Legislative Bargaining with Long Finite Horizons

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Abstract

Institutional rules provide natural deadlines for negotiations in legislative bargaining. In the continuous-time bargaining model framework of Ambrus and Lu (2015), we show that as the time horizon of legislative bargaining increases, equilibrium payoffs with deadline converge to a stationary equilibrium payoffs of the infinite-horizon bargaining game. We provide a characterization of these limit payoffs, and show that under a K-majority rule, the payoffs of the K legislators with the lowest relative recognition probabilities have to be equal to each other when positive. Hence, by varying recognition probabilities, possible limit equilibrium payoffs are constrained to a lower-dimensional subset of the set of all possible allocations. This result contrasts with Kalandrakis' (2006) finding that in the infinite-horizon Baron and Ferejohn (1989) framework, for any discount factor, any division of the surplus can be achieved as a stationary equilibrium payoff through some choice of recognition probabilities.

Keywords: Legislative bargaining, Random recognition

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1 Introduction

The standard workhorse model of legislative bargaining, introduced in Baron and Ferejohn (1989) and used extensively in many applications, assumes an infinite-horizon environment.[1] However, in practice, legislatures face various deadlines that put an end to possible negotiations.

One firm deadline that applies to all negotiations is the end of the mandate of the legislature. There are various other institutional deadlines applying to specific negotiations, mandated by constitutional law or international organizations. These include time limits on government formation after general elections,[2] European Union deadlines for member countries to modify their laws to reflect a common directive,[3] and deadlines imposed by constitutional courts on legislatures to resolve conflicting laws.[4]

In most of the above cases, the deadline is far away relative to how frequently legislators can make proposals and counter-proposals. However, it is not clear whether the qualitative properties of legislative bargaining with long finite horizon (relative to the frequency of proposals) are similar to those with infinite horizon. In fact, Norman (2002) provides a negative result, showing that in discrete-time legislative bargaining games, expected subgame perfect equilibrium payoffs of finite-horizon games do not converge to stationary subgame perfect equilibrium payoffs of the infinite-horizon game.

[1] Applications in political science and economics include Baron (1991, 1996), Winter (1996), Chari *et al.* (1997), Diermeier and Feddersen (1998), Banks and Duggan (2000) and LeBlanc *et al.* (2000).

[2] In the devolved legislatures in Scotland and Wales, the First Minister must be chosen within 28 days of the election. This rule is regarded as an effective limit on the time available to the parties to form a government (see Blackburn *et al.* (2010)).

[3] The Council of the EU set a 19 July 2003 deadline for the legislative bodies of member states to transpose the Racial Equality Directive (see Council Directive 2000/43/EC) and a 2 December 2003 deadline to transpose the Employment Equality Directive (see Council Directive 2000/78/EC).

[4] Article 46, Paragraph 1 of Hungary's *Act XXXII of 1989 on the Constitutional Court* states: "If the Constitutional Court establishes that a law at a higher level than the law promulgating the international treaty conflicts with the international treaty, it invites the legislative organ which concluded the international treaty to resolve the conflict, setting a deadline based on the consideration of circumstances."

We establish a connection between finite and infinite horizon legislative bargaining, using the continuous-time finite-horizon model framework of Ambrus and Lu (2015). In this model, players get the chance to make proposals at random times, according to a Poisson process. A higher recognition rate for a given player means that, in expectation, she can make proposals more frequently. Ambrus and Lu (2015) shows that this model, for general payoff specification, has a unique Markov perfect equilibrium (MPE) payoff vector.[5]

In this paper, we show that in legislative bargaining games, if the time horizon of the game (or, equivalently, the recognition rates) increases, continuous-time MPE payoffs converge to stationary subgame-perfect equilibrium (SSPE) payoffs of the infinite-horizon game. Simply put, if the game is long, the expected division of the surplus is close to an equilibrium allocation in the case of open-ended negotiations. The SSPE payoffs in the infinite-horizon game are unique when all players' recognition rates are strictly positive, but not necessarily unique when some players have zero recognition rates. In the latter specifications, MPE payoffs of finite-horizon games converge to the particular SSPE of the infinite-horizon game, in which players with zero recognition rates receive zero payoff.

We provide an algorithm that finds the above limit division of the surplus for any vector of relative recognition rates. This characterization is particularly simple when we only consider strictly positive recognition rates: if K votes are needed for accepting a proposal, the expected payoffs of the K legislators with the lowest relative recognition rates must be equal to each other. Conversely, any division of the surplus in which the lowest K shares are equalized can be achieved as limit equilibrium payoffs for some vector of recognition rates. We obtain a slightly more complex characterization when allowing for zero recognition rates: a division of the surplus $(x_1, ..., x_n)$, where $x_1 \leq ... \leq x_n$, is feasible as a limit equilibrium payoff if and only if there is $k < K$ such that $x_1 = ... = x_k = 0$, and $x_{k+1} = ... = x_K$.

The results imply that with a long but finite time horizon, the set of equilibrium

[5] It is also shown that any sequence of subgame-perfect equilibria of generic discrete-time games converging in a formal sense to the continuous-time bargaining game converges to the MPE of the latter game. This provides a justification for focusing on the MPE of the continuous-time game.

surplus divisions that can be attained by varying recognition rates is a lower dimensional subset of the set of all possible divisions. Moreover, the restriction that our model places on the payoff division is a particularly simple one.

This finding contrasts with the main result in Kalandrakis (2006), which shows that in a discrete infinite-horizon bargaining framework, if recognition probabilities can be varied, there is no testable implication of SSPE with respect to expected payoffs: for any discount factor, any division of the surplus can be attained by appropriately chosen proposal probabilities.[6] We obtain a different result because there is a multiplicity of equilibria in the infinite-horizon model when there are legislators with recognition probability 0. This multiplicity disappears in a game with a long finite time horizon, where the payoffs of such legislators are pinned down to 0. We consider it an empirical question whether our modeling approach, leading to different implications regarding possible divisions of surplus than in an infinite-horizon game, is valid.

For ease of exposition, in the baseline model, we assume that players receive their payoffs at a pre-specified time at or after the deadline, independently of when they reach an agreement. For example, this pre-specified time can represent the beginning of a new fiscal year. In this version of the model, the urgency to reach an agreement stems only from the approaching deadline and the randomness of proposal opportnities. In Section 4.3, we extend the results to the more standard case where players divide the surplus at the time of reaching an agreement, and they discount future payoffs. In this case, we show that the set of limit equilibria is exactly the same as in the previous model if the time horizon of the game goes to infinity and the discount rate of the legislators goes to 0.

2 Model

Our model of legislative bargaining builds on the framework introduced in Ambrus and Lu (2015). In particular, we consider a set of players $N = \{1, 2, ..., n\}$, and a

[6]Kalandrakis (2006) establishes this not only for the quota rules that we study in the current paper, but also for a large class of voting rules that satisfy a monotonicity requirement.

value function $V : 2^N \to \mathbb{R}_+$ where, for some $K \in \{1, ..., n-1\}$ and every $C \subseteq N$, $V(C) = 1$ if $|C| \geq K$, and $V(C) = 0$ otherwise. We refer to elements of 2^N as *coalitions*. Coalitions consisting of at least K members can pass a legislation regarding how to distribute a pie of size 1 among different members. We refer to K as the quota required to pass a legislation.

The noncooperative bargaining game we investigate is defined as follows. The game is set in continuous time, starting at $T < 0$. There is a Poisson process associated with each player i, with arrival rate $\lambda_i \geq 0$. The processes are independent from each other. For future reference, we define $\lambda \equiv \sum_{i=1}^n \lambda_i$. Whenever the process realizes for a player i, she is recognized and can make an offer $x = (x_1, x_2, ..., x_n)$ to a coalition $C \subseteq N$ satisfying $i \in C$. The offer x must have the following characteristics:

1. $x_j \geq 0$ for all $1 \leq j \leq n$;
2. $\sum_{j=1}^n x_j \leq V(C)$.

Players in $C \backslash \{i\}$ immediately and sequentially accept or reject the offer (the order in which they do so is unimportant). If everyone accepts, the game ends, and all players in N are paid their shares according to x. If an offer is rejected by at least one of the respondents, it is taken off the table, and the game continues with the same Poisson arrival rates. If no offer has been accepted at time 0, the game ends, and all players receive payoff 0.

In most of this paper, we assume that the payoffs are received at time 0, even if an agreement is reached at time $t < 0$. See Section 4.3 for the case where players receive their payoffs immediately after an agreement is reached.

We are interested in MPE of the resulting game, that is subgame-perfect equilibria in which a proposer's strategy depends only on t, and a respondent's strategy depends only on t, the offer on the table and any previous responses to the proposal.

3 Results

For ease of exposition, we assume in this section that all players have strictly positive recognition rates: $\lambda_i > 0$ for all i. Extending the results to the case where some of the rates can be zero is straightforward and discussed in Section 4.

Let $w_i(t)$ denote player i's expected continuation value at time t, and let $w(t) = (w_1(t), ..., w_n(t))$. Ambrus and Lu (2015) show that in their framework, for any value function, $w_i(t)$ is unique in MPE. This implies that in MPE, a proposer at time t approaches a coalition C maximizing $V(C) - \sum_{i \in C} w_i(t)$ among coalitions containing the proposer, offers all other players $i \in C$ their continuation value $w_i(t)$, and that all on-path offers are accepted.

Our first result establishes that at any given time in a MPE, continuation values are weakly increasing in recognition rates.

Proposition 1: If $\lambda_i \leq \lambda_{i'}$, then $w_i(t) \leq w_{i'}(t)\ \forall\ t \leq 0$.

For the proofs of all formal results, see the appendix. The intuition for Proposition 1 is that whenever $w_i(t) > w_{i'}(t)$, player i' would both be included in a proposal weakly more often, and obtain the extra surplus associated with proposing (*i.e.* part of payoff above the continuation value) weakly more often. These facts increase the reservation value of player i' relative to that of player i for τ immediately to the left of t; that is, $w_i(\tau) - w_{i'}(\tau) < w_i(t) - w_{i'}(t)$. However, since continuation value functions are continuous and $w_i(0) = w_{i'}(0)$, the argument implies that $w_i(t) - w_{i'}(t)$ can never be positive in the first place.

Our main result, Theorem 1, shows that the limit expected MPE payoffs as the time horizon goes to infinity converge to the unique SSPE payoff vector of the infinite-horizon game.[7] This payoff vector can be obtained by the simple procedure below.

Without loss of generality, assume that players are ordered such that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$, and that $\lambda = 1$.[8]

For every $j \in \{1, ..., n\}$, define $x^j = (x_1^j, ..., x_n^j)$ as follows:

[7] Baron and Kalai (1993) show the uniqueness of stationary SPNE payoffs in the Baron and Ferejohn (1989) legislative bargaining model with equal proposal probabilities, and Eraslan (2002) extends this result to strictly positive proposal probabilities. Techniques from the latter work can be used in our continuous-time framework to show the uniqueness of stationary SPNE payoffs in the infinite-horizon version of the model with strictly positive recognition rates. We skip the details here.

[8] As discussed near the end of this section, the latter can be obtained by renormalizing time.

Let y_j be the solution to the equation $jy_j + (1-(K-1)y_j)\sum_{i=j+1}^{n} \lambda_i = 1$. Then for every $i \in \{1, ..., j\}$ let $x_i^j = y_j$, and for every $i \in \{j+1, ..., n\}$ let $x_i^j = (1-(K-1)y_j)\lambda_i$. If $j \geq K$ and players $1, ..., j$ are tied for the lowest payoff y_j, x^j is the stationary expected payoff vector with no discounting: players $j+1, ..., n$ would receive payoff $1-(K-1)y_j$ if they are the first to get a chance to propose, and would not be approached when another player proposes.

Let j^* be the smallest $j \in \{1, ..., n\}$ such that $x_{j+1}^j \geq x_j^j \geq 0$ (assume this holds trivially for $j = n$). To simplify notation, let $x_i^* = x_i^{j^*}$ $\forall$ $i \in \{1, ..., n\}$, and let $x^* = x^{j^*}$.

Proposition 2: The unique SSPE payoff vector of the infinite-horizon game is x^*.

Proving Proposition 2 entails checking that $j^* \geq K$ (so that players $i > j^*$ are indeed never approached), and that players $i \leq j^*$ cannot obtain payoff above x_i^* without being approached. The latter is the reason why x^j cannot be an SSPE payoff vector for $j > j^*$, and an argument analogous to the proof of Proposition 1 rules out x^j for $j < j^*$.

Theorem 1: $\lim_{T \to -\infty} w(T) = x^*$. Moreover, there exists $T^* < 0$ such that for all $T < T^*$, $w_i(T) = w_{i'}(T)$ $\forall$ $i, i' \in \{1, ..., j^*\}$.

Theorem 1 implies that if the deadline is sufficiently far away, then the continuation values of the j^* players (where $j^* \geq K$) with the lowest recognition rates are equal, and each of these players is approached with positive probability.[9] The continuation values of the other players are greater, and ordered according to the relative recognition rates. Far away from the deadline, these players are not approached by any other player.

The proof of Theorem 1 first notes that far enough away from the deadline, players $1, ..., K$ must have equal continuation values. This is easy to see when considering

[9] These probabilities are generally not equal. Players with lower recognition rates are approached more frequently - this is what keeps the continuation values of the j^* players equal.

stationary payoffs: if fewer than K players are tied for the lowest payoff, then player 1 would always be approached (and thus receive her continuation payoff) when another player proposes; however, she would also get more than her continuation payoff when she is recognized, which yields a contradiction. The proof also shows that if the q weakest legislators' values (where $q > K$) are equal at some $t \neq 0$, then at all times before t, they are equal as well (and decreasing as time moves toward the deadline). As a result, the number of legislators tied for the lowest continuation value must converge as $t \to -\infty$, and therefore every player's payoff converges as well.

We conclude this section by observing that taking the time horizon of the game to infinity, given a fixed vector of recognition rates, is equivalent to scaling up recognition rates to infinity, for a fixed time horizon. Therefore, our result concerning limit payoffs applies to the case when the time horizon is fixed, but legislators can make proposals more and more frequently.

Observation: Under a rescaling of time, for any $\alpha > 0$, a game with recognition rate vector and time horizon $(\lambda, \alpha T)$ is equivalent to a game with corresponding parameters $(\alpha\lambda, T)$.

Corollary: Let $\overrightarrow{\lambda}$ be a vector in $\mathbb{R}^n_{++}$, and consider games with recognition rate vector $c\overrightarrow{\lambda}$. Then for any $T < 0$, $\lim_{c\to\infty} w(T) = x^*$, the SSPE payoff vector of the infinite-horizon game. Moreover, for any $T < 0$, there exists $c^*(T) > 0$ such that for all $c > c^*(T)$, $w_i(T) = w_{i'}(T)\ \forall i, i' \in \{1, ..., j^*\}$.

4 Discussion

4.1 Zero Recognition Rates

In our model, $\lambda_i = 0$ for some $i \in \{1, ..., n\}$ implies that $w_i(t) = 0$ for all t. To see this, let $p_i(t)$ be the probability that i is included in a proposal conditional on recognition at time t. Then, at almost all t, $w_i'(t) = -\lambda p_i(t)w_i(t) + \lambda w_i(t) \geq 0$. Since $w_i(0) = 0$ and $w_i(.)$ is continuous, we must have $w_i(t) = 0$ for all t. As a result, independently of T, in every MPE, other players offer 0 to i any time they can make a proposal. The presence of a deadline is important: without it, there can be equilibria where, for example, player i is approached and offered $\varepsilon > 0$ at all times.

Assume now that $m > 0$ of the players have zero recognition rates. If $m < K-1$, the payoff functions of players $m+1, ..., n$ are the same as in a game without players $1, ..., m$, and with quota $K - m$. Hence the results of the previous section can be directly applied. If $m \geq K - 1$, the game is trivial: upon the first recognition, which can only happen for a player $i > m$, player i approaches $K - 1$ of players in $\{1, ..., m\}$ and proposes 1 for herself. The limit expected payoffs as $T \to \infty$ are then $(0, ..., 0, \lambda_{m+1}, ..., \lambda_n)$.

4.2 Possible Surplus Divisions

Theorem 1, together with the observations regarding zero recognition rates made in Section 4.1, imply that the main result of Kalandrakis (2006), namely that equilibrium expected payoffs are not restricted in legislative bargaining once the bargaining protocol (probabilities of being selected as a proposer) can be freely chosen, does not hold for limit MPE payoffs in our model:[10] our results imply that there exists $k \in \{1, ..., K\}$ such that $x_i = 0$ for every $i < k$, and $x_i = x_{i'}$ for every $i, i' \in \{k, ..., K\}$. The set of payoff divisions satisfying these restrictions is a lower dimensional subset of the set of all possible payoff divisions, which is the unit simplex in $\mathbb{R}^n$.

The root of this discrepancy lies in the potential presence of players with zero proposal probability in an infinite-horizon versus a finite-horizon model. For example, in 3-player legislative bargaining with equal weights and simple majority rule, the payoff division $(0.5, 0.3, 0.2)$ cannot be achieved as the limit of SPNE payoffs as the time horizon goes to infinity, given our results. This payoff vector can be achieved in stationary equilibrium in the Baron and Ferejohn model only if one of the proposal probabilities is 0. However, in our model, if some player has a 0 recognition rate, then her payoff has to be 0, and if all players have positive recognition rates, then

[10] See also Eraslan (2002), Snyder *et al.* (2005) and Montero (2006) for further investigations of stationary SPNE payoffs in discrete-time legislative bargaining games.

the two lowest limit payoffs have to be equal.

4.3 Payoffs Divided at Time of Agreement

Now assume that if a proposal is accepted at time t, players divide the surplus according to the proposal right at t. Assume also that all legislators discount future payoffs using discount rate $r > 0$. Define $w_i^r(t)$ and $w^r(t)$ analogously to $w_i(t)$ and $w(t)$ in Section 3.

Proposition 1 applies to the new setting, with a very similar proof as before.

We show that limit MPE payoffs when the time horizon is first taken to infinity, and then the legislators become perfectly patient are exactly the same as characterized in Theorem 1. Therefore, even if payoffs are divided at the time of agreement, our model yields similar results as in Theorem 1 provided that players are patient enough.

Theorem 2: If $\lambda_i > 0$ for all i, $\lim\limits_{r \searrow 0} \lim\limits_{T \to -\infty} w^r(T) = x^*$. Moreover, there exists $r^* > 0$ such that whenever $r \in (0, r^*)$, there exists $T^r < 0$ such that for all $T < T^r$, $w_i^r(T) = w_{i'}^r(T)\ \forall\ i, i' \in \{1, ..., j^*\}$.

The proof of Theorem 2 is similar to the proof of Theorem 1. It shows that, if r is sufficiently small relative to λ_1, the limit payoffs solve equations that converge to those of Section 3 as $r \to 0$. The observations of Section 4.1 apply when some players have zero recognition rates: because the existence of a deadline already guarantees zero payoffs for these players, the addition of discounting does not impact them, unlike in the infinite-horizon model.

4.4 Shorter Time Horizons

So far, the focus has been on the case when the time horizon of negotiations is long relative to the expected lag between proposals; below, we make a few simple observations concerning shorter time horizons.

First, close to the deadline, expected payoffs are low, as there is a high probability that the deadline is reached before any player is recognized. Moreover, payoffs are approximately proportional to the recognition rates: if the surplus that a proposer has to offer to other players is low, the relative magnitudes of payoffs are dominated by the relative magnitudes of recognition rates.

Second, we point out that Proposition 1 applies to any time horizon. That is, the expected payoff of a legislator with higher recognition rate is at least weakly higher, no matter how much time is left before the deadline.

Finally, we provide an example showing that a strong legislator's payoff might be nonmonotonic in the time remaining before the deadline.

Consider a legislative bargaining game with $n = 3$ and $K = 2$. Suppose also that $\lambda_1 = 0.15$, $\lambda_2 = 0.25$, and $\lambda_3 = 0.6$. As Figure 1 shows, close to the deadline, $w_1 < w_2 < w_3$. In this region, upon recognition, player 1 would be approached by either player 2 or player 3, while player 2 is approached only by player 1 and not by player 3. Ultimately, this effect dominates the fact that player 2 has a higher recognition rate, and at some point the expected MPE payoff of player 1 becomes equal to that of player 2. Before this time, player 3 approaches both player 1 and player 2 with probability less than 1, in a way that keeps their continuation values equal. The expected continuation payoff of player 3, although she is never approached by any of the other players, remains bounded away from the other players' continuation payoffs, no matter how long the time horizon of the game is. This is due to the high relative recognition rate of player 3, and to the extra surplus from proposing staying bounded away from 0 in this game.

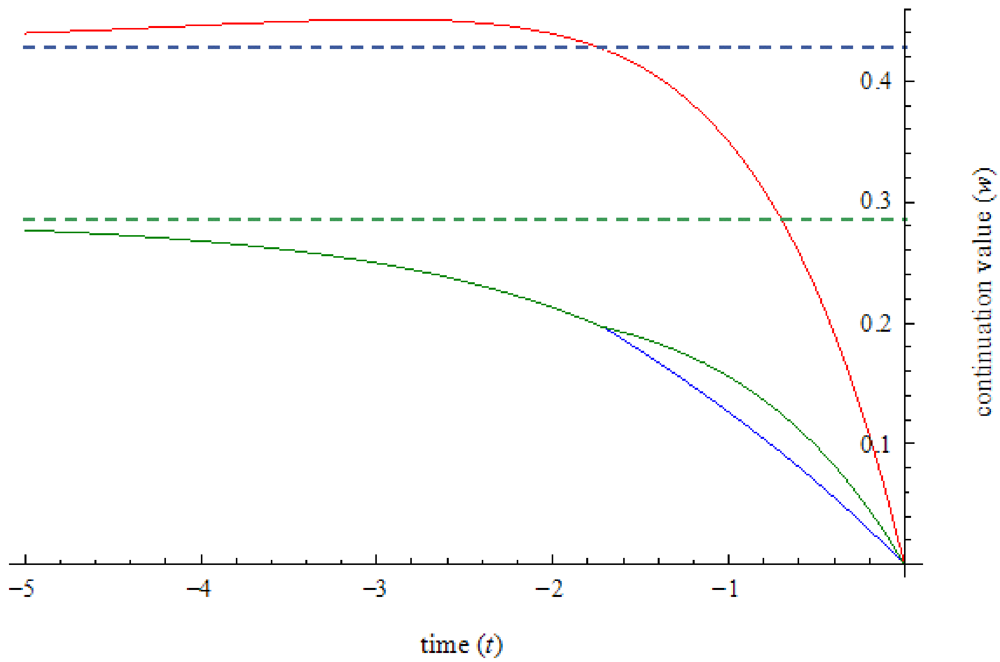

Figure 1. MPE expected payoff functions for $n = 3$, $K = 2$, $\lambda_1 = 0.15$, $\lambda_2 = 0.25$, $\lambda_3 = 0.6$

Player 3's continuation payoff is strictly higher for intermediate values of T than in the limit as $T \to -\infty$. Therefore, if player 3 is the member of the legislature who can set a deadline for negotiations, she would choose an intermediate time horizon even if she were arbitrarily patient. The reason why a strong player might prefer a shorter deadline is that the expected payoffs of weak players increase in the time horizon of negotiations (in fact, weak players always prefer longer deadlines). Therefore, if the deadline is too far out, weak players have to be offered a relatively high share of the surplus.

5 Conclusion

In this paper, we find that in legislative bargaining with a deadline, the relative strength of weak legislators is payoff-relevant only when the deadline is looming. When the deadline is far away, their payoffs are equal. This implies a simple and potentially testable restriction on the set of possible equilibrium payoff divisions for different recognition probabilities.

Appendix: Proofs

Proof of Proposition 1: Ambrus and Lu's (2015) result referenced at the beginning of Section 3 implies that in our legislative bargaining game, at any t in MPE, any player j proposes, if recognized, to the cheapest coalition of size K that includes her. Suppose $w_i(t) > w_{i'}(t)$. Then if any player $j \in N \backslash \{i, i'\}$ approaches i with positive probability, she must approach i' with probability 1. Moreover, if i' approaches i with positive probability, then i approaches i' with probability 1. Since $\lambda_i \leq \lambda_{i'}$, the probability of being included in a proposal at time t for i' (conditional on a proposal at time t), denoted $p_{i'}(t)$, is weakly greater than for i.

Also, $w_i(t) > w_{i'}(t)$ implies that the additional surplus from proposing for i', $\max_{D \ni i'}\{V(D) - \sum_{j \in D} w_j(t)\} \equiv s_{i'}(t)$, is weakly greater than for i.

Combining the above two facts, we have, almost everywhere,

$$
\begin{aligned}
w_i'(t) &= -\lambda_i s_i(t) + \lambda(1 - p_i(t))w_i(t) \\
&\geq -\lambda_{i'} s_{i'}(t) + \lambda(1 - p_{i'}(t))w_{i'}(t) \\
&= w_{i'}'(t)
\end{aligned}
$$

Therefore, the function $w_i(.) - w_{i'}(.)$ must weakly increase (traveling toward the deadline) whenever it is positive. Since $w_i(0) - w_{i'}(0) = 0$, we conclude that such t cannot exist. QED

Proof of Proposition 2: Since an argument similar to that from Eraslan (2002) establishes uniqueness of SSPE payoffs, it suffices to show that x^* is an SSPE payoff vector. For this, we need to show the following:

(i) The assumption that players $i > j^*$ are never approached (so that their payoff is indeed $x_i^* = \lambda_i(1 - (K-1)y_{j^*})$) is accurate. Since we know by construction that $x_i^* > y_{j^*}$, it suffices to show that $j^* \geq K$.

(ii) Players $i \leq j^*$ cannot do better than x_i^* without being approached, *i.e.* $\lambda_i(1-(K-1)x_i^*) \leq x_i^*$. Since the λ_i are ordered, it suffices to show that $\lambda_{j^*}(1-(K-1)y_{j^*}) \leq y_{j^*}$.

By the definition of y_{j^*},

$$j^* y_{j^*} + (1-(K-1)y_{j^*}) \sum_{i=j^*+1}^{n} \lambda_i = 1 \tag{1}$$

Since $\sum_{i=j^*+1}^{n} \lambda_i < 1$, we must have either $j^* y_{j^*} + (1-(K-1)y_{j^*}) > 1$, or $1-(K-1)y_{j^*} \leq 0$. The latter cannot be the case: it implies $y_{j^*} \geq \frac{1}{K-1}$ and $\lambda_{j^*+1}(1-(K-1)y_{j^*}) \leq 0$, so that $x_{j^*+1}^{j^*} = \lambda_{j^*+1}(1-(K-1)y_{j^*}) < y_{j^*} = x_{j^*}^{j^*}$, which violates the definition of j^*. The former simplifies to $j^* y_{j^*} > (K-1)y_{j^*}$; since $y_{j^*} \geq 0$, we must have $j^* > (K-1)$, as desired in (i).

Now suppose (ii) is false, so that $\lambda_{j^*}(1-(K-1)y_{j^*}) > y_{j^*}$. By the definition of j^*, we also have $x_{j^*}^{j^*-1} < x_{j^*-1}^{j^*-1}$, *i.e.* $\lambda_{j^*}(1-(K-1)y_{j^*-1}) < y_{j^*-1}$. Thus, $y_{j^*} - y_{j^*-1} < (K-1)(y_{j^*-1} - y_{j^*})$, which is equivalent to $y_{j^*-1} - y_{j^*} > 0$.

By assumption, $\lambda_{j^*}(1-(K-1)y_{j^*}) - y_{j^*} > 0$. Adding this to (1) gives

$$(j^*-1)y_{j^*} + (1-(K-1)y_{j^*}) \sum_{i=j^*}^{n} \lambda_i > 1$$

And by the definition of y_{j^*-1},

$$(j^*-1)y_{j^*-1} + (1-(K-1)y_{j^*-1}) \sum_{i=j^*}^{n} \lambda_i = 1$$

Taking the difference between the last two equations yields

$$\begin{aligned}
(j^*-1)(y_{j^*} - y_{j^*-1}) - (K-1)(y_{j^*} - y_{j^*-1}) \sum_{i=j^*}^{n} \lambda_i &> 0 \\
\left[(j^*-1) - (K-1)\sum_{i=j^*}^{n} \lambda_i\right](y_{j^*} - y_{j^*-1}) &> 0
\end{aligned}$$

Since $j^* \geq K$ and $\sum_{i=j^*}^{n} \lambda_i \leq 1$, this implies $y_{j^*} - y_{j^*-1} > 0$, which contradicts $y_{j^*-1} - y_{j^*} > 0$. QED

Proof of Theorem 1: First, we establish the following lemma:

Lemma 1: There exists a time $t' < 0$ such that $w_i(t') = w_{i'}(t')\ \forall\, i, i' \in \{1, ..., K\}$.

Proof: By Proposition 1, $w_1(t) = \min_j w_j(t)\ \forall t$, so if there is no t' as in the lemma, then $w_1(t) < w_K(t)\ \forall t < 0$. As a result, at all times, upon recognition of any player, player 1 is included in the coalition with probability 1. Therefore, almost everywhere,

$$\begin{aligned} w_1'(t) &= -\lambda_1 \left[1 - \left(\min_{C \subset N \setminus \{1\}:\ |C| = K-1} \sum_{i \in C} w_i(t) + w_1(t)\right)\right] \\ &= -\lambda_1 \left[1 - \min_{C \subset N:\ |C| = K} \sum_{i \in C} w_i(t)\right] \\ &\leq -\lambda_1 (1 - \frac{K}{n}) \end{aligned}$$

Since $w_1(0) = 0$, $w_1(t) \geq -\lambda_1(1 - \frac{K}{n})t\ \forall t < 0$. Since $K < n$, this implies $w_1(t) > 1$ for sufficiently early t, which is impossible. QED

Let $W_j(t) = \sum_{i=1}^{j} w_i(t)$, and suppose $j \geq K$. Proposition 1 implies that when a sender $i > j$ is recognized, i offers a combined surplus of at most $W_{K-1}(t)$ to players in $\{1, ..., j\}$ (it could be less since a player $k > j$ may be approached if $w_k(t) = w_{K-1}(t)$). Therefore

$$W_j'(t) \geq -\sum_{i=1}^{j} \lambda_i - W_{K-1}(t) \sum_{i=j+1}^{n} \lambda_i + \lambda W_j(t)$$

By Proposition 1, $W_{K-1}(t) \leq \frac{K-1}{j} W_j(t)$. Thus we have

$$W_j'(t) \geq -\sum_{i=1}^{j} \lambda_i - \frac{K-1}{j} W_j(t) \sum_{i=j+1}^{n} \lambda_i + \lambda W_j(t)$$

Therefore,

$$\frac{d}{dt}\frac{W_j(t)}{j} \geq -\frac{1}{j}\left[\sum_{i=1}^{j}\lambda_i + (K-1)\frac{W_j(t)}{j}\sum_{i=j+1}^{n}\lambda_i\right] + \lambda\frac{W_j(t)}{j}$$

Define $w^j(.)$ such that $w^j(0) = 0$ and

$$\frac{d}{dt}w^j(t) = -\frac{1}{j}\left[\sum_{i=1}^{j}\lambda_i + (K-1)w^j(t)\sum_{i=j+1}^{n}\lambda_i\right] + \lambda w^j(t)$$

By construction, we have $w^j(t) \geq \frac{W_j(t)}{j} \geq w_1(t)$, where the latter inequality is due to Proposition 1.

Suppose first that $\lambda_1 < \lambda_K$. Let $t^1 = \max_{t<0}\{t|w_i(t) = w_{i'}(t)\ \forall\ i, i' \in \{1, ..., K\}\}$ be the latest time before the deadline at which the K lowest continuation values are equal. By Lemma 1 and the continuity of continuation values, t^1 is well-defined. Define j^1 such that $w_i(t^1) = w_{i'}(t^1)\ \forall\ i, i' \in \{1, ..., j^1\}$ and, if $j^1 < n$, $w_{j^1}(t^1) < w_{j^1+1}(t^1)$. When $w_1(t) = ... = w_{j^1}(t)$, define $w^*(t)$ as their common value. Since $j^1 \geq K$, every player, if recognized at t^1, approaches $K-1$ other players in $\{1, ..., j^1\}$ and offers $w^*(t^1)$ to each of them.

Consider the auxiliary continuation payoff path w^a defined by

$$\frac{d}{dt}w^a(t) = -\frac{1}{j^1}\left[\sum_{i=1}^{j^1}\lambda_i + (K-1)w^a(t)\sum_{i=j^1+1}^{n}\lambda_i\right] + \lambda w^a(t)$$

with terminal condition $w^a(t^1) = w^*(t^1)$. It corresponds to the payoff path of players $1, ..., j^1$ resulting from all players at any time approaching players in $\{1, ..., j^1\}$ in a way that keeps the payoff of all players in $\{1, ..., j^1\}$ the same. This path is not necessarily feasible: in order to keep payoffs within $\{1, ..., j^1\}$ the same, player 1 may need to be approached "with probability above 1" (*i.e.* have expected payoff exceeding continuation value when another player is recognized), and/or player j^1 may need to be approached "with negative probability" (*i.e.* have negative expected payoff when another player is recognized). That is, along the auxiliary path, we impose that, upon player i being recognized at any time, the sum of the other players' probabilities of being approached by i must be $K-1$, but ignore the constraint that these probabilities must be between 0 and 1.

Lemma 2: For $t' < t^1$, if $w_{j^1}(\tau) < w_{j^1+1}(\tau)$ and $w^a(\tau) \geq w^a(t^1)$ for all $\tau \in (t', t^1)$, then $w_1(t') = ... = w_{j^1}(t') = w^a(t')$.

Proof: When $w_1(t) = ... = w_{j^1}(t) = w^*(t)$, we have

$$w_i'(t) - w_j'(t) = (\lambda_j - \lambda_i)(1 - Kw^*(t)) - \lambda(p_i(t) - p_j(t))w^*(t)$$

for all $i, j \in \{1, ..., j^1\}$, where $p_i(t)$ is defined in the proof of Proposition 1. Therefore, $w_i'(t) \leq w_j'(t)$ is equivalent to

$$\begin{aligned} (\lambda_j - \lambda_i)(1 - Kw^*(t)) &\leq \lambda(p_i(t) - p_j(t))w^*(t) \\ \frac{\lambda_j - \lambda_i}{\lambda}(\frac{1}{w^*(t)} - K) &\leq p_i(t) - p_j(t) \end{aligned}$$

Suppose $\lambda_j > \lambda_i$. Since $j^1 \geq K$, we have $w^*(t) < \frac{1}{K}$. Thus, the left hand side is positive, which implies that $w_i'(t^1) = w_j'(t^1)$ requires a smaller $p_i(t^1) - p_j(t^1) > 0$ than is required for $w_i'(t^1) \leq w_j'(t^1)$. As a result, if $w_1'(t^1) \leq ... \leq w_{j^1}'(t^1)$ is feasible, then $w_1'(t^1) = ... = w_{j^1}'(t^1)$ is also feasible: the required probabilities for the latter are a contraction of the probabilities achieving the former about their mean. Additionally, $w_1'(t) = ... = w_{j^1}'(t)$ is also feasible whenever $w^*(t^1) \leq w^*(t)$ and $w^*(t) = w_{j^1}(t) < w_{j^1+1}(t)$: the required $p_i(t) - p_j(t)$ is smaller than the required $p_i(t^1) - p_j(t^1)$.

By Proposition 1, $w_1(t) \leq ... \leq w_{j^1}(t)$ for all t, in particular just to the right of t^1. Moreover, $w_{j^1+1}(t^1) > w_{j^1}(t^1)$, so $w_{j^1+1}(t) > w_{j^1}(t)$ and thus $\sum_{i=1}^{j^1} p_i(t^1) = \sum_{i=1}^{j^1} p_i(t)$ for t just to the right of t^1. Since $w_1(t^1) = ... = w_{j^1}(t^1)$, by the continuity of w_i' with respect to $(w_1, ..., w_n)$ and of w_i with respect to t, it must feasible to have $w_1'(t^1) \leq ... \leq w_{j^1}'(t^1)$. Combining this observation with the previous paragraph yields the desired result. QED

Observe that $\frac{dw^a(t)}{dt} \lesseqgtr 0$ if $w^a(t) \lesseqgtr \frac{\sum_{i \leq j^1} \lambda_i}{j^1\lambda - (K-1)\sum_{i > j^1} \lambda_i} > 0$; the same holds for $w^{j^1}(t)$.

Therefore, $\lim_{t\to-\infty} w^a(t) = \lim_{t\to-\infty} w^{j^1}(t) = \frac{\sum_{i\leq j^1}\lambda_i}{j^1\lambda-(K-1)\sum_{i>j^1}\lambda_i} = y_{j^1}$, where y_{j^1} is defined in the main text (recall the normalization $\lambda = 1$). Since $w^{j^1}(0) = 0$, $w^{j^1}(t)$ is monotonically decreasing on $(-\infty, 0)$. Thus we have $w^a(t^1) = w_1(t^1) \leq w^{j^1}(t^1) < y_{j^1}$, which implies that $w^a(t)$ is also monotonically decreasing on $(-\infty, t^1)$. Therefore, by Lemma 2, $w_1(t) = ... = w_{j^1}(t) = w^a(t)$ as long as $w_{j^1}(\tau) < w_{j^1+1}(\tau)$ for all $\tau \in (t, t^1)$.

If instead $\lambda_1 = \lambda_i < \lambda_{i+1}$, where $i \geq K$, let $j^1 = i$ and $t^1 = 0$ when defining w^a. Since $w^a(0) = 0$, $\frac{dw^a(t)}{dt} < 0$ for all t, so once again, $w_1(t) = ... = w_{j^1}(t) = w^a(t)$ as long as $w_{j^1}(\tau) < w_{j^1+1}(\tau)$ for all $\tau \in (t, 0)$.

Case 1: $w_1(t) = ... = w_{j^1}(t) < w_{j^1+1}(t)\ \forall\, t < t^1$. Then the above arguments imply $w_1(t) = ... = w_{j^1}(t) = w^a(t)\ \forall\, t < t^1$, which means $\lim_{t\to-\infty} w_i(t) = \lim_{t\to-\infty} w^a(t) = y_{j^1}$ for all $i \in \{1, ..., j^1\}$. For $i \in \{j^1+1, ..., n\}$, since these players are never approached for $t < t^1$ and receive a limit payoff of $1-(K-1)y_{j^1}$ when proposing, standard convergence arguments show that $\lim_{t\to-\infty} w_i(t) = \lambda_i(1-(K-1)y_{j^1})$. It follows that $\lim_{t\to-\infty} w(t) = x^{j^1}$, where x^{j^1} is defined in the main text.

Case 2: At some time $t < t^1$, $w_1(t) = ... = w_{j^1}(t) = w_{j^1+1}(t)$. Let $t^{k+1} = \max_{t<t^k}\{t|w_{j^k}(t) = w_{j^k+1}(t)\}$ and $j^{k+1} = \max_{i\in N}\{i|w_i(t^{k+1}) = w_1(t^{k+1})\}$. Arguments analogous to the above establish that if $k^* \in \mathbb{N}$ is such that either $w_1(t) = ... = w_{j^{k^*}}(t) < w_{j^{k^*}+1}(t)\ \forall t < t^{k^*}$ or $j^{k^*} = n$, then $\lim_{t\to-\infty} w(t) = x^{j^{k^*}}$. Since $j^{k^*} > ... > j^1 \geq K$ and n is finite, such k^* exists.

It remains to be shown that $x^{j^{k^*}} = x^*$. By Proposition 1, $x^{j^{k^*}}_{j^{k^*}+1} \geq x^{j^{k^*}}_{j^{k^*}}$, so the definition of j^* implies $j^{k^*} \geq j^*$. Now assume $j^{k^*} > j^*$; we show below that $y_{j^{k^*}} = y_{j^*}$, which implies the desired result.

Taking the difference of $jy_j + (1-(K-1)y_j)\sum_{i=j+1}^n \lambda_i = 1$ and $(j-1)y_{j-1} + (1-(K-1)y_{j-1})\sum_{i=j}^n \lambda_i = 1$ gives

$$\left(j-(K-1)\sum_{i=j+1}^{n}\lambda_i\right)(y_j - y_{j-1}) = (1-(K-1)y_{j-1})\lambda_j - y_{j-1} \qquad (2)$$

Suppose $j = j^*+1$. Then by the definition of j^*, the right hand side of (2) is nonnegative. Since $j-(K-1)\sum_{i=j+1}^n \lambda_i > 0$, we have $y_j - y_{j-1} \geq 0$.

Subtracting $(1 + (K-1)\lambda_j)(y_j - y_{j-1})$ from both sides of (2) gives

$$\left((j-1) - (K-1)\sum_{i=j}^{n}\lambda_i\right)(y_j - y_{j-1}) = (1-(K-1)y_j)\lambda_j - y_j$$

Since $(j-1) - (K-1)\sum_{i=j}^{n}\lambda_i > 0$ for $j \geq K$, if $y_j - y_{j-1} \geq 0$, then $(1-(K-1)y_j)\lambda_j \geq y_j$.

Since $1 \geq (K-1)y_j$ for $j \geq K$ and $\lambda_{j+1} \geq \lambda_j$, we have $(1-(K-1)y_j)\lambda_{j+1} \geq y_j$. As a result, an inductive argument applies, and $(1-(K-1)y_j)\lambda_j \geq y_j$ for all $j \geq j^*+1$.

Since $j^{k^*} > j^*$, it follows that $(1-(K-1)y_{j^{k^*}})\lambda_{j^{k^*}} \geq y_{j^{k^*}}$. At the same time, player j^{k^*} receives $(1-(K-1)y_{j^{k^*}})$ in the limit when proposing, so her limit payoff $y_{j^{k^*}}$ is bounded below by $\lambda_{j^{k^*}}(1-(K-1)y_{j^{k^*}})$. Thus, $(1-(K-1)y_{j^{k^*}})\lambda_{j^{k^*}} = y_{j^{k^*}}$. The equations above imply that if $y_j > y_{j-1}$ for any $j \geq j^*+1$, we would have $(1-(K-1)y_{j^{k^*}})\lambda_{j^{k^*}} > y_{j^{k^*}}$, which is impossible. Thus, $y_{j^{k^*}} = y_{j^*}$, as desired. QED

Proof of Theorem 2: The proof of Theorem 2 parallels the proof of Theorem 1, except that in certain places, λ is replaced by $\lambda + r$. The changes are presented below. Lemma 3 generalizes Lemma 1 for small r.

Lemma 3: If $r < \lambda_1(1 - \frac{K}{n})$, there exists a time $t' \leq 0$ such that $w_i(t') = w_{i'}(t')$ $\forall\ i, i' \in \{1, ..., K\}$.

Proof: The proof is the same as for Lemma 1, except with $w_1'(t) \leq -\lambda_1(1-\frac{K}{n}) + rw_1(t) \leq -\lambda_1(1-\frac{K}{n})(1-w_1(t)) < -\lambda_1(1-\frac{K}{n})(1-\frac{1}{n}) < 0$. QED

Define $w^j(.)$ such that $w^j(0) = 0$ and

$$\frac{d}{dt}w^j(t) = -\frac{1}{j}\left[\sum_{i=1}^{j}\lambda_i + (K-1)w^j(t)\sum_{i=j+1}^{n}\lambda_i\right] + (\lambda + r)w^j(t)$$

Similar manipulations as in the proof of Theorem 1 show that $w^j(t) \geq \frac{W_j(t)}{j} \geq w_1(t)$.

Suppose first that $\lambda_1 < \lambda_K$. Define t^1, j^1 and $w^*(t)$ as in the proof for Theorem

1, and let w^a be such that

$$\frac{d}{dt}w^a(t) = -\frac{1}{j^1}\left[\sum_{i=1}^{j^1}\lambda_i + (K-1)w^a(t)\sum_{i=j^1+1}^{n}\lambda_i\right] + (\lambda + r)w^a(t)$$

with terminal condition $w^a(t^1) = w^*(t^1)$.

Once again, w^a corresponds to the payoff path of players $1, ..., j^1$ resulting from all players at any time approaching players in $\{1, ..., j^1\}$ in a way that keeps the payoff of all players in $\{1, ..., j^1\}$ the same. The same argument as in the proof of Theorem 1 shows that, for $t < t^1$, as long as $w^a(\tau) \geq w^a(t^1)$ for all $\tau \in (t, t^1)$, we have $w_1(t) = ... = w_{j^1}(t) = w^a(t)$. In particular, $w^a(t)$ is monotonically decreasing on $(-\infty, t^1)$ and now converges to $\frac{\sum_{i \leq j^1}\lambda_i}{j^1(\lambda+r)-(K-1)\sum_{i>j^1}\lambda_i}$ as $t \to -\infty$.

The case $\lambda_1 = \lambda_i < \lambda_{i+1}$, where $i \geq K$, is treated in the same way as for Theorem 1.

Define $y_{j,r}$ as the solution to $jy_{j,r} + \frac{1-(K-1)y_{j,r}}{\lambda+r}\sum_{i=j+1}^{n}\lambda_i = \frac{\lambda}{\lambda+r}$, and let $x_i^{j,r} = y_{j,r}$ for $i \leq j$, and $x_i^{j,r} = \frac{1-(K-1)y_{j,r}}{\lambda+r}\lambda_i$ for $i > j$. Also let j_r^* be the smallest $j \in \{1, ..., n\}$ such that $x_{j+1}^{j,r} \geq x_j^{j,r} \geq 0$ (assume this holds trivially for $j = n$).

Case 1: $w_1(t) = ... = w_{j^1}(t) < w_{j^1+1}(t)$, $\forall\ t < t^1$. By the same argument as in the proof for Theorem 1, $\lim_{t\to-\infty} w_i(t) = \frac{\sum_{i\leq j^1}\lambda_i}{j^1(\lambda+r)-(K-1)\sum_{i>j^1}\lambda_i} = x_i^{j^1,r}$ for $i \in \{1, ..., j^1\}$. For $i \in \{j^1+1, ..., n\}$, standard convergence arguments show that $\lim_{t\to-\infty} w_i(t) = x_i^{j^1,r}$ as well.

Case 2: At some time $t < t^1$, $w_1(t) = ... = w_{j^1}(t) = w_{j^1+1}(t)$. The argument from the proof for Theorem 1 carries through, so $\lim_{t\to-\infty} w_i(t) = x_i^{j^{k^*},r}$ for some $j^{k^*} \geq j^1$.

Manipulations analogous to the rest of the proof of Theorem 1 show that $x_i^{j^{k^*},r} = x_i^{j_r^*,r}$.

It remains to be shown that $x_i^{j_r^*,r} \to x_i^*$ as $r \to 0$ (under the normalization $\lambda = 1$). The equations defining $x_i^{j,r}$ and x_i^j are linear and continuous in r, so $x_i^{j,r} \to x_i^j$ as

$r \to 0$, for all i, j. Since, by definition, $x^j_{j+1} < x^j_j$ for all $j < j^*$, continuity implies $j^*_r \geq j^*$. As a result, either $j^*_r = j^*$ for sufficiently low r, in which case we are done, or $x^{j^*}_{j^*+1} = x^{j^*}_{j^*}$, in which case it is possible to have $x^{j^*,r}_{j^*+1} > x^{j^*,r}_{j^*}$ for arbitrarily low r, and thus $j^*_r > j^*$. The remainder of the proof considers the latter case.

If $x^{j^*}_j = x^{j^*}_{j^*}$ for some $j > j^*$, straightforward algebra shows that $x^j = x^{j^*}$. Therefore, it is sufficient to show that $x^{j^*}_{j^*_r} = x^{j^*}_{j^*}$, which implies $x^{j^*_r} = x^{j^*}$. Suppose instead $x^{j^*}_{j^*_r} > x^{j^*}_{j^*}$. Then there exists $j \in \{j^*, ..., j^*_r - 1\}$ such that $x^{j^*}_{j+1} > x^{j^*}_j = x^{j^*}_{j^*}$. Then, since $x^j = x^{j^*}$, we have $x^j_{j+1} > x^j_j$. By continuity, this implies $j^*_r \leq j$, which contradicts $j < j^*_r$. QED

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