Analytic Combinatorics of Walks with Small Steps in the Quarter Plane

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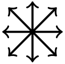
(joint work with Marni Mishna)

Simon Fraser University

May 8, 2012

Introduction

Given: A set of directions $\mathcal{S} \subseteq \{0, 1, -1\}^2 \setminus \{(0, 0)\}$.



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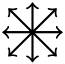

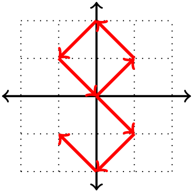

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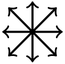

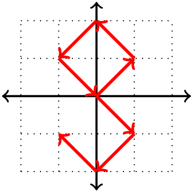

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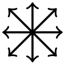

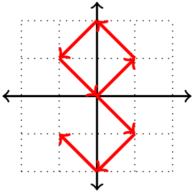

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$$* \quad p_0 = (0, 0),$$

$$* \quad p_i - p_{i-1} \in \mathcal{S}.$$

Motivation

Goal: An asymptotic expression for the number of walks $w(n)$ from the class $\mathcal{W}$ of length n

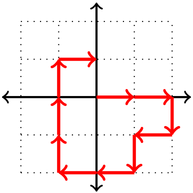
$$w(n) \sim c\beta^n n^s.$$

Why: Walks efficiently model many phenomena in physics, chemistry and probability. Asymptotic expressions are linked closely with properties of these phenomena.

Self Avoiding Walks (SAW) - The Holy Grail

Sequences with the extra restriction

$$p_i \neq p_j \text{ for } i \neq j$$



$$\mathcal{S} = \{\leftarrow, \uparrow, \rightarrow, \downarrow\},\ n = 11$$

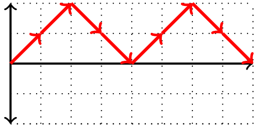
These model linear polymers in solution, and an asymptotic expression is very interesting for chemists. Empirically

$$w(n) \sim c\mu^n n^\gamma$$

where $\mu = 2.638$, $\gamma = 11/32$. No proof is yet known!

Singularity analysis

Example: Dyck paths



$\circlearrowleft$, $n = 8$.

The number of Dyck paths of length $2n$ is $d(n) = \frac{1}{n+1}\binom{2n}{n}$ and the class $\mathcal{D}$ has the generating function

$$\sum_{n \geq 0} d(n) t^{2n} =$$

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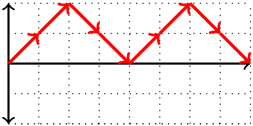

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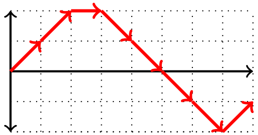
The number of Dyck paths of length $2n$ is $d(n) = \frac{1}{n+1}\binom{2n}{n}$ and the class $\mathcal{D}$ has the generating function

$$\sum_{n \geq 0} d(n) t^{2n} = \frac{1 - \sqrt{1 - 4t}}{2t},$$

with asymptotic expression $d(n) \sim \frac{4^n}{\sqrt{\pi n^3}}$.

Directed Paths

Restricting $\mathcal{S} \subseteq \{1\} \times \{0, 1, -1\}$ gives a subset of SAW which increase to the right at every step



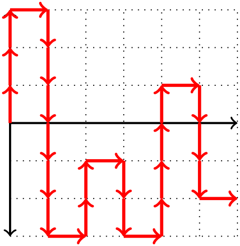
$\mathcal{S} = \{\nearrow, \rightarrow, \searrow\}$, $n = 8$

These model some queuing theory problems and, in a probabilistic relative, sums of discrete random variables. These are solved:

$$W(t) = \frac{1}{1 - 3t}, \quad w(n) = 3^n.$$

Partially Directed Paths

Taking $\mathcal{S} \subseteq \updownarrow\!\rightarrow$ and enforcing self-avoidance gives another subset of SAW.



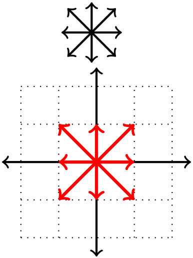
These sometimes correspond to Atomic Force Microscopy (AFM) experiments on linear polymers (as do directed paths).

$$W(t) = \frac{1+t}{1-2t-t^2}, \quad w(n) \sim c(1+\sqrt{2})^n n^s.$$

Self Intersecting Walks

An Easy Place to Start

Let $\mathcal{W}$ be the class of unrestricted walks with steps from

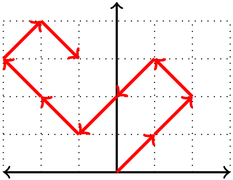


then $w(n) = |\mathcal{S}|^n$ and hence

$$W(t) = \sum_{n \geq 0} w(n) t^n = \frac{1}{1 - |\mathcal{S}| t}.$$

Walks In The Half Plane

To add a boundary, restrict the points to the half plane $y \geq 0$.



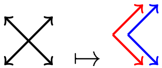
$n = 9$, $\mathcal{S} = \boxtimes$

These model polymers interacting with a boundary, and can give information about adsorbption. Moreover, these will provide enumerative bounds on problems which are restrictions or relaxations of this one.

Enumeration

We define the *horizontal projection*

$$P(y) = S(1, y).$$



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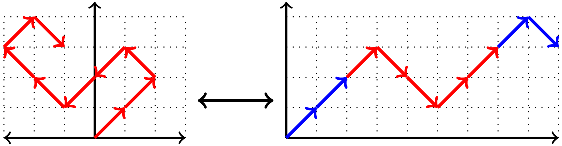

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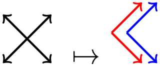

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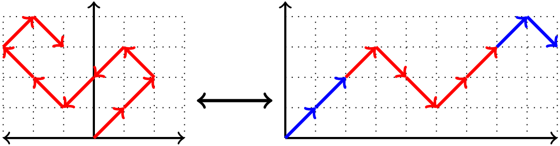
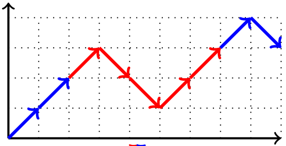


So it suffices in this case to study directed paths in $y \geq 0$.
These are called *directed meanders*.

Directed Meanders

$\mathcal{S}$ is a weighted subset of $\{1\} \times \{1, 0, -1\}$.



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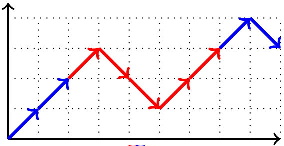

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Good news: Explicit enumeration and asymptotic expressions known.

The kernel method

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Banderier and Flajolet give explicit generating function results for directed meanders by using the kernel method, which:

* relies on an inventory of the step set; and

* gives a generating function which is algebraic, and a simple modification of the full plane model's rational GF.

Our Example

For the set of steps

we have the inventory

$$P(y) = 2y + \frac{2}{y}$$

from which we define the kernel

$$K(y,t) = y - tyP(y) = y - 2ty^2 - 2t.$$

The Result

Theorem: (Banderier/Flajolet) The generating function for a directed meander is given by

$$F(t) = \frac{1 - y(t)}{1 - tP(1)}.$$

Where $y(t)$ is the solution to $K(y,t) = 0$ which is analytic at 0.

Our Example

So, $y - 2ty^2 - 2t = 0$ has solutions

$$y_1(t) = \frac{1 - \sqrt{1 - 16t^2}}{4t}, \quad y_2(t) = \frac{1 + \sqrt{1 - 16t^2}}{4t},$$

of which $y_1(t)$ is analytic at 0. Then

$$F(t) = \frac{2t - 1 + \sqrt{1 - 16t^2}}{2t(1 - 4t)},$$

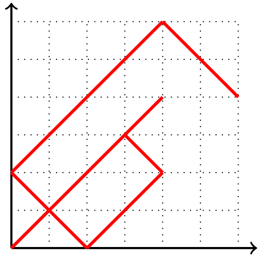
from which we may find the asymptotic expression

$$f(n) \sim C \frac{4^n}{\sqrt{\pi n}}.$$

The Quarter Plane

Walks In The Quarter Plane

We add another boundary by restricting the previous half plane case to $x \geq 0$ and consider $q(n)$, the number of walks of length n in the class $\mathcal{Q}$.



$\mathcal{S} = \boxtimes, n = 16$

Physical interpretation: a polymer or particle interacting with two boundaries, such as the corner of a container.

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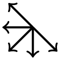
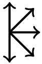
Goal (long term): An analytic theory similar to that of directed paths.

Goal (short term): Asymptotic expressions.

Case Reduction

Let's only consider distinct cases.

We can throw out subsets of

 and 

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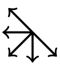
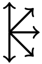
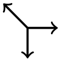
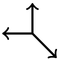
 and 

and notice that reflections across $y = x$ give equivalent models.

 $\leftrightarrow$ 

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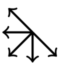
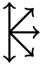
 and 

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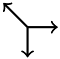
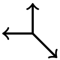
 $\leftrightarrow$ 

Removing previous cases, trivial models and symmetries $\Rightarrow$ 79 non-equivalent quarter plane models.

Tools

We use a two variable inventory of $\mathcal{S}$

$$S(x, y) = \sum_{s \in \mathcal{S}} x^{s_1} y^{s_2}.$$

And define the kernel analogously to the directed models

$$K(x, y, t) = xy - xytS(x, y).$$

The kernel

Define a group $G(\mathcal{S})$ of birational transformations of the plane preserving the kernel

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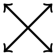

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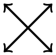

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For 23 cases, $|G(S)| < \infty$ and for the remaining 56 $G(S)$ is an infinite group. This is tied to enumerative results.

Previous Work

Working with the kernel:

- Orbit Sums Method: A series extraction of a rational series (Bousquet-Mélou/Mishna 10) (22 cases).

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- Guess and check (Bostan/Kauers 10) : Gives (likely) asymptotic development for finite group cases.

Selected step sets and their asymptotic behaviour

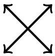
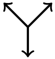
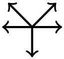
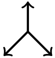
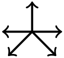
S	δ	c	s	β	Asymptotic Estimate
	0	$\frac{2}{\pi}$	-1	4	$\frac{2}{\pi} \cdot \frac{4^n}{n}$
	+	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	3	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})} \cdot \frac{3^n}{\sqrt{n}}$
	+	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	5	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})} \cdot \frac{5^n}{\sqrt{n}}$
	-	$\frac{24\sqrt{2}}{\pi}$	-2	$2\sqrt{2}$	$\frac{24\sqrt{2}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	-	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi}$	-2	$2(1+\sqrt{2})$	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi} \cdot \frac{(2(1+\sqrt{2}))^n}{n^2}$

Table: The number of walks grows asymptotically as $cn^s\beta^n$.

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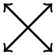
Naively: $q(n) \leq w(n) = |\mathcal{S}|^n$, and $q(n) \sim c\beta^n n^s$ for some β and s. Taking $\lim \frac{1}{n}\log$ of both gives

$$\log(\beta) \leq \log|\mathcal{S}| \Rightarrow \beta \leq |\mathcal{S}|.$$

Our example

We can apply this method to our example



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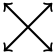

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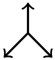

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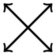

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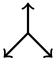


Experiments tell us that the associated $\beta = 2\sqrt{2}$, so we need something better.

Selected step sets and their asymptotic behaviour

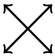
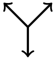
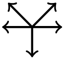
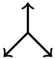
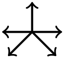
S	δ	c	s	β	Asymptotic Estimate
	0	$\frac{2}{\pi}$	-1	4	$\frac{2}{\pi} \cdot \frac{4^n}{n}$
	+	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	3	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})} \cdot \frac{3^n}{\sqrt{n}}$
	+	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	5	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})} \cdot \frac{5^n}{\sqrt{n}}$
	-	$\frac{24\sqrt{2}}{\pi}$	-2	$2\sqrt{2}$	$\frac{24\sqrt{2}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	-	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi}$	-2	$2(1+\sqrt{2})$	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi} \cdot \frac{(2(1+\sqrt{2}))^n}{n^2}$

Table: The number of walks grows asymptotically as $cn^s\beta^n$.

Bounding The Exponential Growth

Observation: The interaction with the x-axis is what matters. By removing the y-axis as a boundary, we relax the problem and find

$$q(n) \leq f(n).$$

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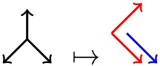
we can take $\lim \frac{1}{n} \log$ on both sides to get

$$\beta \leq \gamma.$$

A Negative Drift Example

Take $\mathcal{S}$ as shown below with the associated directed meander.



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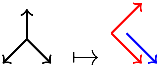

From the inventory $P(y) = y + \frac{2}{y}$, we find

$$F(t) = \frac{2t - 1 + \sqrt{1 - 8t^2}}{2t(1 - 3t)}.$$

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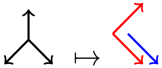

From the inventory $P(y) = y + \frac{2}{y}$, we find

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Which has a dominant singularity at the branch point $t = \frac{1}{2\sqrt{2}}$, as desired.

Infinite group cases

Taking the horizontal projection provides a tight (experimentally) upper bound for all finite group cases. What about infinite groups?

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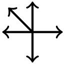

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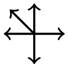

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Moral: We can use this to bound the dominant singularity on IG step sets, as long as we use more care in choosing the direction of projection.

Lower Bounds

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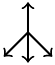
So, what if we look at walks returning to the origin?

The origin

Take as an example the step set which we call the trident



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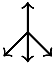

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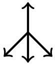

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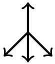

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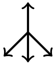

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$$T(n) \sim 2^{2n} \sqrt{3}^{2n} = (2\sqrt{3})^{2n}.$$

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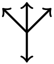
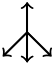
 and 

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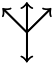
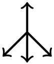
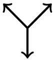
 and 

Walks to the origin on both step sets are equinumerous, so the upward drift version needs a larger lower bound.
How can we fix this?

Bootstrapping

One of the solved sporadic cases is



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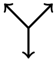
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Inserting a $\uparrow$ step at any point gives some of the walks on the upward trident,

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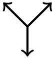
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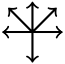
We can similarly add the steps $\leftrightarrow$ to bootstrap to


,

as long as the walk on the inserted steps is a meander.

Recap

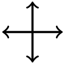
* Simplifications and relaxations can directly and indirectly provide methods for proving harder problems.

Allowing walks to self intersect provides a bound (albeit too large) on the egf of SAW.

We need to add boundaries to make something interesting. Two boundary case still uses ad-hoc methods, and an analytic theory, similar to the single boundary case, is desirable.

Future Work

Other methods of counting: interleaving meanders of paired steps. Take



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$$q(n) = \sum_{k \geq 0} \binom{n}{k} H_n V_{n-k}.$$

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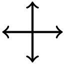

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The theory on directed meanders comes from a more general theory of walks on $\mathcal{S} \subseteq \{1\} \times \mathbb{Z}$ with $|\mathcal{S}| < \infty$. More boundaries is harder.

Question: Is there an analogous theory for these undirected models, two boundary models?

THANKS FOR LISTENING!

References

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A. Bostan, M. Kauers, *Automatic Classification of Restricted Lattice Walks.*

M. Mishna, A. Rechnitzer, *Two non-holonomic lattic walks in the quarter plane.*