IN MEMORY OF PETER BORWEIN

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Peter Borwein passed away on August 23rd, 2020, with his wife Jenny at his side. It was and continues to be a big loss for his family, our university, our department and the mathematics community as a whole. I lost my closest friend, my mentor and one of the most important people in my career. With the Covid pandemic, it has been especially sad that none of us are able to attend any ceremony to honour Peter's impact in our lives and express our deepest condolences to his family.

I first met Peter in Seattle at the meeting, "In Celebration of the Centenary of the Proof of the Prime Number Theorem" in 1996 when I was in the last year of my doctoral studies. After spending one year in the Institute for Advanced Study, I started a postdoctoral fellowship under the tutelage of Peter, and David Boyd, at SFU and UBC respectively. My memory is still very clear on the call from Peter offering me this job, and the moment when I accepted without any hesitation. This is how my long enjoyable collaboration with Peter began.

The postdoctoral period with Peter and David was fruitful as we produced several papers together. Afterwards, I spend two years at another postdoctoral job at the University of Hong Kong in 1999. Peter visited me in Hong Kong and he encouraged me to apply the job opening at SFU. I subsequently started my assistant professorship at SFU in September 2001. Since then, Peter and I have always worked together closely. We met everyday at offices, had at least two to three coffee breaks each day, shared rides to UBC for number theory seminars, savored Dim Sum gatherings with Vancouver number theorists and travelled to Seattle for many Pacific Northwest Number Theory Meetings. We have also travelled to many other conferences in Canada and the US, particularly in Banff. Together we published 16 joint papers and one book (of course, that itself is only a small amount from Peter's remarkable 180+ publications). We shared supervision of postdoctoral and graduate students. We co-organized 7 conferences. Peter had special talent in initiating collaborations with not just the mathematics in his specialities, but also in other research areas for unique cross-functional insights. Peter and his brother, Jonathan, established the Centre for Experimental and Constructive Mathematics (CECM), in the late 1990s, and in the early 2010s, was able to secure the funding of 11 million dollars for the building and running of the IRMACS (Interdisciplinary Research in the Mathematical and Computational Sciences) Centre, a visionary project with the purpose to, in Peter's words, "host any scientist who uses computers as a tool in their research." Peter was the true heart and soul of the IRMACS Centre and a larger-than-life presence in every team and room.

One of my long term research projects with Peter has been on polynomials with the coefficients $+1$ or -1. As Littlewood raised a number of these questions and studied a lot on these polynomials in [13], Peter suggested calling these polynomials Littlewood polynomials in [2]. Since then, the term *Littlewood polynomials* has been

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widely used for these polynomials. For convenience, we denote $\mathfrak{L}_n$ to be the set of all 2^n Littlewood polynomials of degree $n-1$. When building up a Littlewood polynomial with a larger degree from two Littlewood polynomials $P(x)$ and $Q(x)$, a natural way is $P(\pm x)Q(\pm x^n)$ and $Q(\pm x)P(\pm x^m)$ for $P \in \mathfrak{L}_n$ and $Q \in \mathfrak{L}_m$. We showed in [2] that it is the only way to construct cyclotomic Littlewood polynomials for even degrees. More precisely, suppose N is odd. A Littlewood polynomial, $P(x)$, of degree $N-1$ is cyclotomic if and only if

$$P(x) = \pm\Phi_{p_1}(\pm x)\Phi_{p_2}(\pm x^{p_1}) \cdots \Phi_{p_r}(\pm x^{p_1 p_2 \cdots p_{r-1}}) \tag{1}$$

where $N = p_1 p_2 \cdots p_r$ and the p_i are primes, not necessarily distinct. We also conjectured (1) is true for even N and showed (1) for some special cases. This conjecture is still unsolved.

For any Littlewood polynomial $P(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ with $a_\ell = \pm 1$, it is natural to associate $P(x)$ to a finite binary sequences $(a_\ell)_{\ell=0}^{n-1}$ of length n. In the study of finite binary sequences, the term merit factor $M(P)$ of finite binary sequences $(a_\ell)_{\ell=0}^{n-1}$ has been studied widely in communication engineering and information theory (see [10]). People are interested in finding finite binary sequences with large $M(P)$ and such sequences are called *easily distinguishable binary sequences* in spread spectrum radio communication, which has vast applications in information theory. In the setting of Littlewood polynomials, the merit factor, in fact, can be expressed in terms simply of the L_4 norm, $\|P\|_4$, over the unit circle, namely,

$$M(P) = \frac{\|P\|_2^4}{\|P\|_4^4 - \|P\|_2^4} = \frac{n^2}{\|P\|_4^4 - n^2}$$

because $\|P\|_2^2 = \sum_{\ell=0}^{n-1} |a_\ell|^2 = n$ for $P \in \mathfrak{L}_n$. Here $\|P\|_p := \left(\int_0^1 |P(e^{2\pi i\theta})|^p d\theta\right)^{\frac{1}{p}}$ for $p > 0$. Therefore, we are interested in finding Littlewood polynomials with small L_4 norm. Littlewood in [13] asked whether it is possible to find Littlewood polynomial P of degree $n-1$ such that

$$C_1\sqrt{n} \le |P(z)| \le C_2\sqrt{n} \tag{2}$$

for all z of modulus 1 and for two constants C_1, C_2 independent of n.

Erdös conjectured that there is a $\varepsilon > 0$ such that for any Littlewood polynomial P_n of degree $n-1$, we have

$$(1+\varepsilon)\sqrt{n} < \|P\|_\infty$$

where $\|P\|_\infty := \max\{|P(z)| : |z| = 1\}$.

Peter and I conjectured that there is a $\varepsilon > 0$ such that for any Littlewood polynomial P of degree $n-1$, we have

$$(1+\varepsilon)\sqrt{n} < \|P\|_4.$$

In view of monotonicity of L_p norm, our conjecture implies the Erdös conjecture. Since the trivial lower bound for $\|P\|_4$ is $\|P\|_2 = \sqrt{n}$ for any $P \in \mathfrak{L}_n$ because $\|P\|_4 \ge \|P\|_2$, our conjecture is slightly better than the trivial lower bound. However, even with a much weaker conjecture, for any Littlewood polynomial P of degree $n-1$, we have

$$\sqrt{n^2 + 2(n-1)} < \|P\|_4$$

which implies that there are only finitely many Barker sequences - a longstanding open question in combinatorics. Except for Littlewood's conjecture, all other conjectures are still unsolved. Very recently, Balister, Bollobaás, Morris, Sahasrabudhe and Tiba shows the existence of a sequence of Littlewood polynomials satisfying (2) in [1], which solves Littlewood's conjecture.

One of the natural Littlewood polynomials is constructed by the Legendre or Jacobi symbols in number theory. We let $f_p(x) := \sum_{\ell=0}^{p-1} \left(\frac{\ell}{p}\right) x^\ell$ for any prime p where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol and f_p is called the Fekete polynomial. (Fekete polynomials are not exactly Littlewood polynomials because the constant is zero but only one term does not affect the asymptotic behaviour of f_p.) The asymptotic formula for the L_4 norm of f_p and its rotations was obtained by Høhodt and Jensen in [11] (in terms of merit factor). Peter and I gave an exact evaluation for the L_4 norms of f_p and its rotations. In [5], we show that for any $1 \le r \le (p+1)/2$

$$\|f_p^r(z)\|_4^4 = \frac{1}{3}(5p^2+3p+4)+8r^2-4pr-8r-\frac{8}{p^2}\left(1-\frac{1}{2}\left(\frac{-1}{p}\right)\right)\left(\sum_{i=1}^{p-1} i\left(\frac{i+r}{p}\right)\right)^2$$

where

$$f_p^r(z) := \sum_{\ell=0}^{p-1}\left(\frac{\ell+r}{p}\right) z^\ell.$$

If we take the optimal choice of r to be $[p/4]$ where $[x]$ is the integral part of x, we get

$$\|f_p^{[p/4]}(z)\|_4^4 = \frac{7}{6}p^2 - p - \frac{1}{6} - \gamma_p \tag{3}$$

where

$$\gamma_p := \begin{cases} h(-p)(h(-p)-4) & \text{if } p \equiv 1, 5 \pmod 8, \\ 12(h(-p))^2 & \text{if } p \equiv 3 \pmod 8, \\ 0 & \text{if } p \equiv 7 \pmod 8, \end{cases}$$

and $h(-d)$ is the class number of the imaginary quadratic field of $\mathbb{Q}(\sqrt{-d})$. We also obtain analogous results for polynomials with Jacobi symbols coefficients in [4] and Dirichlet's character coefficients in [3].

The rotated Fekete polynomials in (3) have small asymptotic L_4 norms and it was conjectured that it is the smallest possible asymptotic L_4 norm (see Golay conjecture of the merit factor is > 6). Until Jonathan Jedwab, Peter and I, based on the numerical data, suggested in [8] that a new sequence of Littlewood polynomials that is constructed by appending and rotating the Fekete polynomials of degree $n-1$ will have a L_4 norm $< \left(1+\frac{1}{6.34\cdots}\right)n^2 = (1.157\cdots)n^2 < \frac{7}{6}n^2$ for sufficiently large n. We consider $f_p^{(r,t)}(x)$ to be the polynomial of the first $[tp]$ of the rotated Fekete polynomials $f_p^{[rp]}(x)$ for any $t, r, \in [0,1)$ and the polynomial $F_p^{(r,t)}(x) = f_p^{[rp]}(x) + x^p f_p^{(r,t)}(x)$. Based on the numerical data, we conjectured that for any $0 \le r \le 1$ and $0 < t \le 1$,

$$\|f_p^{(r,t)}\|_4^4 = \left(1 + \frac{1}{g(r,t)} + o(1)\right)[tp]^2 \tag{4}$$

and

$$\|F_p^{(r,t)}\|_4^4 = \left(1 + \frac{2(g(r,t)+t^2)+g(r+t,1-t)}{(1+t)^2} + o(1)\right)[(1+t)p]^2 \tag{5}$$

where $g(r,t) = t^2(1-4/3t) + h(r,t)$ and $h(r+1/2,t) = h(r,t)$ for $0 \le r \le 1/2$ and $0 < t \le 1$ and for $0 < t \le 1/2$

$$h(r,t) := \begin{cases} 0 & \text{for } 0 \le r \le 1/2 - t, \\ 4(r-1/2+t)^2 & \text{for } 1/2 - t \le r \le (1-t)/2, \\ 4(r-1/2)^2 & \text{for } (1-t)/2 \le r \le 1/2, \end{cases}$$

and for $1/2 < t \le 1$,

$$h(r,t) := \begin{cases} 4(r-1/2+t)^2 & \text{for } 0 \le r \le (1-t)/2, \\ 4(r-1/2)^2 & \text{for } (1-t)/2 \le r \le 1-t, \\ 8(r-3/4+t/2)^2 + 2(t-1/2)^2 & \text{for } 1-t \le r \le 1/2. \end{cases}$$

By choosing optimal $\hat{r} = \frac{1}{4} - \frac{\hat{t}}{2}$ and $\hat{t}$ ($\sim 0.0578\cdots$) the real root of $4t^3 + 12t^2 - 18t + 1 = 0$, from (5), we get

$$\|F_p^{(\hat{r},\hat{t})}\|_4^4 = (1.157\cdots + o(1))[(1+\hat{t})p]^2.$$

Our conjecture implies the existence of a sequence of Littlewood polynomials with smaller asymptotic L_4 norm than anything else known at that time.

Subsequently, conjectures (4) and (5) were proved by J. Jedwab, D.J. Katz and K.-U. Schmidt in [12] in 2013. This is currently the best record.

We also extend Byrnes and Newman's result on the expected values of L_4 norms in [9]. In [6], we showed that

$$\frac{1}{2^n} \sum_{P \in \mathfrak{L}_n} \|P\|_4^4 = 2n^2 - n \quad \text{(Byrnes and Newman)}$$

$$\frac{1}{2^n} \sum_{P \in \mathfrak{L}_n} \|P\|_6^6 = 6n^3 - 9n^2 + 4n$$

and

$$\frac{1}{2^n} \sum_{P \in \mathfrak{L}_n} \|P\|_8^8 = 24n^4 - 66n^3 + 58n^2 - 9n - 3 + 3(-1)^n.$$

(Note that the formulae in [6] are for $\frac{1}{2^{n+1}} \sum_{P \in \mathfrak{L}_{n+1}} \|P\|_p^p$.) The analogous formulae for polynomials with coefficients $0, \pm 1$ in [6] and $0, 1$ in [7] are also obtained.

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