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ON SUMS OF THREE SQUARES

S.K.K. CHOI, A.V. KUMCHEV, AND R. OSBURN

Abstract. Let $r_3(n)$ be the number of representations of a positive integer n as a sum of three squares of integers. We give two distinct alternative proofs of a conjecture of Wagon concerning the asymptotic value of the mean square of $r_3(n)$.

1. Introduction

Problems concerning sums of three squares have a rich history. It is a classical result of Gauss that

$$n = x_1^2 + x_2^2 + x_3^2$$

has a solution in integers if and only if n is not of the form $4^a(8k+7)$ with $a, k \in \mathbb{Z}$. Let $r_3(n)$ be the number of representations of n as a sum of three squares (counting signs and order). It was conjectured by Hardy and proved by Bateman [1] that

$$r_3(n) = 4\pi n^{1/2}\mathfrak{S}_3(n), \tag{1}$$

where the singular series $\mathfrak{S}_3(n)$ is given by (16) with $Q = \infty$.

While in principle this exact formula can be used to answer almost any question concerning $r_3(n)$, the ensuing calculations can be tricky because of the slow convergence of the singular series $\mathfrak{S}_3(n)$. Thus, one often sidesteps (1) and attacks problems involving $r_3(n)$ directly. For example, concerning the mean value of $r_3(n)$, one can adapt the method of solution of the circle problem to obtain the following

$$\sum_{n\le x} r_3(n) \sim \frac{4}{3}\pi x^{3/2}.$$

Moreover, such a direct approach enables us to bound the error term in this asymptotic formula. An application of a result of Landau [9, pp. 200–218] yields

$$\sum_{n\le x} r_3(n) = \frac{4}{3}\pi x^{3/2} + O\left(x^{3/4+\epsilon}\right)$$

for all $\epsilon > 0$, and subsequent improvements on the error term have been obtained by Vinogradov [19], Chamizo and Iwaniec [3], and Heath-Brown [6].

In this note we consider the mean square of $r_3(n)$. The following asymptotic formula was conjectured by Wagon and proved by Crandall; see [4] (see also [2]) .

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Theorem. *Let $r_3(n)$ be the number of representations of a positive integer n as a sum of three squares of integers. Then*

$$\sum_{n\le x} r_3(n)^2 \sim \frac{8\pi^4}{21\zeta(3)} x^2. \tag{2}$$

Apparently, at the time they proposed this conjecture Crandall and Wagon were unaware of the earlier work of Müller [11, 12]. He obtained a more general result which, in a special case, gives

$$\sum_{n\le x} r_3(n)^2 = Bx^2 + O\big(x^{14/9}\big),$$

where B is a constant. However, since in Müller's work B arises as a specialization of a more general (and more complicated) quantity, it is not immediately clear that $B = \frac{8}{21}\pi^4/\zeta(3)$. The purpose of this paper is to give two distinct proofs of this fact: one that evaluates B in the form given by Müller and a direct proof using the Hardy–Littlewood circle method.

2. A direct proof: the circle method

Our first proof exploits the observation that the left side of (2) counts solutions of the equation

$$m_1^2 + m_2^2 + m_3^2 = m_4^2 + m_5^2 + m_6^2$$

in integers $m_1, \dots, m_6$ with $|m_j| \le x$. This is exactly the kind of problem that the circle method was designed for.

Set $N = \sqrt{x}$ and define

$$f(\alpha) = \sum_{m\le N} e\big(\alpha m^2\big),$$

where $e(z) = e^{2\pi i z}$. Then for an integer $n \le x$, the number $r^*(n)$ of representations of n as a sum of three squares of *positive* integers is

$$r^*(n) = \int_0^1 f(\alpha)^3 e(-\alpha n)\, d\alpha.$$

Since $r_3(n) = 8r^*(n) + O(r_2(n))$, where $r_2(n)$ is the number of representations of n as a sum of two squares, we have

$$\sum_{n\le x} r_3(n)^2 = 64 \sum_{n\le x} r^*(n)^2 + O\big(x^{3/2+\epsilon}\big). \tag{3}$$

Therefore, it suffices to evaluate the mean square of $r^*(n)$. Let

$$P = N/4 \qquad \text{and} \qquad Q = N^{1/2}.$$

We introduce the sets

$$\mathfrak{M}(q,a) = \big\{\alpha \in \big[PN^{-2}, 1 + PN^{-2}\big] : |q\alpha - a| \le PN^{-2}\big\}$$

and

$$\mathfrak{M} = \bigcup_{q\le Q} \bigcup_{\substack{1\le a\le q \\ (a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = \big[PN^{-2}, 1 + PN^{-2}\big] \setminus \mathfrak{M}.$$

We have

$$(4)\qquad r^*(n) = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) f(\alpha)^3 e(-\alpha n)\,d\alpha$$
$$= r^*(n,\mathfrak{M}) + r^*(n,\mathfrak{m}), \qquad \text{say.}$$

We now proceed to approximate the mean square of $r^*(n)$ by that of $r^*(n,\mathfrak{M})$. By (4) and Cauchy's inequality,

$$(5)\qquad \sum_{n\le x} r^*(n)^2 = \sum_{n\le x} r^*(n,\mathfrak{M})^2 + O\Big((\Sigma_1\Sigma_2)^{1/2} + \Sigma_2\Big),$$

where

$$\Sigma_1 = \sum_{n\le x} |r^*(n,\mathfrak{M})|^2, \qquad \Sigma_2 = \sum_{n\le x} |r^*(n,\mathfrak{m})|^2.$$

By Bessel's inequality,

$$(6)\qquad |\Sigma_2| = \sum_{n\le x} \left| \int_{\mathfrak{m}} f(\alpha)^3 e(-\alpha n)\,d\alpha \right|^2 \le \int_{\mathfrak{m}} |f(\alpha)|^6\,d\alpha.$$

By Dirichlet's theorem of diophantine approximation, we can write any real α as $\alpha = a/q + \beta$, where

$$1 \le q \le N^2P^{-1}, \qquad (a,q) = 1, \qquad |\beta| \le P/(qN^2).$$

When $\alpha \in \mathfrak{m}$, we have $q \ge Q$, and hence Weyl's inequality (see Vaughan [18, Lemma 2.4]) yields

$$(7)\qquad |f(\alpha)| \ll N^{1+\epsilon}\Big(q^{-1} + N^{-1} + qN^{-2}\Big)^{1/2} \ll N^{1+\epsilon}Q^{-1/2}.$$

Furthermore, we have

$$(8)\qquad \int_0^1 |f(\alpha)|^4\,d\alpha \ll N^{2+\epsilon},$$

because the integral on the right equals the number of solutions of

$$m_1^2 + m_2^2 = m_3^2 + m_4^2$$

in integers $m_1,\dots,m_4 \le N$. For each choice of m_1 and m_2, this equation has $\ll N^\epsilon$ solutions. Combining (6)–(8) and replacing ϵ by $\epsilon/3$, we obtain

$$(9)\qquad \Sigma_2 \ll N^{4+\epsilon}Q^{-1}.$$

Furthermore, another appeal to Bessel's inequality and appeals to (8) and to the trivial estimate $|f(\alpha)| \le N$ yield

$$(10)\qquad \Sigma_1 \le \int_{\mathfrak{M}} |f(\alpha)|^6\,d\alpha \le \int_0^1 |f(\alpha)|^6\,d\alpha \ll N^{4+\epsilon}.$$

We now define a function f^* on $\mathfrak{M}$ by setting

$$f^*(\alpha) = q^{-1}S(q,a)v(\alpha - a/q) \qquad \text{for } \alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M};$$

here

$$S(q,a) = \sum_{1\le h\le q} e\big(ah^2/q\big), \qquad v(\beta) = \frac{1}{2}\sum_{m\le x} m^{-1/2}e(\beta m).$$

Our next goal is to approximate the mean square of $r^*(n, \mathfrak{M})$ by the mean square of the integral

$$R^*(n) = \int_{\mathfrak{M}} f^*(\alpha)^3 e(-\alpha n)\, d\alpha.$$

Similarly to (5),

$$\sum_{n \le x} r^*(n, \mathfrak{M})^2 = \sum_{n \le x} R^*(n)^2 + O\big(\Sigma_3 + (\Sigma_1 \Sigma_3)^{1/2}\big), \tag{11}$$

where

$$\Sigma_3 = \sum_{n \le x} \left| \int_{\mathfrak{M}} \left[f(\alpha)^3 - f^*(\alpha)^3 \right] e(-\alpha n)\, d\alpha \right|^2 \le \int_{\mathfrak{M}} \left| f(\alpha)^3 - f^*(\alpha)^3 \right|^2 d\alpha, \tag{12}$$

after yet another appeal to Bessel's inequality. By [18, Theorem 4.1], when $\alpha \in \mathfrak{M}(q, a)$,

$$f(\alpha) = f^*(\alpha) + O\big(q^{1/2+\epsilon}\big).$$

Thus,

$$\int_{\mathfrak{M}(q,a)} \left| f(\alpha)^3 - f^*(\alpha)^3 \right|^2 d\alpha \ll q^{1+2\epsilon} \int_{\mathfrak{M}(q,a)} \big(|f(\alpha)|^4 + q^{2+4\epsilon} \big)\, d\alpha,$$

whence

$$\int_{\mathfrak{M}} \left| f(\alpha)^3 - f^*(\alpha)^3 \right|^2 d\alpha \ll Q^{1+2\epsilon} \int_0^1 |f(\alpha)|^4\, d\alpha + PQ^{4+6\epsilon} N^{-2}.$$

Bounding the last integral using (8) and substituting the ensuing estimate into (12), we obtain

$$\Sigma_3 \ll QN^{2+2\epsilon} + PQ^4 N^{-2+3\epsilon} \ll QN^{2+2\epsilon}. \tag{13}$$

Combining (5), (9)–(11), and (13), we deduce that

$$\sum_{n \le x} r^*(n)^2 = \sum_{n \le x} R^*(n)^2 + O\big(N^{4+\epsilon} Q^{-1/2} + N^{3+\epsilon} Q^{1/2}\big). \tag{14}$$

We now proceed to evaluate the main term in (14). We have

$$\int_{\mathfrak{M}(q,a)} f^*(\alpha)^3 e(-\alpha n)\, d\alpha = q^{-3} S(q, a)^3 e(-an/q) \int_{\mathfrak{M}(q,0)} v(\beta)^3 e(-\beta n)\, d\beta,$$

so

$$R^*(n) = \sum_{q \le Q} A(q, n) I(q, n),$$

where

$$A(q, n) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} q^{-3} S(q, a)^3 e(-an/q), \qquad I(q, n) = \int_{\mathfrak{M}(q,0)} v(\beta)^3 e(-\beta n)\, d\beta.$$

Hence,

$$\sum_{n \le x} R^*(n)^2 = \sum_{n \le x} I(n)^2 \mathfrak{S}_3(n, Q)^2 + O\big((\Sigma_4 \Sigma_5)^{1/2} + \Sigma_5\big), \tag{15}$$

where

$$\mathfrak{S}_3(n,Q) = \sum_{q\le Q} A(q,n), \qquad I(n) = \int_{-1/2}^{1/2} v(\beta)^3 e(-\beta n)\, d\beta, \tag{16}$$

$$\Sigma_4 = \sum_{n\le x} I(n)^2 \Big(\sum_{q\le Q} |A(q,n)| \Big)^2, \qquad \Sigma_5 = \sum_{n\le x} \Big(\sum_{q\le Q} |A(q,n)(I(n) - I(q,n))| \Big)^2.$$

By [18, Theorem 2.3] and [18, Theorem 4.2],

$$I(n) = \Gamma(3/2)^2 \sqrt{n} + O(1) = \frac{\pi}{4}\sqrt{n} + O(1), \qquad A(q,n) \ll q^{-1/2}. \tag{17}$$

Furthermore, since $A(q,n)$ is multiplicative in q, [18, Lemma 4.7] yields

$$\begin{aligned} \sum_{q\le Q} |A(q,n)| &\le \prod_{p\le Q} \big(1 + |A(p,n)| + |A(p^2,n)| + \cdots\big) \\ &\ll \prod_{p\le Q} \big(1 + c_1(p,n)p^{-3/2} + 3c_1 p^{-1}\big) \ll (nQ)^{\epsilon}, \end{aligned} \tag{18}$$

where $c_1 > 0$ is an absolute constant. In particular, we have

$$\Sigma_4 \ll N^{4+\epsilon}. \tag{19}$$

We now turn to the estimation of Σ_5. By Cauchy's inequality and the second bound in (17),

$$\Sigma_5 \ll (\log Q) \sum_{n\le x} \sum_{q\le Q} |I(n) - I(n,q)|^2$$

Another application of Bessel's inequality gives

$$\sum_{n\le x} |I(n) - I(n,q)|^2 \le \int_{P/qN^2}^{1/2} |v(\beta)|^6\, d\beta.$$

Using [18, Lemma 2.8] to estimate the last integral, we deduce that

$$\Sigma_5 \ll \log Q \sum_{q\le Q} \big(q^2 N^4 P^{-2} + 1\big) \ll N^2 Q^{3+\epsilon}.$$

Substituting this inequality and (19) into (15), we conclude that

$$\sum_{n\le x} R^*(n)^2 = \sum_{n\le x} I(n)^2 \mathfrak{S}_3(n,Q)^2 + O\big(N^{3+\epsilon} Q^{3/2}\big). \tag{20}$$

We then use (17) and (18) to replace $I(n)$ on the right side of (20) by $\frac{\pi}{4}\sqrt{n}$. We get

$$\sum_{n\le x} I(n)^2 \mathfrak{S}_3(n,Q)^2 = \frac{\pi^2}{16} \sum_{n\le x} n\mathfrak{S}_3(n,Q)^2 + O\big(N^{3+\epsilon}\big).$$

Together with (14) and (20), this leads to the asymptotic formula

$$\sum_{n\le x} r^*(n)^2 = \frac{\pi^2}{16} \sum_{n\le x} n\mathfrak{S}_3(n,Q)^2 + O\big(N^{4+\epsilon} Q^{-1/2} + N^{3+\epsilon} Q^{3/2}\big). \tag{21}$$

Finally, we evaluate the sum on the right side of (21). On observing that $\mathfrak{S}_3(n, Q)$ is in fact a real number, we have

$$\sum_{n\le t} \mathfrak{S}_3(n, Q)^2 = \sum_{q_1,q_2\le Q} \sum_{\substack{1\le a_1\le q_1\\(a_1,q_1)=1}} \sum_{\substack{1\le a_2\le q_2\\(a_2,q_2)=1}} (q_1q_2)^{-3} S(q_1, a_1)^3 S(q_2, -a_2)^3 \sum_{n\le t} e\big((a_1/q_1 - a_2/q_2)n\big).$$

As the sum over n equals $t + O(1)$ when $a_1 = a_2$ and $q_1 = q_2$ and $O(q_1q_2)$ otherwise, we get

$$\sum_{n\le t} \mathfrak{S}_3(n, Q)^2 = t \sum_{q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} q^{-6}|S(q, a)|^6 + O\big(\Sigma_6^2\big),$$

where

$$\Sigma_6 = \sum_{q\le Q} \sum_{\substack{1\le a\le q\\(a,q)=1}} q^{-2}|S(q, a)|^3 \ll Q^{3/2}.$$

We find that

$$\sum_{n\le t} \mathfrak{S}_3(n, Q)^2 = B_1t + O\big(tQ^{-1} + Q^3\big),$$

with

$$B_1 = \sum_{q=1}^{\infty} \sum_{\substack{1\le a\le q\\(a,q)=1}} q^{-6}|S(q, a)|^6.$$

Thus, by partial summation,

$$\sum_{n\le x} n\mathfrak{S}_3(n, Q)^2 = (B_1/2)x^2 + O\big(x^2Q^{-1} + xQ^3\big).$$

Combining this asymptotic formula with (21), we deduce that

$$\sum_{n\le x} r^*(n)^2 = \frac{\pi^2}{32} B_1x^2 + O\big(x^{15/8+\epsilon}\big).$$

Recalling (3), we see that (2) will follow if we show that

$$B_1 = \frac{8\zeta(2)}{7\zeta(3)}.$$

This, however, follows easily from the well-known formula (see [7, §7.5])

$$|S(q, a)| = \begin{cases} \sqrt{q} & \text{if } q \equiv 1 \pmod 2, \\ \sqrt{2q} & \text{if } q \equiv 0 \pmod 4, \\ 0 & \text{if } q \equiv 2 \pmod 4. \end{cases} \tag{22}$$

Indeed, (22) yields

$$B_1 = \frac{4}{3} \sum_{q \text{ odd}} q^{-3}\phi(q) = \frac{8\zeta(2)}{7\zeta(3)},$$

where the last step uses the Euler product of $\zeta(s)$. This completes the proof of our theorem.

3. Second Proof of Theorem

Rankin [13] and Selberg [17] independently introduced an important method which allows one to study the analytic behavior of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where $a(n)$ are Fourier coefficients of a holomorphic cusp form for some congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$. Originally the method was for holomorphic cusp forms. Zagier [20] extended the method to cover forms that are not cuspidal and may not decay rapidly at infinity. Müller [11, 12] considered the case where $a(n)$ is the Fourier coefficient of non-holomorphic cusp or non-cusp form of real weight with respect to a Fuchsian group of the first kind. It is this last approach we wish to discuss. Note that if we apply a Tauberian theorem to the above Dirichlet series, we then gain information on the asymptotic behavior of the partial sum

$$\sum_{n \le x} a(n).$$

We now discuss Müller's elegant work. For details regarding discontinuous groups and automorphic forms, see [8, 10, 11, 14, 15, 16]. Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half plane and $G = SL(2, \mathbb{R})$ the special linear group of all 2×2 matrices with determinant 1. G acts on $\mathbb{H}$ by

$$z \mapsto gz = \frac{az + b}{cz + d}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We write $y = y(z) = \Im(z)$. Thus we have

$$y(gz) = \frac{y}{|cz + d|^2}.$$

Let $dx\,dy$ denote the Lebesgue measure in the plane. Then the measure

$$d\mu = \frac{dx\,dy}{y^2}$$

is invariant under the action of G on $\mathbb{H}$. A discrete subgroup Γ of G is called a *Fuchsian group of the first kind* if its fundamental domain $\Gamma\backslash\mathbb{H}$ has finite volume. Let Γ be a Fuchsian group of the first kind containing $\pm I$ where I is the identity matrix. Let $\mathcal{F}(\Gamma, \chi, k, \lambda)$ denote the space of (non-holomorphic) automorphic forms of real weight k, eigenvalue $\lambda = \frac{1}{4} - \rho^2$, $\Re(\rho) \geq 0$, and multiplier system χ. For $k \in \mathbb{R}$, $g \in SL(2, \mathbb{R})$ and $f : \mathbb{H} \to \mathbb{C}$, we define the stroke operator $|_k$ by

$$(f|_k g)(z) := \left(\frac{cz + d}{|cz + d|} \right)^{-k} f(gz)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The transformation law for $f \in \mathcal{F}(\Gamma, \chi, k, \lambda)$ is then

$$(f|_k g)(z) = \chi(g) f(z)$$

for all $g \in \Gamma$. Automorphic forms $f \in \mathcal{F}(\Gamma, \chi, k, \lambda)$ have a Fourier expansion at every cusp κ of Γ, namely

$$A_{\kappa,0}(y) + \sum_{n \neq 0} a_{\kappa,n} W_{(sgn\, n)\frac{k}{2},\rho}(4\pi|n + \mu_\kappa|y) e((n + \mu_k)x),$$

where μ_κ is the cusp parameter and $a_{\kappa,n}$ are the Fourier coefficients of f at κ. The functions $W_{\alpha,\rho}$ are Whittaker functions (see [11, §3]), $A_{\kappa,0}(y) = 0$ if $\mu_k \neq 0$ and

$$A_{\kappa,0}(y) = \begin{cases} a_{\kappa,0}y^{1/2+\rho} + b_{\kappa,0}y^{1/2-\rho} & \text{if } \mu_\kappa = 0, \rho \neq 0, \\ a_{\kappa,0}y^{1/2} + b_{\kappa,0}y^{1/2}\log y & \text{if } \mu_\kappa = 0, \rho = 0. \end{cases}$$

An automorphic form f is called a cusp form if $a_{\kappa,0} = b_{\kappa,0} = 0$ for all cusps κ of Γ. Now consider the Dirichlet series

$$S_\kappa(f, s) = \sum_{n>0} \frac{|a_{\kappa,n}|^2}{(n + \mu_k)^s}.$$

This series is absolutely convergent for $\Re(s) > 2\Re(\rho)$ and has been shown [12] to have meromorphic continuation in the entire complex plane. In what follows, we will only be interested in the case f is not a cusp form. If f is not a cusp form and $\Re(\rho) > 0$, then $S_\kappa(f, s)$ has a simple pole at $s = 2\Re(\rho)$ with residue

$$\beta_\kappa(f) = \operatorname*{res}_{s=2\Re(\rho)} S_\kappa(f, s) = (4\pi)^{2\Re(\rho)} b^+(k/2, \rho) \sum_{\iota\in K} \varphi_{\kappa,\iota}(1 + 2\Re(\rho))|a_{\iota,0}|^2, \tag{23}$$

where K denotes a complete set of Γ-inequivalent cusps, $\varphi_{\kappa,\iota}(1 + 2\Re(\rho)) > 0$ and $b^+(\frac{k}{2}, \rho) > 0$ if $\rho + \frac{1}{2} \pm \frac{k}{2}$ is a non-negative integer. For the definition of the functions $\varphi_{\kappa,\iota}$ and b^+, see Lemma 3.6 and (69) in [12]. This result (23) and a Tauberian argument then provide the asymptotic behaviour of the summatory function

$$\sum_{n\leq x} |a_{\kappa,n}|^2 |n + \mu_\kappa|^r.$$

Precisely, we have (see [11, Theorem 2.1] or [12, Theorem 5.2]) that

$$\sum_{n\leq x} |a_{\kappa,n}|^2 |n + \mu_\kappa|^r = \sum_{z\in R} \operatorname*{res}_{s=z} S_\kappa(f, s) \frac{x^{r+s}}{r + s} + O(x^{r+2\Re\rho-\gamma}(\log x)^g), \tag{24}$$

where $2\Re(\rho)+r \geq 0$, $R = \{\pm 2\Re(\rho), \pm 2i\Im(\rho), 0, -r\}$, $\gamma = (2+8\Re\rho)(5+16\Re\rho)^{-1}$, and $g = \max(0, b-1)$; b denotes the order of the pole of $S_\kappa(f, s)(r + s)^{-1}x^{r+s}$ at $s = 2\Re\rho$ $(0 \leq b \leq 5)$.

We now consider an application of (24). Let $Q \in \mathbb{Z}^{m\times m}$ be a non-singular symmetric matrix with even diagonal entries and $q(\mathbf{x}) = \frac{1}{2}Q[\mathbf{x}] = \frac{1}{2}\mathbf{x}^T Q\mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^m$, the associated quadratic form in $m \geq 3$ variables. Here we assume that $q(\mathbf{x})$ is positive definite. Let $r(Q, n)$ denote the number of representations of n by the quadratic form Q. Now consider the theta function

$$\theta_Q(z) = \sum_{\mathbf{x}\in\mathbb{Z}^m} e^{\pi izQ[\mathbf{x}]}.$$

By [11, Lemma 6.1], the Dirichlet series associated with the automorphic form θ_Q is

$$(4\pi)^{-m/4}\zeta_Q(\tfrac{m}{4} + s)$$

where

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \frac{r(Q, n)}{n^s} = \sum_{\mathbf{x}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} q(\mathbf{x})^{-s}$$

for $\Re(s) > m/2$. Using (24), Müller proved the following (see [11, Theorem 6.1])

Theorem (Müller). *Let $q(\mathbf{x}) = \frac{1}{2}Q[\mathbf{x}] = \frac{1}{2}\mathbf{x}^T Q\mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^m$ be a primitive positive definite quadratic form in $m \geq 3$ variables with integral coefficients. Then*

$$\sum_{n\leq x} r(Q,n)^2 = Bx^{m-1} + O\left(x^{(m-1)\frac{4m-5}{4m-3}}\right)$$

where

$$B = (4\pi)^{m/2}\frac{\beta_\infty(\theta_Q)}{m-1}$$

and $\beta_\infty(\theta_Q)$ is given by (23).

We are now in a position to prove our theorem in page 2.

Proof. We are interested in the case $q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$ and so $r(Q,n) = r_3(n)$ counts the number of representations of n as a sum of three squares. By Müller's Theorem above,

$$\sum_{n\leq x} r_3(n)^2 = Bx^2 + O\left(x^{14/9}\right)$$

where B is a computable constant. Specifically, we have by (23) (with $k = 3/2$ and $\rho = 1/4$)

$$B = \frac{4\pi^2}{3-1} b^+(3/4, 1/4) \sum_{\iota\in K} \varphi_{\infty,\iota}(3/2)|a_{\iota,0}|^2,$$

where K denotes a complete set of $\Gamma_0(4)$-inequivalent cusps and $a_{\iota,0}$ is the 0-th Fourier coefficient of $\theta_Q(z)$ at a rational cusp ι. Choose $K = \{1, \frac{1}{2}, \frac{1}{4}\}$. Then by p. 145 and (67) in [11], we have

$$|a_{\iota,0}|^2 = W_\iota^3 |G(S_\iota)|^2$$

where $\iota = u/w$, $(u,w) = 1$, $w \geq 1$, W_ι is width of the cusp ι, and

$$|G(S_\iota)|^2 = 2^{-3}w^{-3}\Big|\sum_{x=1}^{w} e(\frac{u}{w}x^2)\Big|^6.$$

As $W_{1/4} = W_{1/2} = 1$, $W_1 = 4$, we have $|a_{1,0}|^2 = 1$, $|a_{1/2,0}|^2 = 0$, and $|a_{1/4,0}|^2 = 1$. An explicit description of the functions $\varphi_{\infty,\iota}(s)$ in the case $\Gamma_0(4)$ is given by (see (1.17) and p. 247 in [5])

$$\varphi_{\infty,1/4}(s) = 2^{1-4s}(1-2^{-2s})^{-1}\pi^{1/2}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

$$\varphi_{\infty,1/2}(s) = \varphi_{\infty,1}(s) = 2^{-2s}(1-2^{-2s})^{-1}(1-2^{1-2s})\pi^{1/2}\frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

Thus for $s = 3/2$, we have

$$\varphi_{\infty,1/4}(3/2) = 2^{-5}(1-2^{-3})^{-1}\pi^2\frac{\zeta(2)}{\Gamma(3/2)\zeta(3)},$$

$$\varphi_{\infty,1/2}(3/2) = \varphi_{\infty,1}(3/2) = 2^{-3}(1-2^{-3})^{-1}(1-2^{-2})\pi^2\frac{\zeta(2)}{\Gamma(3/2)\zeta(3)}.$$

Now, from p. 65 in [12], we have

$$b^+(3/4, 1/4) = G^*_{1/4,1/4}(3/2).$$

By Lemma 3.3 and (16) in [12],

$$G^*_{1/4,1/4}(s) = \Gamma(s + 1/2)^{-1}$$

and so $b^+(3/4, 1/4) = \Gamma(2)^{-1}$. In total,

$$\begin{aligned} B &= \frac{(4\pi)^2}{(3-1)}\frac{1}{\Gamma(2)}\left(2^{-3}(1-2^{-3})^{-1}(1-2^{-2})\pi^{1/2}\frac{\zeta(2)}{\Gamma(3/2)\zeta(3)}\right. \\ &\quad \left. + 2^{-5}(1-2^{-3})^{-1}\pi^{1/2}\frac{\zeta(2)}{\Gamma(3/2)\zeta(3)}\right) \\ &= \frac{8\pi^4}{21\zeta(3)}. \end{aligned}$$

Thus

$$\sum_{n\le x} r_3(n)^2 \sim \frac{8\pi^4}{21\zeta(3)}x^2.$$

□

Remark. Müller's Theorem can also be used to obtain the mean square value of sums of $N > 3$ squares. Precisely, if $r_N(n)$ is the number of representations of n by $N > 3$ squares, then a calculation similar to the second proof of our theorem yields (compare with Theorem 3.3 in [2])

$$\sum_{n\le x} r_N(n)^2 = W_N x^{N-1} + O\left(x^{(N-1)\frac{4N-5}{4N-3}}\right)$$

where

$$W_N = \frac{1}{(N-1)(1-2^{-N})}\frac{\pi^N}{\Gamma(N/2)^2}\frac{\zeta(N-1)}{\zeta(N)}.$$

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Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6
E-mail address: `kkchoi@cecm.sfu.ca`

Department of Mathematics, 1 University Station C1200, The University of Texas at Austin, Austin, TX 78712, U.S.A.
E-mail address: `kumchev@math.utexas.edu`

Max-Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `osburn@mpim-bonn.mpg.de`