetic sequence are multiples of the common difference manifested as a false theorem-in-action.

Interviewer: Okay. One more. Would you please consider the following sequence: 8, 15, 22, 29. So far it's an arithmetic sequence, how would you continue?

Leah: 36?

Interviewer: And …

Leah: 43.

Interviewer: Okay. And how about the number 704?

Leah: I'm going to check and see if 7 is a factor of 704, (pause) no …

Interviewer: No for what?

Leah: Um, 704 is not going to be in this sequence because 7 is not a factor of 704.

Interviewer: Okay. How about 700?

Leah: Yeah, um, 7 is a factor of 700, so I think it's going to be in the sequence. 7 x 100 is 700.

Leah claims that the number 700 is an element in a sequence 8, 15, 22, … because 7 is a factor of 700. In the case below, Sue is considering the sequence 2, 5, 8, … and makes a similar false claim.

Interviewer: Could you please give me an example of a number, and I would like a relatively big number, like three-digit number or four-digit number, that you're sure will be listed in this sequence [2, 5, 8, …] ?

Sue: Mmm, okay, I guess it has to be a multiple of 3, because it's common difference, so um 333, maybe?

Interviewer: So you think that 333 will be listed in this sequence?

Sue: I think so.

Sue holds that an element in an arithmetic sequence is a multiple of the common difference. In such cases, the student's theorem-in-action was challenged by the interviewer by pointing out contradictory evidence. As a result of these types of challenges, some participants refined their scheme by limiting it to certain kind of situations, while others, such as Sue below, regressed to previously successful strategies, such as using the formula.

Sue: Hmm, wait a minute, 360 is a multiple of 3, yet I just said that it didn't go in, right …

Interviewer: You did …

Sue: So then this might not go in there, I don't know, um, (pause) I'm not sure (laugh). I think I'll have to guess a couple, I'll have to do trial and error to figure it out.

Interviewer: And what do you mean by trial and error here?

Sue: Like um, I'm going to start with pick a couple of numbers that I think would work and then put it back into this formula …

Interviewer: Okay …

Sue: To see if I get a whole number …

Interviewer: For?

Sue: For N.

In total, there were nine participants who believed, at least temporarily, that multiples of *d* would generate elements in any arithmetic sequence. This strategy was applied in situations of testing membership as well as in situations of generating elements. Consideration of multiples of *d* for any arithmetic sequence can be seen as students' attempts to extend a previously established scheme to a new situation, without realizing the difference between the structure of arithmetic sequence of multiples and arithmetic sequence of non-multiples. A similar phenomenon, referred to as 'difference product' or 'direct proportion,' was observed by researchers investigating middle school students' generalization of repeating patterns (Orton & Orton, 1999; Stacey, 1989).

Considering non-multiples

Classifying arithmetic sequences as 'multiples' and 'non-multiples' presents a dichotomy that may be both useful and problematic for students. In the following excerpt, Sally considers the sequence 8, 15, 22, … and the number 704.

Interviewer: So 704 is not divisible by 7, none of these elements in this sequence you believe will be divisible by 7, so can you draw conclusions from what you have now?

Sally: It's, it's um very possibly in this set.

Interviewer: Um hm. What, what will convince you?

Sally: (laugh) Well just because it's not divisible by 7, doesn't mean it's in the set, right?

Interviewer: Can you give me an example of a number that you know for sure that is not in this arithmetic sequence?

Sally: Um hm, um 700 …

Interviewer: Another one …

Sally: Um, 77.

Interviewer: Okay. And how about 78?

Sally: It may be in the set, but it's not divisible by 7 …

Interviewer: (laugh) So 77 you're sure is not, 78 you're not sure.

Sally: Right.

Interviewer: 79?

Sally: Could be …

Interviewer: Could be. 80?

Sally: Could be …

Sally is confident that multiples of 7 are not elements in the given sequence, but she believes that any number that is not a multiple of 7 'could be' in the sequence. Similarly, for Leah a possible element in the given sequence is a randomly picked number that is not divisible by 7.

> Interviewer: Okay, so can you give me an example of a number that you believe is not in the sequence and an example of a number that you believe is, or could be in the sequence?
>
> Leah: Um, I don't think 714 would be in the sequence, um, a number that could be, I would just pick a number that hasn't a factor of 7, so like 511 possibly, or something.
>
> Interviewer: And you are saying possibly because …
>
> Leah: Just, I just picked a number that wasn't, didn't have 7 as a factor.

This implicit theorem-in-action—'Every element in an arithmetic sequence is a multiple of *d*'—has been discussed in a previous section. This theorem-in-action holds true for a situations in which an arithmetic sequence under consideration is a sequence of multiples [see Dave, above]. However, it is false when extended and applied to a sequence of non-multiples [see Sue, above].

Attention to multiples and non-multiples restricts the previously used theorem-in-action to a specific class of situations. At this stage, students are able to differentiate and note that previously generated theorems-in-action are not fruitful in a new situation. However, they have not yet revised their theorems-in-action to generate rules of action for the new class of situation. Whereas a number's property of 'being a multiple' gives a clear indication of its belonging to a sequence of multiples and non-belonging to a sequence of non-multiples, the property of 'being a non-multiple' identifies that a number doesn't belong to a sequence of multiples, but gives no explicit hint with respect to the number's membership in a given sequence of non-multiples. Therefore, Leah and Sally in the excerpts above clearly claim that any given multiple of 7 is not an element in a sequence of 'non-multiples.' Nevertheless, they are not able to draw a definite conclusion when testing a membership of the number that is not a multiple of 7. Their expressions 'quite possible' or 'could be' suggest that they have identified the dichotomy between multiples and non-multiples. They are aware of the multiplicative structure in the sequence of multiples; however, they are not attending to the inherent multiplicative structure of the arithmetic sequence of non-multiples.

The main problem here—and this is the place to return to our mathematical analysis—is that there is one sequence of multiples of *d* (or, in a more formal way, identical sequences from a certain place on) while there are *d-1* sequences of non-multiples of *d*. Therefore, *for d>2,* the property of a number of being non-multiple of *d* doesn't give a clear indication to which of the *d-1* sequences the element belongs.

A note on multiples and non-multiples as concepts-in-action

We interpret Vergnaud's use of 'concepts' as established conventional mathematical concepts, whereas 'concepts-in-action' are mathematical concepts as they are constructed in an individual student's mind. Though concepts and concepts-in-action may have the same lexical reference, a student's concepts-in-action are dynamic and represent his or her understanding, at times erratic or incomplete, of a given mathematical concept. The concept of multiple is our natural example in this discussion. In the following example, Eve claims that all the numbers in the given arithmetic sequence 17, 34, 51, 68, … are multiples of 17; however, she is unable to generate a four-digit element in this sequence.

> Interviewer: How would you decide whether 204 is an element in this sequence?

Eve: Okay, (pause) okay I guess I would use 204 and divide by the first number in here, because it looks like, when I'm looking at this sequence it looks like um all these numbers are multiples of 17, so if 204 is a multiple of 17 which means that it will also occur in this sequence, so in order to be a multiple of 17, 204 divided by 17 must give us a result of a whole number and no decimal places.

Interviewer: Okay …

Eve: So 204 divided by 17, that gives us 12, okay it's 12, this whole number, so it's a number in this sequence.

Interviewer: Okay. Can you please give me an example of any four-digit number in this sequence?

Eve: I could just randomly pick any, okay …

Interviewer: Yes, please pick any, but convince me that it is in the sequence.

Eve: Okay. 17, um, (pause) I just keep on adding 17 to get um this sequence up 85, 102, 119, 136 and 153 …

Interviewer: Yeah, this is a pretty hard work …

Eve: Yeah …

Interviewer: If I want a four-digit number, it will take you quite a while to get that …

Eve: Oh, you want a four-digit number …

Interviewer: Yeah …

Eve: Umm, (pause) I don't know how to do this.

We suggest that Eve's perception of a multiple is entirely additive. Multiples for her are lists of numbers created by adding on and no connection is made between repeated adding on and multiplication. In this case, an appropriate concept-in-action was clearly identified, but a related theorem-in-action relied solely on listing the elements.

Hazzan and Zazkis (1999) report a similar phenomenon, where an explicitly stated property of divisibility didn't direct students to the inherent multiplicative relationship. In their research, participants were asked to give an example of a five-digit number divisible by 17. A frequent strategy was to pick a number at random and check its divisibility with a calculator. It was also observed that the degree of freedom—that is, a possibility of many correct answers, presented an obstacle for some students as they were looking for 'the right one.'

Furthermore, when Leah identified a common property of numbers in the sequence 8, 15, 22, … as 'none of these numbers on the list have 7 as a factor,' the interviewer decided to question this claim for numbers not currently listed.

Leah: Because none of these numbers on the list have 7 as a factor.

Interviewer: Isn't there a chance that as we go on and add on 7's to the numbers and continue this sequence, we eventually will hit some number which is a multiple of 7?

Leah: If you keep adding 7's?

Interviewer: Um hm …

Leah: (pause) You might. Um, well 7's, you know, it generally ends in, if something is a multiple of 7 it can end 7, 14, (pause) well it could end in almost anything. (pause) 28, so I, yeah you could, I think.

Leah believes that eventually a multiple of 7 could appear in the given sequence. Her belief is based on observing the last digits of numbers in the sequence. We suggest that though appropriate concepts-in-action were identified, the inherent relationships were not a part of her scheme. The inherent relationship in this case can be described as 'every seventh number is a multiple of 7.' This relationship was also overlooked by several participants in Zazkis and Campbell's (1996) study. In their investigation, one of the questions posed to participants was to determine whether there was a number divisible by 7 between 12,358 and 12,368. Rather than considering the frequency of appearance of multiples of 7, a preferred strategy of 14 out of 21 participants was to find such a number by performing division.

Considering multiples and adjusting

Any arithmetic sequence of whole numbers can be considered as a translation along the number line of a corresponding sequence of multiples. (For a sequence of multiples, this can be seen as translation by zero units). This view provides a method to deal with the interview situations through means other than listing the elements and applying formulas.

Interviewer: Number 360, do you believe it is an element in this sequence? [2, 5 ,8 …]

Dave: No, I don't think so.

Interviewer: Could you please explain why?

Dave: The (pause), any number in this sequence is going to be, in this case the difference between all the sequences is 3, any number in this sequence is going to be some multiple of 3 plus the first element in the sequence, so some multiple of 3 plus 2; 360 is a multiple of 3, but every element in the sequence must be a multiple of 3 plus 2 …

Interviewer: Oh, so can you please give me an example of a number that you think is an element in this sequence. Of a big number.

Dave: A large number then. Uh, (pause) sure, 3,000,002.

Interviewer: 3,000,002. (Laugh) Another one.

Dave: Bigger than that, or should, can we go a little smaller?

Interviewer: We can go a little smaller.

Dave: 335.

Interviewer: 335. And how do you get 335?

Dave: I know that 333 is a multiple of 3 and 2 + 333 is 335.

Identifying the invariant among the elements as ‘multiples of 3 plus 2’ appears at first glance as a mature theorem-in-action. Though the choice of numbers 3 and 2 as difference and first element is explicitly stated, their specific roles are implicit. They emerge in the next stage of the interview:

Interviewer: Okay. Let me give you another sequence please, and it is 8, 15, 22, 29, and so on. 704 is the number

Dave: No.

Interviewer: Why not?

Dave: The Nth member of the sequence is going to be (N - 1) x 7 + the first member of the sequence, which is 8.

Interviewer: Um hm …

Dave: So I can (pause), and any number in the sequence is going to, you’re going to be able to name it that way, that terminology. 700 is a multiple of 7, 704 is not a multiple, any multiple of 7 plus 8 …

Interviewer: Um hm …

Dave: It’s only 4 more than 7, so no matter how you slice it, it’s not going to turn out to be a multiple of 7 plus 8.

In considering the sequence 8, 15, 22, … Dave describes the general form of each element as ‘multiple of 7 plus 8.’ He is consistently attending to this form in claiming that 704 is not an element and in constructing several ‘large’ elements of the sequence. It could be the case that rather than attending to the invariant structure of the elements, ‘multiples of 7 plus 8’ is Dave’s interpretation of the formula for the Nth element. We have exemplified earlier participants’ routine application of formulas that didn’t include interpretation of the meaning of the formula. Dave seems to be able to capture and describe the essence of what the formula provides. However, seeing the structure through the formula presented an obstacle for Dave, as described below:

Interviewer: I have another one for you here, let’s look at the following: 15, 28, 41, 54, and I would like to ask you about the number 1,302 .

Dave: *(pause)* No.

Interviewer: Why not?

Dave: The constant difference in the sequence is 13, and any number of the sequence is going to be a multiple of 13 plus 15, which is the first element.

[…]

Dave: (pause) I’m making an assumption based on, based on knowing that 1,300 is, is a multiple of 13 and that 1,302 is, is only 2 away from that, it just doesn’t, it just doesn’t seem likely now …

[…]

Interviewer: Can you please give me an example of a number that would be an element in this sequence?

Dave: 1,315.

Interviewer: 1,315. Okay. Another one please?

Dave: (pause) 106.

Interviewer: 106. Would you please explain how you got each one of those?

Dave: I did the math right, uh, I took 7 x 13 and added 15 to that …

Interviewer: Okay, and here?

Dave: This is 100 x 13 and add 15 …

In this final interview excerpt, Dave makes a wrong claim that 1,302 is not an element in the sequence 15, 28, 41, … . His scheme is unchanged—the sequence is described as 'multiples of d plus the first element,' in this case, 'multiples of 13 plus 15.' However, with this structure in mind, 1,315 is clearly fitting Dave's pattern, where 1,302 is not. Therefore the next important step in developing individual scheme is recognizing the invariant multiplicative structure of elements in an arithmetic sequence of non-multiples, referred to as 'multiples adjusted,' where the adjustment is not necessarily the first element.

Interviewer: Can you give me an example of a bigger number, like let's say a four-digit number that is an element in this sequence?

Lily: So, 2,999 …

Interviewer: Thank you. Can you give me another one please?

Lily: Okay, um, (pause) 1,002, one behind 1,001.

Interviewer: Could you please describe your strategy, how did you find the numbers?

Lily: The first, I'm making sure that the number is divisible by 3, by um the sum of the digits in the number are divisible by 3 and the sum of these digits is 3, so 1,002 is divisible by 3, but in this particular sequence the number will be one less than a multiple of 3, therefore one less than 1,002 is 1,001.

[...]

Interviewer: Let's take one more. 8, 15, 22, 29 another sequence.

Lily: Okay, so this is a difference of 7 …

Interviewer: How about the number 704?

Lily: (pause) 704, so (pause) this is, these numbers are plus 1 of multiples of 7, multiples of 7 plus 1 …

Consideration of multiples and 'adjustment' where necessary clearly equips Lily with a powerful scheme. Such an 'adjustment' is expressed by Megan in a more mathematical way as she considers division with remainder.

Interviewer: And what about 704?

Megan: No, because that's got a remainder of (pause) 4, not 1 … it needs to have a remainder of 1.

Interviewer: So can you please describe for me your general strategy? How would you decide whether a number I give you does belong to this sequence or doesn't belong to it?

Megan: Um, if it's divisible by 7, with the remainder of 1, then it does belong to the set.

Megan's reference to a division with reminder is a logical extension of her previous scheme. Considering 3, 6, 9, … and 17, 34, 51, …, Michele described the relationship of divisibility, rather than all elements being multiples of the same number. On the other hand, Lily referred to these sequences as 'multiples;' therefore the adjustment of multiples is a consistent extension of her scheme.

Among the 20 participants in this study, only 2 explicitly mentioned division with remainder, whereas six participants eventually succeeded in suggesting some adjustment of either multiples or numbers divisible by *d*.

Development of schemes

Development of students' schemes is guided by identifying invariants as well as by identifying differences between classes of situations. In what follows, we exemplify both cases. Further, we describe a possible path through which an individual may proceed in developing his or her scheme.

Identifying differences

Lack of attention to differences between two classes of situations causes students to apply the same theorem-in-action for both classes. Several examples are discussed above in which students consider an arithmetic sequence with a common difference of *d*, without noting that consideration of multiples of *d* is applicable only to a specific type of arithmetic sequences.

Identifying differences between two types of sequences (or two classes of situations) results in the realization that the same theorem-in-action cannot be used for both. At this stage, students restrict their theorem-in-action to one specific class of situations and seek extension of their schemes in order to accommodate a new class of situation in a different way. In the excerpt below, Connie explains the difference between the two classes of situations.

Interviewer: Good. So here [pointing to sequence 6, 9, 12, …] you said immediately, yes, or immediately no, over here [pointing to sequence 2, 5, 8, …] you had to do more work. Can you please explain to me what is the difference? Because, both tasks appear very similar.

Connie: Right. So in both of these sequences the common difference is 3 …

Interviewer: Um hm …

Connie: However, in the first sequence there is no other common relationship between the elements, other than they have a difference of 3. However, in this sequence they have a difference of 3 and they're also multiples of 3, beginning at 3 x 2, that's the first element …

Connie's observation of 'no other common relationship between the elements' in addition to the common difference is typical in this group of students. In the sequence of multiples, Connie identified two invariants: additive invariant of ' common difference of 3' and multiplicative invariant of 'multiples of 3.' However, in the sequence of non-multiples, she is aware of only the additive invariant.

Identifying similarities

By identifying invariant structure in a certain class of situations, students develop theorems-in-action that guide their strategies. Most participants were aware of the multiplicative invariant structure of elements in arithmetic sequences of multiples as they came to the interview. During the interview, several participants identified the invariants (1) in the structure of elements in a sequence of 'non-multiples' and (2) in the structure of two classes of sequences.

1. Multiplicative invariants in the sequence of 'non-multiples'

At the beginning of the interview, Sally believed that 704 'could be' in the sequence 8, 15, 22, … because it was not divisible by 7. She clearly classified the sequence in the discussion as non-multiples of 7, but this information was not sufficient to deal with the situation. Observing the relative ease in which Sally discussed sequences of multiples, the interviewer returned to previously unanswered questions.

Interviewer: 8, 15, 22, 29, and so on

Sally: Um hm …

Interviewer: Can you give me an example of a big number, like a three-digit number, that you would believe is an element of this sequence, if it's possible to find such an example without going and counting up.

Sally: Um, it's more difficult …

Interviewer: I agree with you. (pause) What makes it more difficult?

Sally: Because you don't know what, what the common divisor is, the common factor …

Interviewer: Maybe there isn't any …

Sally: Maybe there isn't, no, so in my thinking what you could do is, is find all, a three or four-digit number that wasn't …

Interviewer: Um hm …

Sally: (pause) Oh wait a second here, that wasn't divisible by 7 and that would have, (pause) ohhh, every element is one more or one less, every element is one more than a multiple of 7, I just saw that now …

Interviewer: There was this exclamation, 'ohhh.'

Sara: (laugh) I don't know why I didn't see it before. 8 is one more than 1 x 7, 15 is one more than 2 x 7 …

Interviewer: Okay, so …

Sally: Um, (pause) well you could just go 7 x any number you like plus 1, and that would give you a number that was in there.

Interviewer: Okay. Uh, if we go back to our question about 704 …

Sally: Um hm …

Interviewer: Does your new insight, new finding, help you with the previous question, whether 704 is a member of this sequence?

Sally: Um, (pause) um hm, because 704 is 4 more than a multiple of 7, so you know it's not in there because everything has to be one more than a multiple of 7…

Identifying invariant structure of the elements as 'one more than a multiple of 7' enables Sally to approach with a relative ease a situation that was problematic for her just a few minutes earlier.

In the following excerpt, Alice makes a connection between the structure of sequence of multiples 3, 6, 9, … and the sequence of non-multiples 2, 5, 8, … .

Interviewer: Can you please describe for me why the second one was much easier for you than the first one?

Alice: Because they're numbers that you play with all the time, I suppose, and they're just multiples of 3 and this one, these ones are actually … , oh these ones are just … , ooh, these ones are just uh, these are multiples of 3 minus 1, which I didn't really notice before for some reason…

Alice identifies a relationship between the two sequences by referring to them as 'just multiples of 3' and 'multiples of 3 minus 1,' a relationship that guides her in dealing with the posed problems. Alice achieves a much clearer formulation of this relationship in the next question, considering the sequence 8, 15, 22, … and the number 704.

Alice: So all of these numbers would be divisible, or are divisible by 7 with a remainder of 1, okay, so that means that this number [704] has to be divisible by 7 with a remainder of 1, okay, so that would mean that if we take 1 from 704, if we take the remainder, so 703, it means that 7 has to divide that evenly. 7 does not divide 703, so no, it's not an element, not an element of the sequence.

In considering the sequence 8, 15, 22, … Sally generates a sequence of numbers divisible by 7, a sequence that was not discussed previously in her interview, and makes a reference to this sequence in her solution. Similarly for Alice, connecting 'numbers you play with all the time' with the new sequence of numbers is a clear asset. Alice also makes a transition in her deliberation from consideration of 'multiples-adjusted' to consideration of division with remainder. She may be just a step away from identifying the remainder of zero in the sequence of multiples and, with that, unifying the structure of the two classes of situations.

2. *Multiplicative invariants in the structure of two different types of sequences*

Considering the invariant of 'multiples' in the sequences of multiples and invariant of 'multiples adjusted' in the sequences of 'non-multiples' provides sufficient tools to deal efficiently with the interview situations. Identifying similarities, other than lexical, between the two classes, can be a next step in scheme development. In the next excerpt, Lily describes a general strategy applicable for both types of sequences.

Interviewer: So in terms of general strategy, if I give you a number, how would you decide if my number is an element in a sequence?

Lily: First determine what is the constant difference. And then look to see where we're starting in the um, in the sequence, to see the, how to adjust. So if this is the first multiple, this is times 1, so we can start with this, we don't have to adjust the number, … understand what I say?

Using Lily's approach, one should first determine the common difference between the pairs of consecutive elements and then, based on the first element of the sequence, determine the adjustment, if necessary. This scheme treats 'multiples' and 'non-multiples' as 'multiples adjusted,' allowing the adjustment to be zero. Identifying invariant structure in two previously-treated-as-different multiplicative invariants supports the development of this scheme. However, the interview situations provided little opportunity for participants to extend their scheme in a way that it could accommodate both classes.

An individual's scheme and its possible evolution

'The theoretical importance of schemes comes from the fact that operational invariants are more or less adequate: the relevance of concepts in action and the truth of theorems-in-action are essential conditions of the efficiency of schemes' (Vergnaud, 1997, p. 27). Individual's scheme is dynamic, and its evolution is guided by the identification of invariants as well as differences in problem situations. In what follows, we outline a possible progression in the development of an individual's scheme (Figure 1). Although there was no single participant in our study that proceeded through all the stages, every sequential subset in the outlined scheme development was displayed by students.

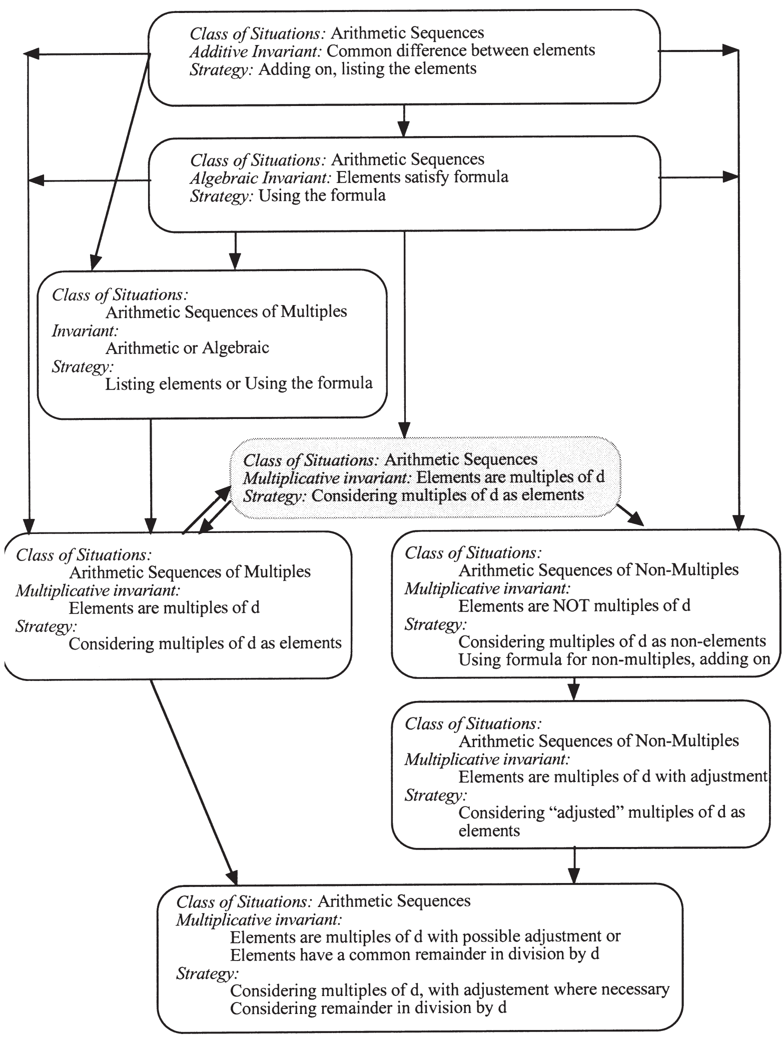

Figure 1: A possible evolution of a scheme

Confronting situations of testing membership and providing examples of large elements in arithmetic sequences, the learner initially identifies the additive invariant of common difference and builds on this invariant in approaching the problems. Then, depending on the learner's prior exposure and experience, he or she may also identify the algebraic invariant—that is, all elements satisfy the formula $a_i=a_1+(n-1)d$. This stage, though helpful in dealing with the presented situations, is not essential, since the problems can be solved successfully without relying on any algebraic formalism.

An individual's scheme is functional—that is, aimed at achieving a goal. Therefore, an individual will seek more efficient strategies that could be achieved by further identification of invariants. In the next step of the development of the scheme, an individual may identify a subset of arithmetic sequences as sequences of multiples (the left side of Figure 1). This leaves the complement of other arithmetic sequences, denoted in this study by sequences of non-multiples (the right side of Figure 1). However, a reference to multiples does not necessarily imply identification of a multiplicative invariant. Eve (above, in section 'A note on multiples and non-multiples as concepts-in-action) identified the sequence 17, 34, 51, … as multiples of 17; however, her strategy for generating multiples relied on additive invariant.

Identification of the multiplicative invariant—every element is a multiple of *d*, that is, of the form *md* for some whole number *m*—in the sequences of multiples leads to more efficient strategies in testing membership and generating elements. This multiplicative invariant could be temporarily overgeneralized to any arithmetic sequence. Alternatively, this multiplicative invariant could first be incorrectly identified for any arithmetic sequence and then restricted to hold only for sequences of multiples. (This stage of incorrect identification or overgeneralization is distinguished in Figure 1 with a grey background). In any case, the learner may find herself/himself in a position where a clear distinction exists between two classes of situations—sequences of multiples and sequences of non-multiples; however, invariant multiplicative structure is identified for one class only. This equips the learner with tools to deal with situations in the class of multiples; however, these tools are insufficient when applied to non-multiples. As shown above, Leah (in section 'Considering non-multiples') claims that multiples of 7 are not elements in 8, 15, 22, … , but has no tools, other than formula or 'adding on,' to test membership of numbers that are not divisible by 7.

Further, the invariant structure in the sequences of non-multiples is identified—every element is a multiple of *d* with 'adjustment,' that is, of the form $md + c$, where *c* is the common reminder in division of elements by *d*. At this stage, the learner sees multiples and non-multiples as two separate classes that each has a different invariant structure, and different schemes are therefore invoked in dealing with situations. However, identifying a multiplicative invariant within non-multiples is essential for the development of a unified scheme. It is a further sophistication to consider multiples with adjustment (that can be zero) or common remainder (that can be zero) in division by *d* as the invariant that unifies both classes of situations and allows an individual to invoke the same scheme for any arithmetic sequence.

Discussion

In this study, Vergnaud's theory of conceptual fields provided a useful language to describe and analyze the students' attempts to deal with arithmetic sequence–related problems. The growth in students' understanding has been outlined as a development of their schemes related to a particular class of situations. Our data show that scheme development is guided by identifying invariants as well as identifying differences in classes of situations. Thus, the data appear consistent with the refinement of the theory suggested earlier. Further research should provide additional empirical examination of the proposed theoretical considerations.

The findings of this study suggest that students distinguish between two separate classes of infinite arithmetic sequences of whole numbers. One, the sequences of multiples, are perceived as amicable and orderly, while the other, sequences of non-multiples, are perceived as unfriendly and at times 'sporadic.' The participants in our study were much more successful in testing membership and generating elements in the sequences of multiples than in performing the same tasks on the sequences of non-multiples. As an indication of success, we take not only the ability to generate correct answers, but also the relative ease with which students approached the situation and the efficiency of the strategies they chose. Moreover, the students had difficulty identifying the unifying multiplicative property inherent in any arithmetic sequence, thereby integrating the two classes. Out of the 20 interviewees, only one was able to make this connection. How the problem is situated, the problematics associated with the link between additive and multiplicative structures, and the limitations of a clinical interview could be among the factors responsible for these difficulties.

From a pedagogical perspective, arithmetic sequences are often introduced using their inherent additive property and therefore are temporarily situated within the conceptual field of additive structures. However, the emphasis is eventually shifted to the formula-based procedural component. This focus on the formula situates the problem in the conceptual field of elementary algebra (see Vergnaud, 1996, for elaboration on conceptual fields of elementary algebra, multiplicative structures and additive structures). Thus, when students are presented with an arithmetic sequence, they default to the tools found within this conceptual field. As long as these tools are adequate for the treatment of the problems, students have no motivation to invoke the schemes situated in other conceptual fields. Our data further support the finding of 'apparent relationship between students' use of solution strategies and certain contextual features of the problem situation' (Baranes, Perry, & Stigler, 1989) and the 'relationship between situation-based sense making and mathematical problem solving' (Silver, Shapiro, & Deutsch, 1993, p.132).

In order to bridge to other conceptual fields, the students need to identify additional invariant structures that are not inherent within the conceptual fields in which the problem is initially situated. The situations of exploring arithmetic sequences served as an appropriate tool for examining students' ability to make these constructs. As already stated, arithmetic sequences may be treated solely within the conceptual field of additive structures, or solely within the conceptual field of elementary algebra. Within our study, 17 out of the 20 students chose to use the formula as their initial strategy and two initially chose to focus on the additive property. All but two students were able to identify 'multiples' and to move to a conceptual field of multiplicative structures. However, only eight out of the 20 participants, when prompted for alternative strategies, settled on a strategy involving treatment of non-multiples as either a sequence of multiples with an additive adjustment or as a sequence of elements having a common remainder in division by *d.* These strategies involve a coordinated use of both the multiplicative and the additive structures. The rules of action and the theorems-in-action cued by this bridging of conceptual fields may be too underdeveloped for the students to invoke effectively. This is consistent with findings of Campbell and Zazkis (1994), who, when studying pre-service teachers' understanding of the distributive property, 'found evidence to suggest that significant obstacles to conceptual understanding of divisibility involved a lack of understanding of the relationship between additive and multiplicative structure' (p. 268). This lack of understanding also manifested as a recurring error in considering, at least temporarily, any sequence with a common difference of *d* as a sequence of multiples of *d.*

The students' strategies evolved during the interview. The interview excerpt with Sally (in section 'Multiplicative invariants in the sequence of 'non-multiples') clearly shows this student's 'Aha!' experience as she identifies the structure in the sequence 8, 15, 22, … . Probing for alternate treatments of the problem, as well as pointing out conflicts or inconsistencies in one's claims, invited students to reconsider their strategies and, therefore, seemed to aid the development of students' schemes. Although the clinical interview process was helpful for students in extending and clarifying their ideas, the time limitations may not have presented students with a sufficient variety of situations to identify the unifying invariant structure of arithmetic sequences.

We suggest that a traditional instructional treatment of arithmetic sequences is missing an aspect. This missing aspect is the lack of attention to common structure of elements in any sequence. Pedagogical attention to structure and pattern recognition is crucial for any learner of mathematics. Moreover, such focus could be especially beneficial for pre-service elementary school teachers, who are more likely to engage in the activity of pattern recognition than in algebraic manipulations during their teaching careers.

Cai and Silver (1993), studying students' inability to deal with division-with-remainder problems in a contextualized setting, found that these problems 'provide an interesting context in which to consider students' mathematical thinking and reasoning' (p. 491). We add to this claim that division-with-remainder embedded in the arithmetic sequence of whole numbers is an interesting context not only in a contextualized but also in a decontextualized setting.

The use of arithmetic sequence as a tool to study students' abilities to bridge conceptual fields has not been exhausted. Possible extensions of this study may involve students' notions of the last digit patterns as a means of justifying inclusion or exclusion of elements within a given sequence. Another possible extension is an exploration of students' robust treatment of sequences of multiples as being easier. Would they feel the same if, in the presented situations, the first element were not equal to the common difference (e.g., 36, 42, 48, …)? Would students treat multiples with relative ease when considering multiples of 'large' numbers (e.g., 157, 314, 471, …)? Is there a preferred sequence of situations to promote the development of students' schemes?

Greer (1992) proposed that the analysis of the relationship between the conceptual fields of additive and multiplicative structures is a long-term objective on the agenda for further research in mathematics education. Our research is a step in this direction. We have shown that arithmetic sequences can serve a dual purpose: first as a research tool to investigate students' connections between multiplicative and additive structures, and second, as a pedagogical tool, or the core of a didactical situation, in helping students link their additive and their multiplicative schemes.

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