Walks with Small Steps in the Quarter Plane

Sam Johnson and Steve Melczer
(Joint work with Marni Mishna)

Simon Fraser University

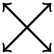
October 23, 2011

Introduction

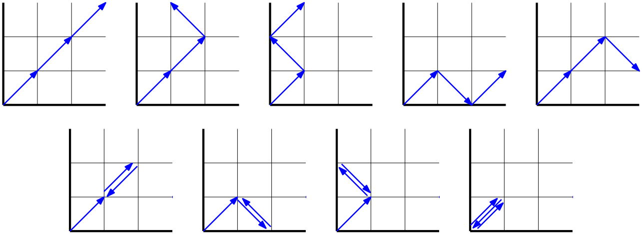
Given: A set of directions

Count: Number of integer lattice walks in the first quadrant using these steps.

For instance, given the step set $S = \{NE, SE, NW, SW\}$



there are 9 walks of length 3:



Selected step sets and their asymptotic behaviour

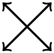
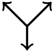
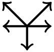
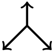
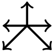
S	c	s	β	Asymptotic Estimate
	$\frac{2}{\pi}$	-1	4	$\frac{2}{\pi} \cdot \frac{4^n}{n}$
	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	3	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})} \cdot \frac{3^n}{\sqrt{n}}$
	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	5	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})} \cdot \frac{5^n}{\sqrt{n}}$
	$\frac{24\sqrt{2}}{\pi}$	-2	$2\sqrt{2}$	$\frac{24\sqrt{2}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi}$	-2	$2(1+\sqrt{2})$	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi} \cdot \frac{(2(1+\sqrt{2}))^n}{n^2}$

Table: The number of walks grows asymptotically as $cn^s\beta^n$.

If we can classify the generating function of a walk as algebraic or holonomic (satisfies a nice linear ODE) then we will know the form of its growth.

Tools

Two methods of reasoning (both are interconnected):

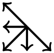
Combinatorial: Drift Arguments
Projection onto axis'
'Squeezing' with known cases (Catalan, etc.)
Useful for determining growth factors

Analytic: Uses multivariable generating function
Exploit functional equation for GF
Useful for classifying GF of a walk

Some history and case reduction

Step sets which are subsets of

are a subset of half space problems which have been solved [Banderier,Flajolet 01] and are not considered.
Also, subsets of



will never leave the origin, so these are also not considered.

In fact, only 79 step sets are considered. Of these 23 are 'nice', and the remaining 56 are 'mysterious'.

‘Nice’ Case

What makes it 'nice'?

The step set possesses a vertical symmetry **OR**
The vector sum of the step set vectors is 0.

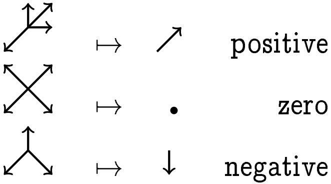
All are holonomic (many are algebraic).
([Bousquet-Melou,Mishna 09] and [Bostan,Kauers 09])

The asymptotics are estimated experimentally to great accuracy. [Bostan,Kauers 09]

Many are susceptible to arguments or reductions which are largely *combinatorial*.

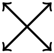
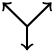
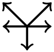
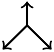
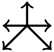
Tools - Drift

We take the vector sum of the step set, and consider the direction of the resulting vector:



This is the direction that walks on these step sets will tend in. Drift into the quarter plane is called *positive* and drift out of the quarter plane is *negative*. This parameter plays a role in the asymptotics.

Selected step sets and their asymptotic behaviour

S	δ	c	s	β	Asymptotic Estimate
	0	$\frac{2}{\pi}$	-1	4	$\frac{2}{\pi} \cdot \frac{4^n}{n}$
	+	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	3	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})} \cdot \frac{3^n}{\sqrt{n}}$
	+	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}$	$-\frac{1}{2}$	5	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})} \cdot \frac{5^n}{\sqrt{n}}$
	-	$\frac{24\sqrt{2}}{\pi}$	-2	$2\sqrt{2}$	$\frac{24\sqrt{2}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2}$
	-	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi}$	-2	$2(1+\sqrt{2})$	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi} \cdot \frac{(2(1+\sqrt{2}))^n}{n^2}$

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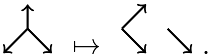
A point of interest

If the drift is negative, take a horizontal projection of the step set



From the new GF $P(u) = u + \frac{2}{u}$, define

$$\tau \; : \; P'(\tau) = 0 \;\Rightarrow\; \tau = \sqrt{2}.$$

Then the exponential growth factor is equal to

$$P(\tau) = P(\sqrt{2}) = \sqrt{2} + \frac{2}{\sqrt{2}} = 2\sqrt{2}.$$

Observations

Non-negative drift implies full exponential growth.

Distinct strategies seem necessary for each case: squeezing will work with non negative drift.

One strategy allows us to shift our focus to the asymptotics of a subset of all walks (MPEP) [Janse van Rensburg,Rechnitzer 01].

Negative drift walks share asymptotic properties with negative drift directed walks à la Banderier and Flajolet.

‘Mysterious’ Case

Step Sets with Nonholonomic GFs

GFs in this second category are all conjectured to be non-holonomic (in t).

First 2 proven non-holonomic by Mishna and Rechnitzer in '09 using an 'Iterated Kernel Method'.

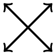
As seen below, this argument utilizes a functional equation and is largely *analytic* in nature.

Tools - Generating Functions

The generating function

$$Q(x, y; t) = \sum_{n,i,j \geq 0} q_{ijn} x^i y^j t^n$$

which counts the number of walks of length n ending at (i, j) satisfies an obvious functional equation. For example, with the previous step set

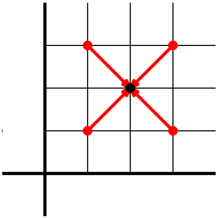


we use the step generating function

$$S(x, y) = xy + \frac{y}{x} + \frac{1}{xy} + \frac{x}{y}$$

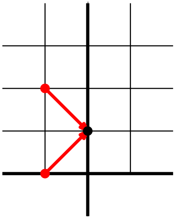
A functional equation

$$Q(x,y;t) \quad = \quad 1 + tS(x,y)Q(x,y;t)$$



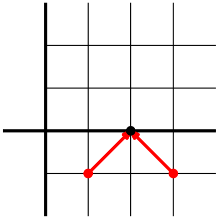
A functional equation

$$Q(x,y;t) = 1 + tS(x,y)Q(x,y;t) - t\left(\frac{y}{x} + \frac{1}{xy}\right)Q(0,y;t)$$



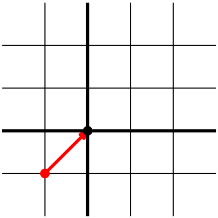
A functional equation

$$\begin{aligned} Q(x,y;t) &= 1 + tS(x,y)Q(x,y;t) \\ &\quad - t\left(\frac{y}{x} + \frac{1}{xy}\right)Q(0,y;t) \\ &\quad - t\left(\frac{x}{y} + \frac{1}{xy}\right)Q(x,0;t) \end{aligned}$$



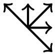
A functional equation

$$\begin{aligned} Q(x,y;t) &= 1 + tS(x,y)Q(x,y;t) \\ &\quad -t(\frac{y}{x} + \frac{1}{xy})Q(0,y;t) \\ &\quad -t(\frac{x}{y} + \frac{1}{xy})Q(x,0;t) \\ &\quad +\frac{t}{xy}Q(0,0;t). \end{aligned}$$



Example of Iterated Kernel Method

For the step set



Begin with with functional equation:

$$Q(x,y) = 1 + t\left(\frac{y}{x} + y + xy + x + \frac{x}{y}\right)Q(x,y) - \frac{ty}{x}Q(0,y) - \frac{tx}{y}Q(x,0)$$

Regroup to give the form

$$\boxed{K(x,y)Q(x,y) = xy - ty^2Q(y,0) - tx^2Q(x,0)}$$

where K is the *kernel*

$$K(x,y) = xy - t\left(y^2 + xy^2 + x^2y^2 + x^2y + x^2\right).$$

Example of Iterated Kernel Method

Let $Y_\pm$ be the roots of $K(x,y)$ in y:

$$Y_\pm = \frac{x}{2t\left(1+x+x^2\right)}\left(1 - tx \mp \sqrt{1-2tx-3t^2x^2-4t^2x-4t^2}\right).$$

Define Y_n to be Y_+ composed with itself n times.
Substituting $y = Y_1(x)$ back into the functional equation gives

$$0 = xY_1(x) - tY_1(x)^2 Q\left(Y_1(x), 0\right) - tx^2 Q\left(x, 0\right).$$

Repeatedly set $x = Y_1(x)$ and take an alternating sum to get

$$\boxed{Q(x,0) = \frac{1}{x^2 t}\sum_{n\geq 0}(-1)^n Y_n(x) Y_{n+1}(x)}$$

Example of Iterated Kernel Method

This gives an equation for the generating function of the number of walks:

$$W(t) = \frac{1 - 2tQ(1,0)}{1 - 5t} = \frac{1 - 2\sum_{n\geq 0}(-1)^n Y_n(1) Y_{n+1}(1)}{1 - 5t}$$

If we can show each summand has distinct singularities, we have shown W is not holonomic.

Example of Iterated Kernel Method

This is achieved by

1) Relating $1/Y_-$ and $1/Y_+$ (as roots of a quadratic)

   In this case $\dfrac{1}{Y_+(x)} + \dfrac{1}{Y_-(x)} = \dfrac{1}{tx} - 1$

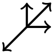
2) Using $(Y_- \circ Y_+)(x) = x$ to find a recurrence for $1/Y_n$

3) explicitly solving the recurrence for $1/Y_n$

we can then (theoretically) use arguments from Complex Analysis and Calculus to show an infinite source of singularities.

Problems with Method

Don't always get convergence in series with Y_n (or hard to show).

Can't always solve for Y_n (this fails for the step set  for example).

Usually not easy to find singularities of Y_n
(MM-AR did walks with 3 steps)

Conclusion

Conclusions

There is a large interplay between Combinatorial and Analytic arguments

Combinatorial arguments are useful for reducing to simpler cases and finding growth factors

Analytic arguments are useful for classifying walk GFs as algebraic, holonomic, non-holonomic, etc.

Open questions

'Nice' $\Rightarrow$ holonomic GF. We'd like asymptotic forms combinatorially. For positive drift, this comes down to finding clever lower bounds.

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Explicit integral representations of some GFs have been found by considering this as a BVP [Raschel 11], but the techniques are quite complicated.

References

K. Raschel, *Counting walks in a quadrant: a unified approach via boundary problems*, 2011.

M. Mishna, A. Rechnitzer, *Two non-holonomic lattice walks in the quarter plane*, 2009.

A. Bostan, M. Kauers, *Automatic classification of restricted lattice walks*, 2009.

M. Bousquet-Melou, M. Mishna, *Walks with small steps in the quarter plane*, 2009.

C. Banderier, P. Flajolet, *Basic analytic combinatorics of directed lattice paths*, 2001.

E. J. Janse van Rensburg, A. Rechnitzer, *Adsorbing and collapsing directed animals*, 2001.

THANK YOU