Applications of Lagrangian: Kuhn Tucker Conditions

Utility Maximization with a simple rationing constraint

Consider a familiar problem of utility maximization with a budget constraint:

$$\begin{array}{ll} \text{Maximize} & U = U(x,y) \\ & \\ \text{subject to} & B = P_x x + P_y y \\ \text{and} & \overline{x} \geqslant x \end{array}$$

But where a ration on x has been imposed equal to $\overline{x}$. We now have two constraints. The Lagrange method easily allows us to set up this problem by adding the second constraint in the same manner as the first. The Lagrange becomes

$$\underset{x,y}{Max} \qquad U(x,y) + \lambda_1(B - P_x x - P_y y) + \lambda_2(\overline{x} - x)$$

However, in the case of more than one constraint, it is possible that one of the constraints is nonbinding. In the example we are using here, we know that the budget constraint will be binding but it is not clear if the ration constraint will be binding. It depends on the size of $\overline{x}$.

The two possibilities are illustrated in figure one. In the top graph, we see the standard utility maximization result with the solution at point E. In this case the ration constraint, $\overline{x}$, is larger than the optimum value x^*. In this case the second constraint could have been ignored.

In the bottom graph the ration constraint is binding. Without the constraint, the solution to the maximization problem would again be at point E. However, the solution for x violates the second constraint. Therefore the solution is determined by the intersection of the two constraints at point *E'*

Procedure:

This type of problem requires us to vary the first order conditions slightly. Cases where constraints may or not be binding are often referred to as *Kuhn-Tucker* conditions.

The Kuhn-Tucker conditions are

$$\begin{array}{ll} L_x = U_x - P_x\lambda_1 - \lambda_2 = 0 & x \geq 0 \\ L_y = U_y - P_y\lambda_1 = 0 & y \geq 0 \\ and & \\ L_{\lambda_1} = B - P_x x - P_y y \geq 0 & \lambda_1 \geq 0 \\ L_{\lambda_2} = \overline{x} - x \geq 0 & \lambda_2 \geq 0 \end{array}$$

Now let us interpret the Kuhn-Tucker conditions for this particular problem. Looking at the Lagrange

$$U(x,y) + \lambda_1(B - P_x x - P_y y) + \lambda_2(\overline{x} - x)$$

Figure 1:

We require that

$$\lambda_1(B - P_x x - P_y y) = 0$$

therefore either

$$\lambda_1 = 0$$
$$or$$
$$B - P_x x - P_y y = 0$$

If we interpret $\lambda_1$as the marginal utility of the budget (Income), then if the budget constraint is not met the marginal utility of additional B is zero ($\lambda_1 = 0$).

(2) Similarly for the ration constraint, either

$$\overline{x} - x = 0$$
$$or$$
$$\lambda_2 = 0$$

λ_2 can be interpreted as the marginal utility of relaxing the ration constraint.

Solving by Trial and Error

Solving these types of problems is a bit like detective work. Since there are more than one possible outcomes, we need to try them all. But before you start, it is important to think about the problem and try to make an educated guess as to which constraint is more likely to be nonbinding. In this example we can be sure that the budget constraint will always be binding, therefore we only need to worry about the effects of the ration constraint.

Step one: Assume $\lambda_2 = 0, \lambda_1 > 0$ (simply ignore the second constraint) the first order conditions become

$$\begin{aligned}
L_x &= U_x - P_x\lambda_1 - \lambda_2 = 0 \\
L_y &= U_y - P_y\lambda_1 = 0 \\
L_{\lambda_1} &= B - P_x x - P_y y = 0
\end{aligned}$$

Find a solution for x^* and y^* then check if you have violated the constraint you ignored. If you have, go to step two.

Step two: Assume $\lambda_2 > 0, \lambda_1 > 0$ (use both constraints, assume they are binding) The first order conditions become

$$\begin{aligned}
L_x &= U_x - P_x\lambda_1 - \lambda_2 = 0 \\
L_y &= U_y - P_y\lambda_1 = 0 \\
L_{\lambda_1} &= B - P_x x - P_y y = 0 \\
L_{\lambda_2} &= \overline{x} - x = 0
\end{aligned}$$

In this case, the solution will simply be where the two constraints intersect.

Step three: Assume $\lambda_2 > 0, \lambda_1 = 0$ (use the second constraint, but ignore the first constraint)

Numerical example

$$Maximize \quad U = xy$$

subject to:

$$\begin{gathered}
100 \geq x + y \\
and \\
x \leq 40
\end{gathered}$$

The Lagrange is

$$xy + \lambda_1(100 - x - y) + \lambda_2(40 - x)$$

and the Kuhn-Tucker conditions become

$$\begin{array}{ll}
L_x = y - \lambda_1 - \lambda_2 = 0 & x \geq 0 \\
L_y = x - \lambda_1 = 0 & y \geq 0 \\
L_{\lambda_1} = 100 - x - y \geq 0 & \lambda_1 \geq 0 \\
L_{\lambda_2} = 40 - x \geq 0 & \lambda_2 \geq 0
\end{array}$$

Which gives us four equations and four unknowns: x, y, λ_1 and λ_2.

To solve, we typically approach the problem in a stepwise manner. First, ask if any λ_i could be zero Try $\lambda_2 = 0$ ($\lambda_1 = 0$ does not make sense, given the form of the utility function), then

$$x - \lambda_1 = y - \lambda_1 \quad or \quad x = y$$

from the constraint $100 - x - y$ we get $x^* = y^* = 50$ which violates our constraint $x \leq 40$. Therefore $x^* = 40$ and $y^* = 60$, also $\lambda_1^* = 40$ and $\lambda_2^* = 20$

War-Time Rationing

Typically during times of war the civilian population is subject to some form of rationing of basic consumer goods. Usually, the method of rationing is through the use of redeemable coupons used by the government. The government will supply each consumer with an allotment of coupons each month. In turn, the consumer will have to redeem a certain number of coupons at the time of purchase of a rationed good. This effectively means the consumer "pays" two "prices" at the time of the purchase. He or she pays both the coupon price and the monetary price of the rationed good. This requires the consumer to have both sufficient funds and sufficient coupons in order to buy a unit of the rationed good.

Consider the case of a two-good world where both goods, x and y. are rationed. Let the consumer's utility function be $U = U(x, y)$. The consumer has a fixed money budget of B and faces the money prices P_x and P_y. Further, the consumer has an allotment of coupons, denoted C, which can be used to purchase both x or y at a coupon price of c_x and c_y. Therefore the consumer's maximization problem is

Maximize

$$U = U(x, y)$$

Subject to

$$B \geq P_x x + P_y y$$

and

$$C \geq c_x x + c_y y$$

in addition, the non-negativity constraint $x \geq 0$ and $y \geq 0$.

The Lagrangian for the problem is

$$Z = U(x, y) + \lambda(B - P_x x - P_y y) + \lambda_2(C - c_x x + c_y y)$$

where λ, λ_2 are the Lagrange multiplier on the budget and coupon constraints respectively. The Kuhn-Tucker conditions are

$$\begin{array}{ll} Z_x = U_x - \lambda_1 P_x - \lambda_2 c_x = 0 & \\ Z_y = U_y - \lambda_1 P_y - \lambda_2 c_y = 0 & \\ Z_{\lambda_1} = B - P_x x - P_y y \geq 0 & \lambda_1 \geq 0 \\ Z_{\lambda_2} = C - c_x x - c_y y \geq 0 & \lambda_2 \geq 0 \end{array}$$

Numerical Example

Let's suppose the utility function is of the form $U = x \cdot y^2$. Further, let $B = 100, P_x = P_y = 1$ while $C = 120$ and $c_x = 2, c_y = 1$.

The Lagrangian becomes

$$Z = xy^2 + \lambda_1(100 - x - y) + \lambda_2(120 - 2x - y)$$

The Kuhn-Tucker conditions are now

$$\begin{array}{lll} Z_x = y^2 - \lambda_1 - 2\lambda_2 \leq 0 & x \geq 0 & x \cdot Z_x = 0 \\ Z_y = 2xy - \lambda_1 - \lambda_2 \leq 0 & y \geq 0 & y \cdot Z_y = 0 \\ Z_{\lambda_1} = 100 - x - y \geq 0 & \lambda_1 \geq 0 & \lambda_1 \cdot Z_{\lambda_1} = 0 \\ Z_{\lambda_2} = 120 - 2x - y \geq 0 & \lambda_2 \geq 0 & \lambda_2 \cdot Z_{\lambda_2} = 0 \end{array}$$

Solving the problem:

Typically the solution involves a certain amount of trial and error. We first choose one of the constraints to be non-binding and solve for the x and y. Once found, use these values to test if the constraint chosen to be non-binding is violated. If it is, then redo the procedure choosing another constraint to be non-binding. If violation of the non-binding constraint occurs again, then we can assume both constraints bind and the solution is determined only by the constraints.

Step one: Assume $\lambda_2 = 0, \lambda_1 > 0$

By ignoring the coupon constraint, the first order conditions become

$$\begin{aligned} Z_x &= y^2 - \lambda_1 = 0 \\ Z_y &= 2xy - \lambda_1 = 0 \\ Z_{\lambda_1} &= 100 - x - y = 0 \end{aligned}$$

Solving for x and y yields

$$x^* = 33.33 \quad y^* = 66.67$$

However, when we substitute these solutions into the coupon constraint we find that

$$2(33.33) + 66.67 = 133.67 > 120$$

The solution violates the coupon constraints.

Step two: Assume $\lambda_1 = 0, \lambda_2 > 0$

Now the first order conditions become

$$\begin{aligned} Z_x &= y^2 - 2\lambda_2 = 0 \\ Z_y &= 2xy - \lambda_2 = 0 \\ Z_{\lambda_1} &= 120 - 2x - y = 0 \end{aligned}$$

Solving this system of equations yields

$$x^* = 20 \quad y^* = 80$$

When we check our solution against the budget constraint, we find that the budget constraint is just met. In this case, we have the unusual result that the budget constraint is met but is not binding due to the particular location of the coupon constraint. The student is encouraged to carefully graph the solution, paying careful attention to the indifference curve, to understand how this result arose.

Peak Load Pricing

Peak and off-peak pricing and planning problems are common place for firms with capacity constrained production processes. Usually the firm has invested in capacity in order to target a primary market. However there may exist a secondary market in which the firm can often sell its product. Once the capital has been purchased to service the firm's primary market, the capital is freely available (up to capacity) to be used in the secondary market. Typical examples include: schools and universities who build to meet day-time needs (peak), but may offer night-school classes (off-peak); theatres who offer shows in the evening (peak) and matinees (off-peak); or trucking companies who have dedicated routes but may choose to enter "back-haul" markets. Since the capacity price is a factor in the profit maximizing decision for the peak market and is already paid, it normally, should not be a factor in calculating optimal price and quantity for the smaller, off-peak market. However, if the secondary market's demand is close to the same size as the primary market, capacity constraints may be an issue, especially given that it is common practice to price discriminate and charge lower prices in off-peak periods. Even though the secondary market is smaller than the primary, it is possible at the lower (profit maximizing) price that off-peak demand exceeds capacity. In such cases capacity choices maust be made taking both markets into account, makeing the problem a classic application of Kuhn-Tucker.

Consider a profit maximizing Company who faces two demand curves

$$\begin{array}{ll} P_1 = D^1(Q_1) & \text{in the day time (peak period)} \\ P_2 = D^2(Q_2) & \text{in the night time (off-peak period)} \end{array}$$

to operate the firm must pay b per unit of output, whether it is day or night. Furthermore, the firm must purchase capacity at a cost of c per unit of output. Let K denote total capacity measured in units of Q. The firm must pay for capacity, regardless if it operates in the off peak period. Question: Who should be charged for the capacity costs? Peak, off-peak, or both sets of customers? The firm's maximization problem becomes

$$\underset{Q_1,Q_2,K}{Maximize} \quad P_1Q_1 + P_2Q_2 - b(Q_1 - Q_2) - cK$$

Subject to

$$\begin{array}{l} K \geq Q_1 \\ K \geq Q_2 \end{array}$$

Where

$$\begin{array}{l} P_1 = D^1(Q_1) \\ P_2 = D^2(Q_2) \end{array}$$

The Lagrangian for this problem is:

$$Z = D^1(Q_1)Q_1 + D^2(Q_2)Q_2 - b(Q_1 + Q_2) - cK + \lambda_1(K - Q_1) + \lambda_2(K - Q_2)$$

The Kuhn-Tucker conditions are

$$\begin{array}{lll} Z_1 = D^1 + Q_1\frac{\partial D^1}{\partial Q_1} - b - \lambda_1 = 0 & & (MR_1 - b - \lambda_1 = 0) \\ Z_2 = D^2 + Q_2\frac{\partial D^2}{\partial Q_2} - b - \lambda_2 = 0 & & (MR_2 - b - \lambda_2 = 0) \\ Z_K = -c + \lambda_1 + \lambda_2 = 0 & & (c = \lambda_1 + \lambda_2) \\ Z_{\lambda_1} = K - Q_1 \geq 0 & \lambda_1 \geq 0 & \\ Z_{\lambda_2} = K - Q_2 \geq 0 & \lambda_2 \geq 0 & \end{array}$$

Assuming that $Q_1, Q_2, K > 0$ the first-order conditions become

$$\begin{array}{ll} MR_1 = b + \lambda_1 = b + c - \lambda_2 & (\lambda_1 = c - \lambda_2) \\ MR_2 = b + \lambda_2 & \end{array}$$

Figure 2:

Finding a solution:

Step One: Since $D^2(Q_2)$ is smaller than $D^1(Q_1)$ try $\lambda_2 = 0$

Therefore from the Kuhn-Tucker conditions

$$\begin{aligned} MR_1 &= b + c - \lambda_2 = b + c \\ MR_2 &= b + \lambda_2 = b \end{aligned}$$

which implies that $K = Q_1$. Then we check to see if $Q_2^* \leq K$. If true, then we have a valid solution. Otherwise the second constraint is violated and the assumption that $\lambda_2 = 0$ was false. Therefore we proceed to the next step.

Step Two: if $Q_2^* > K$ then $Q_1^* = Q_2^* = K$ and

$$\begin{aligned} MR_1 &= b + \lambda_1 \\ MR_2 &= b + \lambda_2 \end{aligned}$$

Since $c = \lambda_1 + \lambda_2$ then λ_1 and λ_2 represent the share of c each group pays. Both cases are illustrated in figure 2

Numerical Example Suppose the demand during peak hours is

$$P_1 = 22 - 10^{-5}Q_1$$

and during off-peak hours is

$$P_2 = 18 - 10^{-5}Q_2$$

To produce a unit of output per half-day requires a unit of capacity costing 8 cents per day. The cost of a unit of capacity is the same whether it is used at peak times only or off-peak also. In addition to the costs of capacity, it costs 6 cents in operating costs (labour and fuel) to produce 1 unit per half day (both day and evening)

If we assume that the capacity constraint is binding ($\lambda_2 = 0$), then the Kuhn-Tucker conditions (above) become

$$\begin{aligned} &\lambda_1 = c = 8 \\ &\overbrace{22 - 2 \times 10^{-5}Q_1}^{MR} \; \overbrace{= b + c}^{MC} \; = 14 \\ &18 - 2 \times 10^{-5}Q_2 \; = b \; = 6 \end{aligned}$$

Solving this system gives us

$$\begin{aligned} Q_1 &= 40000 \\ Q_2 &= 60000 \end{aligned}$$

which violates the assumption that the second constraint is non-binding ($Q_2 > Q_1 = K$).

Therefore, assuming that both constraints are binding, then $Q_1 = Q_2 = Q$ and the Kuhn-Tucker conditions become

$$\begin{aligned} \lambda_1 + \lambda_2 &= 8 \\ 22 - 2 \times 10^{-5} Q &= 6 + \lambda_1 \\ 18 - 2 \times 10^{-5} Q &= 6 + \lambda_2 \end{aligned}$$

which yields the following solutions

$$\begin{aligned} Q &= K = 50000 \\ \lambda_1 &= 6 \qquad \lambda_2 = 2 \\ P_1 &= 17 \qquad P_2 = 13 \end{aligned}$$

Since the capacity constraint is binding in both markets, market one pays $\lambda_1 = 6$ of the capacity cost and market two pays $\lambda_2 = 2$.

Problems

1. Suppose in the above example a unit of capacity cost only 3 cents per day.
   (a) What would be the profit maximizing peak and off-peak prices and quantitites?
   (b) What would be the values of the Lagrange multipliers? What interpretation do you put on their values?

2. Skippy lives on an island where she produces two goods, x *and* y, according the the production possibility frontier $200 \geq x^2 + y^2$, and she consumes all the goods herself. Her utility function is
$$u = x \cdot y^3$$
   Skippy also faces and environmental constraint on her total output of both goods. The environmental constraint is given by $x + y \leq 20$
   (a) Write down the Kuhn Tucker first order conditions.
   (b) Find Skippy's optimal x *and* y. Identify which constaints are binding.

3. An electric company is setting up a power plant in a foreign country and it has to plan its capacity. The peak period demand for power is given by $p_1 = 400 - q_1$ and the off-peak is given by $p_2 = 380 - q_2$. The variable cost to is 20 per unit (paid in both markets) and capacity costs 10 per unit which is only paid once and is used in both periods.

(a) write down the lagrangian and Kuhn-Tucker conditions for this problem

(b) Find the optimal outputs and capacity for this problem.

(c) How much of the capacity is paid for by each market (i.e. what are the values of λ_1 and λ_2)?

(d) Now suppose capacity cost is 30 per unit (paid only once). Find quantities, capacity and how much of the capacity is paid for by each market (i.e. λ_1 and λ_2)?