The exponential growth of restricted lattice paths

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Introduction

A *planar lattice path* model is a combinatorial class, $\mathcal{R}_{\mathfrak{S}}$, defined by a **region**, $R \subseteq \mathbb{Z}^2$, and **direction set** $\mathfrak{S} \subseteq \{0, 1, -1\} \times \{0, 1, -1\}$.

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For example, $\mathcal{Q}_{⅄}$ is the class of walks in the first quadrant with steps from $\mathfrak{S} = \{(0, 1), (1, -1), (-1, -1)\}$.

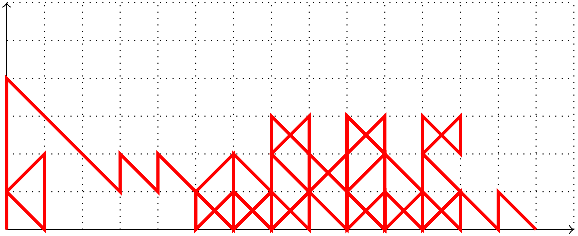

Figure: A quarter plane walk of length 100 taken on $\mathfrak{S} = ⅄$.

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Goal: Given a model $\mathcal{R}_{\mathfrak{S}}$, find $\beta_{\mathfrak{S}}$.

This is the *exponential growth factor*, and we write

$$r_{\mathfrak{S}}(n) \bowtie \beta_{\mathfrak{S}}.$$

Motivation

* In statistical mechanical applications, the exponential growth is the **limiting free energy**, linked to the entropy of the system.

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* Although we can estimate $\beta_{\mathfrak{S}}$ with series computations, we prefer an approach that is **direct, systematic and combinatorial**.

The drift

Each step set $\mathfrak{S}$ carries a parameter called the *drift*. This is the vector sum of a step set:

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\delta(\{\uparrow, \downarrow, \leftarrow, \rightarrow\}) &= 0.
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If the drift vector has at most one non-zero component, we call the drift *simple*.

History

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* **Full plane,** $\mathcal{W}_{\mathfrak{S}}$: trivially enumerated, $\beta_{\mathfrak{S}} = |\mathfrak{S}|$.

* **Half plane,** $\mathcal{H}_{\mathfrak{S}}$: drift dependent, explicit enumerative formulae and asymptotic growth via singularity analysis of generating functions.

* **Quarter plane,** $\mathcal{Q}_{\mathfrak{S}}$: experimental results from series computations, several enumerative strategies. Some sporadic cases solved. Work is ongoing.

We prove exponential growth factors for quarter plane models with simple drift.

Conjectured exponential growth

$\mathfrak{S}$	δ	$\beta_{\mathfrak{S}}^{conj}$		$\mathfrak{S}$	δ	$\beta_{\mathfrak{S}}^{conj}$		$\mathfrak{S}$	δ	$\beta_{\mathfrak{S}}^{conj}$
	$\cdot$	4			$\cdot$	4			$\cdot$	3
	$\cdot$	6			$\cdot$	8			$\cdot$	6
	$\uparrow$	3			$\uparrow$	5			$\cdot$	3
	$\uparrow$	4			$\uparrow$	6			$\cdot$	3
	$\uparrow$	5			$\uparrow$	7			$\cdot$	6
	$\downarrow$	$2\sqrt{2}$			$\downarrow$	$2(1+\sqrt{2})$			$\cdot$	4
	$\downarrow$	$2\sqrt{3}$			$\downarrow$	$2(1+\sqrt{3})$			$\cdot$	4
	$\downarrow$	$2\sqrt{6}$			$\downarrow$	$2(1+\sqrt{6})$				

Table: The number of walks of length n has growth $q_{\mathfrak{S}}(n) \bowtie \beta_{\mathfrak{S}}^{conj}$, Bostan and Kauers.

Proving the exponential growth.

Isolating the exponential growth

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We can isolate the exponential growth of $q_{\mathfrak{S}}(n)$ by taking

$$\lim_{n} \frac{1}{n} \log\left(\kappa \beta_{\mathfrak{S}}^{n} n^{\alpha}\right) = \lim_{n} \frac{\log \kappa}{n} + \lim_{n} \log \beta_{\mathfrak{S}} + \lim_{n} \frac{\alpha \log n}{n}$$

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&= \log \beta_{\mathfrak{S}}.
\end{aligned}$$

If we can bound $q_{\mathfrak{S}}(n)$ by sequences with the same exponential growth, we can use this technique to prove the value of $\beta_{\mathfrak{S}}$ by squeezing.

Upper bounds

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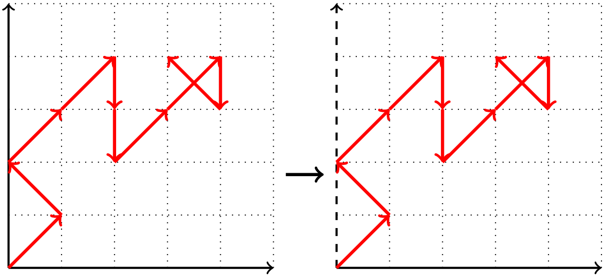
Question: What if we consider the class $\mathcal{H}_{\mathfrak{S}}$ with region $H = \{y \geq 0\}$?

Relaxing the constraints

It turns out that $h_{\mathfrak{S}}(n) \bowtie \beta_{\mathfrak{S}}^{conj}$.

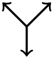


Since $q_{\mathfrak{S}}(n) \leq h_{\mathfrak{S}}(n)$,

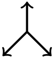
$$\beta_{\mathfrak{S}} \leq \beta_{\mathfrak{S}}^{conj}.$$

Lower bounds

We consider introducing restrictions to produce lower bounds, eg. returning to the origin.



$\leftrightarrow$

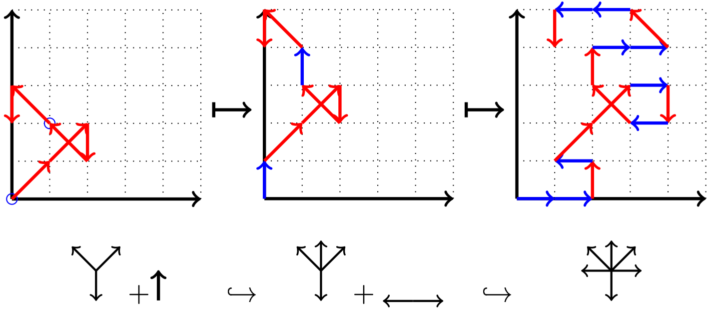


β ≤ 3 but β $\leq 2\sqrt{2}$

The same restriction will not work for all cases, so we are forced into a case analysis.

Lower bounds: the idea

We can insert new steps into old ones to import exponential growth.



Lower bounds

We can reduce to 11 base cases, using the following lemma.

Lemma: Let $d(j)$ be the number of Dyck prefixes of length j and let $q(i) \sim \kappa\beta^i i^\alpha$, where $\alpha \leq 0, \kappa, \beta \in \mathbb{R}^+$. Then:

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$$q'(n) \quad = \quad \sum_{i \geq 0} \binom{n}{i} q(i) \bowtie \beta + 1;$$

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$$q'(n) = \sum_{i \geq 0} \binom{n}{i} q(i) \bowtie \beta + 1; \qquad (1)$$

$$q''(n) = \sum_{i \geq 0} \binom{n}{i} q(i) d(n-i) \bowtie \beta + 2. \qquad (2)$$

Base cases and their children

Base Case	Children
	
	
	
	
	
	
	None
	None
	None
	None
	

Table: A list of base cases and their children

The end results.

$\mathfrak{S}$	δ	$\beta_{\mathfrak{S}}$		$\mathfrak{S}$	δ	$\beta_{\mathfrak{S}}$		$\mathfrak{S}$	δ	$\beta_{\mathfrak{S}}$
	$\cdot$	4			$\cdot$	4			$\cdot$	3
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Table: Conjectured values are proven, Johnson and Mishna.

Perspective.

Drift with two components: upper bounds

We can apply our methods to the larger family of sets $\mathfrak{S}$ with drift in two directions:

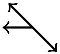
$$\textbf{eg.}\ \nwarrow\uparrow\leftarrow\searrow \;\mapsto\; \nwarrow.$$

When producing upper bounds, we must take care in choosing which boundary to remove.
Let $H^y = \{y \geq 0\}$ and $H^x = \{x \geq 0\}$. Then

$$h^x_{\nwarrow\uparrow\leftarrow\searrow} \bowtie 1 + 2\sqrt{2} \quad \text{but} \quad h^y_{\nwarrow\uparrow\leftarrow\searrow} \bowtie 4.$$

Drift with two components: lower bounds

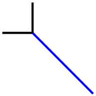
For the same example,  , our lemma isn't as helpful for producing lower bounds. Our only choice of step to remove is $\uparrow$, giving:



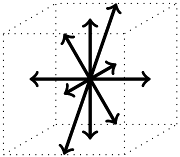
which is a trivial QP model.
This just means that this has to be a base case. We are also working on another approach to build families in this larger class of sets $\mathfrak{S}$.

Generalisations

There are two generalisations to our methods that immediately come to mind.



Larger Steps



Higher Dimensional Regions

Our upper bounds rely on removing a boundary, and our lemma relies on taking the shuffle product of some walks. These are not dependent on step size or lattice dimension.

Larger steps

HP walks on steps of any size are enumerated already, and so this first conjecture is low hanging fruit.

Conjecture: Let $\mathfrak{S}$ be a set of steps of any size, and let $\mathcal{Q}_{\mathfrak{S}}$ be the quarter plane model on $\mathfrak{S}$. Then removing the appropriate boundary will give a half plane model $\mathcal{H}_{\mathfrak{S}}$ with the same exponential growth as $\mathcal{Q}_{\mathfrak{S}}$.

Conjecture: Let $\mathfrak{S}$ be a set of steps of any size, and $\mathcal{Q}_{\mathfrak{S}}$ be the quarter plane model on $\mathfrak{S}$ with exponential growth $\beta_{\mathfrak{S}}$. Then if s_0 is a step of any size not towards a boundary, or $\{s_1, s_2\}$ is a pair of steps of any size with drift not towards a boundary, we can apply our Lemma to get

$$\begin{aligned} \beta + 1 &\leq \beta_{\mathfrak{S} \cup \{s_0\}}, \\ \beta + 2 &\leq \beta_{\mathfrak{S} \cup \{s_1, s_2\}}. \end{aligned}$$

Higher dimensional regions

Conjecture: Let $\mathfrak{S}$ be a set of small steps in $\mathbb{Z}^d$, and let $\mathcal{O}_{\mathfrak{S}}$ be the first orthant model on $\mathfrak{S}$. Then removing the appropriate boundary will give a model $\mathcal{O}_{\mathfrak{S}}^+$ with the same exponential growth as $\mathcal{O}_{\mathfrak{S}}$

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Other future endeavours

* Removing case analysis from the method entirely: producing an automatic, unified method of producing lower bounds.

* Extending our method to subexponential factors: understanding the singular structure of the associated generating functions, classifying singularities.

THANKS FOR LISTENING!

References

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