The Evolution of Risk Attitudes
with Fertility Thresholds*

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Abstract We examine the evolutionary selection of attitudes toward aggregate risk in an age structured population. Aggregate shocks perturb the population's consumption possibilities. Consumption is converted to fertility via a technology that exhibits first increasing and then decreasing returns to scale, captured in the simplest case by a fertility threshold. We show that evolution will select preferences that exhibit arbitrarily high aversion to aggregate risks with even very small probabilities of sufficiently low outcomes. These findings complement the familiar result that evolution will select for greater aversion to aggregate than idiosyncratic risks by identifying circumstances under which the difference can be extreme.

The Evolution of Risk Attitudes with Fertility Thresholds

1   Introduction

Robson [17] is the point of departure for a literature in economics built on the observation that evolution will select for preferences that are more averse to aggregate risk than to idiosyncratic risk.[1] In this paper we identify conditions under which such evolved preferences can exhibit *arbitrarily* higher aversion to aggregate risk than to idiosyncratic risk. We do so in in a continuous-time dynamic model in which the pure rate of time preference and attitudes to idiosyncratic risk remain at moderate levels.

A direct approach to the evolutionary foundations of risk attitudes supposes that evolution induces preferences over lotteries, where the outcomes of these lotteries are expected numbers of offspring. With the simplest discrete-time life history, Robson [17] shows that evolution will then select for preferences over idiosyncratic lotteries that maximize the expected number of offspring, and will select for preferences over aggregate risks that maximize the expected (natural) log of offspring. This gives risk-neutral preferences over idiosyncratic risks and constant-relative-risk-aversion preferences over aggregate risks, with a coefficient of relative risk aversion equal to 1.

The intended interpretation of evolutionary models of risk preferences is not that people consciously choose among lotteries over offspring, but rather that people choose between lotteries with material outcomes that affect reproduction. For most of our evolutionary history, these material outcomes involved resources such as food, shelter, safety from predators, and access to mates, while in our contemporary environment, income may serve as a useful proxy for the suite of relevant material rewards. In this paper, we explicitly incorporate the technology by which material rewards affect reproduction into the model. We examine a relationship between material input and reproduction that is inspired by an empirical literature in biology, supposing that the reproduction technology initially exhibits increasing returns to scale followed by subsequent decreasing returns.

Sections 2-3 consider a simple discrete-time model. Here, the forces behind our most striking finding, that evolution selects for arbitrarily high aversion to sufficiently severe aggregate risk, emerge readily. Robatto and

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[1]See also Robatto and Szentes [16], Robson [18], Robson and Samuelson [20, 21], Sinn and Weichenrieder [24], and Zhang, Brennan and Lo [31].

Szentes [16] suggest caution in drawing conclusions about aggregate uncertainty from discrete models. They show that if age is not demographically relevant, aggregate and idiosyncratic risks are equivalent in continuous time. Expanding on their analysis, Robson and Samuelson [21] show that, once differentiation by age arises, the equivalence again fails. In light of this interchange, it is important that Section 4 obtains similar results from a continuous-time model of an age-structured population.

Section 5 builds a resource allocation problem into the model, allowing us to capture some essential features of human life histories. A "grandmother effect" now appears, in that evolution selects for life histories that invest in the survival of people who themselves have no fertility. Moreover, we find that evolution responds to adverse aggregate shocks by contracting the period of fertility, and to advantageous shocks by expanding this range. As before, we find extreme aversion to sufficiently severe aggregate risk.

Our work generalizes that of Robson and Orr [?], who derived similar results in static and two-period models. Our continuous time generalization allows us to incorporate an age-structured population and to consider resource allocation. The qualitative result that evolution will select for extreme aversion to aggregate risks with low outcomes remains, while new results emerge, notably the attenuation of this effect induced by interaction between shocks and the period of fertility. Robson and Orr suggested that these results provide a potential resolution of the equity premium puzzle—evolution might imply a high degree of aversion to aggregate risk, such as asset market risk, while aversion to idiosyncratic risk remains moderate. Our concluding discussion returns to this point.

2 Idiosyncratic and Aggregate Risk

We begin with the simplest possible model, presented informally. Consider a population in which each individual lives for a single period. During that period, the agent selects a lottery from a set of feasible lotteries. The selected lottery draws a material reward c from a cumulative distribution denoted by F. This reward in turn allows the agent to produce $g(c)$ offspring, where g is convex-concave, as described precisely later in this section.

Our interpretation of this setting is that the choice of lottery corresponds (in our evolutionary environment) to a choice of what food to hunt and how to pursue it, what food to gather, where to live, what alliances to form,

and so on. The selected lottery induces a randomly determined consumption c, which in turn leads to offspring $g(c)$. Individuals are characterized by the preferences that induce (or, in a revealed-preference interpretation, that describe) their choices of lotteries. These preferences are heritable, and evolution selects for the preferences that maximize the population's growth rate.

It is a familiar result (cf. Robson [17]) that if the uncertainty facing the agents in this setting is idiosyncratic, meaning that the lotteries chosen by the various agents are independent, then evolution will select for preferences inducing choices that maximize the expected number of offspring. Hence, an agent will choose from the set of feasible lotteries that which maximizes

$$\int_0^\infty g(c)dF(c), \tag{1}$$

inducing a growth rate given by

$$\ln\left(\int_0^\infty g(c)dF(c)\right). \tag{2}$$

The intuition is that as the population gets large, the law of large numbers will ensure that in the population as a whole, the number of offspring emerging from the various lotteries will be very close to the expected number, so that maximizing expected offspring maximizes the population growth rate.

Suppose instead that the uncertainty is aggregate, meaning that the outcomes of all agents choosing a given lottery are perfectly correlated. Familiar arguments (cf. Robson [17]) establish that evolution will then select for preferences that maximize

$$\int_0^\infty \ln g(c)dF(c), \tag{3}$$

which is the relevant long run growth rate in this case.

Much will depend on the nature of the function g that converts consumption into offspring. If g is linear, as in the common case in which c is measured directly in terms of offspring, then evolution will select for risk neutrality for idiosyncratic risks and will select the constant relative risk aversion utility function $\ln c$ for aggregate risks.

We assume that g is increasing and bounded—more consumption leads to more offspring—with $g(0) = 0$. In addition, we assume that the second derivative g'' is (at least weakly) positive for small values of c and negative

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for larger values of c. A particularly tractable limiting case is to assume that $g(c) = 0$ for all x below some threshold $\underline{c}$, after which $g(c)$ is concave.

Assuming that g'' is positive for small values of c and negative for large values of c gives us the first-convex-then-concave form typically ascribed to production functions in intermediate-economics discussions of competitive markets. Our preferred interpretation, literally applicable when g is zero up to some threshold $\underline{c}$, is that the shape of g reflects the existence of a survival threshold or a minimal resource level required for fertility. Such thresholds commonly appear in models of foraging under risk, and there is evidence that human fertility virtually disappears at low nutrition levels.[2]

If the uncertainty is idiosyncratic, then the utility maximization problem given by (1) will cause the agent to be adverse to risks over a range of higher outcomes and to seek risks over a lower range.[3] The preferences over aggregate risk captured by (3) can give rise to significantly more risk aversion that the references over idiosyncratic risk captured by (1). Any lottery that attaches a mass point to 0 will give an arbitrarily small payoff under (3), as long as $g(0) = 0$. Under the specification that $g(c) = 0$ for all $c \leq \underline{c}$, the same will be true of any lottery that puts positive mass below $\underline{c}$. Suppose that g is smooth and that the lottery has no mass points, with (3) being equivalent to the criterion $\int_0^\infty \ln g(c) f(c) dc$ for some density f. Then a lottery will generate arbitrarily negative payoffs, and hence extreme risk aversion, if the distribution F concentrates enough mass near zero. This will be the case, for example, if f is continuous and $f(0)$ sufficiently large. In all of these cases, evolution will select for preferences that are arbitrarily averse to such risks.

3   Risk Preferences and Threshold Fertility

This section explores the risk implications of the basic model, by analyzing an example incorporating a fertility threshold.

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[2]See Ball, Barnes and Visscher [2], Gopalan and Naidu [7], Stephens and Krebs [27], and Real and Caraco [15].

[3]Clotfelter and Cook [?] show that poorer people spend a larger fraction of income on lottery tickets than do richer people. This is also reminiscent of Friedman and Savage's [6] observation that people commonly buy both insurance and lottery tickets. The details differ, however, in that their utility function is first concave and then convex. The convex-then-concave shape invoked here also plays a prominent role in prospect theory (Kahneman and Tversky [11], Wakker [29]), where the inflection point in prospect theory reflects a reference level that can shift depending on the circumstances and framing of the decision.

For comparison, suppose first the function $g(c)$ does *not* exhibit a fertility threshold, and is given by

$$g(c) = k(1 - (1 + c)^{-\alpha}),$$

for all $c \geq 0$, where $k > 0$, $\alpha > 0$. This captures an upper bound on the number of offspring given by k, which is approached as c gets arbitrarily large. The function g is strictly increasing and concave, with $g(0) = 0$.

For idiosyncratic risks, for which the objective given by (1) is applicable, we can calculate the coefficients of absolute and relative risk aversion to be

$$-\frac{g''(c)}{g'(c)} = (\alpha + 1)\frac{1}{1 + c}$$
$$-\frac{cg''(c)}{g'(c)} = (\alpha + 1)\frac{c}{1 + c}.$$

As is intuitive, absolute risk aversion decreases to zero as c increases. This is expected, as $g(c)$ becomes nearly constant.

Now suppose we are concerned with aggregate risk, so that (3) is applicable. Then we have coefficients of absolute and relative risk aversion given by

$$-\frac{(\ln g(c))''}{(\ln g(c))'} = \frac{(\alpha + 1) - (1 + c)^{-\alpha}}{1 - (1 + c)^{-\alpha}}\frac{1}{1 + c}$$
$$-\frac{c(\ln g(c))''}{(\ln g(c))'} = \frac{(\alpha + 1) - (1 + c)^{-\alpha}}{1 - (1 + c)^{-\alpha}}\frac{c}{1 + c}.$$

Because $\frac{(\alpha+1)-(1+c)^{-\alpha}}{1-(1+c)^{-\alpha}} > (\alpha + 1)$, both measures of risk aversion are larger in the case of aggregate rather than idiosyncratic risk. As c grows arbitrarily large, both measures of aversion to aggregate risk converge to those of idiosyncratic risk. The log transformation of the (here) nearly constant function g has little effect. As c approaches zero the coefficient of absolute risk aversion explodes to infinity, while the coefficient of relative risk aversion approaches one.

We thus get larger measures of relative risk aversion when dealing with aggregate as opposed to idiosyncratic risk, but not immensely larger.

The contrast is more striking once we incorporate a fertility threshold. To capture this in a simple form, suppose the reproduction technology is given by

$$g(c) = \begin{cases} 0 & c \leq \underline{c} \\ k(1 - (1 + (c - \underline{c}))^{-\alpha}) & c \geq \underline{c} \end{cases}, \tag{4}$$
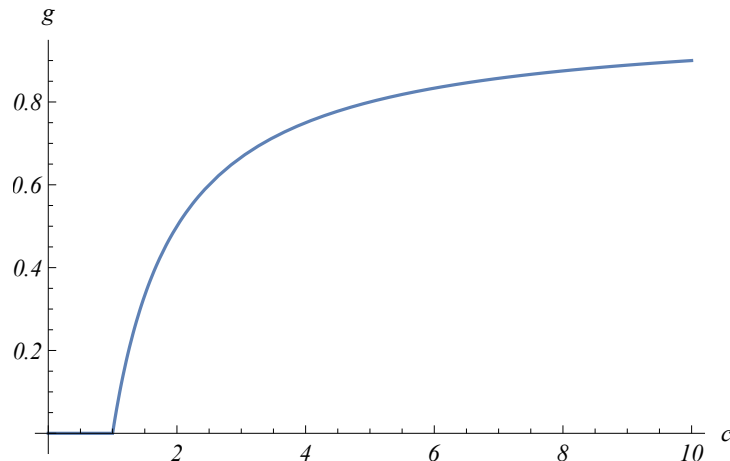
Figure 1: Fertility function (4), giving fertility g as a function of consumption c (with $k = \underline{c} = \alpha = 1$).

where $\underline{c}$ is the fertility threshold. Figure 1 illustrates.

If we restrict attention to values of c above $\underline{c}$, we find risk attitudes that are more averse to aggregate than to idiosyncratic risk, with the difference now becoming dramatic at consumption levels close to the fertility threshold. To keep the notation uncluttered, adopt the normalization that $\underline{c} = 1$. The coefficient of relative risk aversion to idiosyncratic risk is given by

$$-\frac{cg''(c)}{g'(c)} = \alpha + 1, \tag{5}$$

giving us a constant relative risk aversion utility function, for which conventional estimates would place the value of α not too far from one. The coefficient of relative risk aversion to aggregate risk is given by

$$-\frac{c(\ln g(c))''}{(\ln g(c))'} = \frac{(\alpha + 1) - c^{-\alpha}}{1 - c^{-\alpha}}. \tag{6}$$

The latter is larger than the former. More importantly, no matter what the value of $\alpha \geq 0$, the coefficient of relative risk aversion to aggregate risks now explodes to infinity as the value of consumption approaches 1, or more generally approaches the fertility threshold $\underline{c}$. Figure 2 illustrates.

We can illustrate the source of the extreme aversion to aggregate lotteries that place mass near the fertility threshold. Given the fertility function (4),
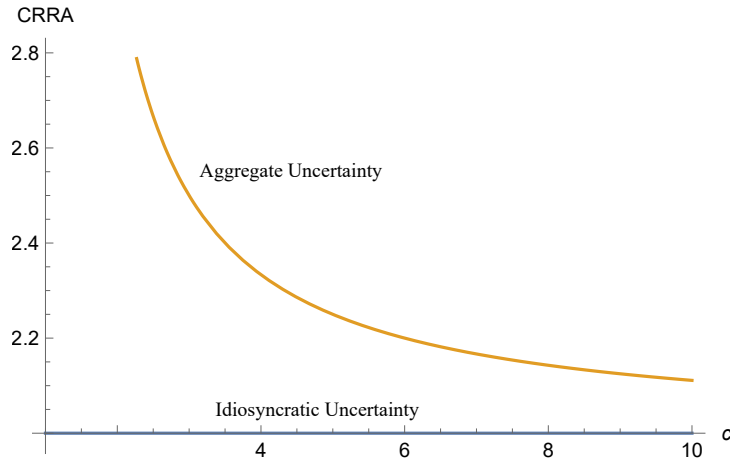
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Figure 2: Graph of coefficient of relative risk aversion (CRRA, vertical axis) as a function of consumption c, for the case of idiosyncratic uncertainty (equation (5), horizontal line) and aggregate uncertainty (equation (6), curve, approaching infinity as c approaches 1), where g is given by (4) with $k = \underline{c} = \alpha = 1$.

consider a lottery over consumption levels whose outcomes are governed by a uniform distribution. Fixing the mean of this distribution, let the support of the distribution expand so that its lower boundary falls below the fertility threshold $\underline{c}$. If the lottery is idiosyncratic, then the expanding spread will reduce the population growth rate, reflecting aversion to the induced risk, but this rate will remain finite. As the lower bound of the distribution moves past the fertility threshold $\underline{c}$, the lottery confronts each individual with the possibility of zero offspring, but the average number of offspring remains positive and hence the population growth rate remains bounded below. In contrast, under aggregate uncertainty, the population growth rate approaches negative infinity as the lower bound of the lottery's support moves past the consumption threshold. Once positive mass slips below the threshold, the population faces eventual extinction.

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4 Age-Structured Population, Continuous Time

Motivated by the results of Robatto and Szentes [16] and Robson and Samuelson [21] (see Section 1), we extend the analysis to a continuous-time model with a general age-structured population. This section assumes there is an exogenous trajectory for consumption which is subject to aggregate shocks. Section 5 allows the consumption trajectory to be endogenously determined as an optimal tradeoff between resources allocated to increase fertility and those to reduce mortality. Resources are then subject to aggregate shocks.

4.1 The Evolutionary Setting

To construct a more general model, let time be continuous. Each individual survives from birth until age $A > 0$. Individuals may reproduce at ages younger than A, but all reproduction ceases at age A. We can view individuals as dying at age A, or as living forever after, or as facing any intermediate pattern of death, all with equivalent implications for the population growth rate and hence for the evolutionary selection of preferences. We could also incorporate an idiosyncratic, age-dependent risk of death at each age $a \in [0, A]$, complicating the notation but leaving the results intact.[4]

We let the scalar w denote the state of the environment. Consumption at age a in state w is given by $wc(a)$, where $c : [0, A] \to (0, \infty)$ is continuously differentiable. We view the function $c(a) > 0$ as fixed, reflecting a combination of technological factors and behavioral factors selected by evolution.[5]

A twice continuously differentiable fertility function $\hat{g}(c, a)$ gives the fertility of an individual of age $a \in [0, A]$ who consumes $c \geq 0$. We assume that $\hat{g}(c, a) = 0$ for all $c \geq 0$ and all $a < M$ where $M < A$ is the age of first reproduction. Further, if $a \in [M, A]$, then $g(c, a) = 0$, for all $c \leq \bar{c}_a$. This captures that there is an age-dependent consumption threshold, $\bar{c}_a > 0$,

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[4]More precisely, idiosyncratic mortality before age A can be subsumed in fertility. That is, the expected number of offspring at any age a in the present formulation can be reinterpreted as the product of the probability of survival to that age and the actual expected number of offspring produced at that age conditional on survival.

[5]As the next section shows, the evolutionarily optimal choice of c should take into account how resources promote fertility and survival, with survival being important in promoting future fertility. For example, resources should be devoted to the survival of individuals younger than the age of first reproduction, even though they currently produce no offspring.

required for reproduction at age $a \in [M, A]$. Suppose $\bar{c}_a$ is continuously differentiable in a. For all $c > \bar{c}_a$ at any age $a \in [M, A]$, we assume $\hat{g}_c(c, a) > 0$ (so that $\hat{g}(c, a) > 0$ for $a \in [M, A]$) and $\hat{g}_{cc}(c, a) < 0$. Above the threshold for reproduction, for ages $a \in [M, A]$, increasing consumption gives rise to increasing offspring, at a decreasing rate.

It is convenient to redefine fertility as a function of the state of the environment w and age, letting $g(w, a) = \hat{g}(wc(a), a)$. If $a \in [M, A]$, then $w_a c(a) = \bar{c}_a$ has a unique continuously differentiable solution for $w_a > 0$. It follows that $g(w, a) = 0, \forall w \leq w_a$, and $g(w, a) > 0, g_w(w, a) > 0, g_{ww}(w, a) < 0, \forall w > w_a$. That is, the function g inherits the qualitative features of the $\hat{g}$ function. We assume that for any state w, we have $g(w, a) > 0$ on an interval $(\underline{a}(w), \bar{a}(w)) \subseteq (M, A)$.

If the state of the environment is fixed at some w, then the growth rate $\lambda(w)$ of the population solves the Euler-Lotka equation (Charlesworth [3, p. 23]; see Robson and Samuelson [21, Proposition 6] for a discrete foundation):

$$\int_M^A g(w, a)e^{-\lambda a}da = 1. \tag{7}$$

The population converges to a steady-state age structure exhibiting growth rate $\lambda(w)$.

We now incorporate aggregate uncertainty in the state of the environment by assuming that the value of w, common across all members of the population, is determined by a lottery G, in the sense that a Poisson process generates "arrivals," and that at each such arrival a new value of w is drawn according to the cumulative distribution G. This value of w then persists until the next arrival. Each possible lottery G induces a long-run population growth rate. Evolution will select for preferences over lotteries, with lottery G preferred to lottery $\tilde{G}$ if the former gives rise to a larger long-run population growth rate. For example, different methods of foraging or food production may induce different aggregate lotteries, and evolution will shape preferences over such lotteries so as to maximize the population growth rate.

To simplify the analysis of long-run population growth rates, we follow Robson and Samuelson [21] in focusing on an approximation obtained by examining the limit as the Poisson arrivals heralding changes in the aggregate state become arbitrarily infrequent. We view this an an analytically convenient approximation of the case in which changes in the aggregate state are rare.

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Let the set W of possible environments be finite. Suppose that $\max_{w \in W} \underline{a}(w) < \min_{w \in W} \overline{a}(w)$ and $A - M < \min_{w \in W} \overline{a}(w)$.[6] Hence, there are some ages assured of reproduction. The Appendix builds on Proposition 1 of Robson and Samuelson [21] to prove:

Proposition 1 *In the limit as the Poisson arrival rate of a new aggregate state becomes arbitrarily infrequent, the long-run population growth rate induced by lottery G is given by*

$$\int_{w \in \mathbb{R}_+} \lambda(w) dG(w), \tag{8}$$

where $\lambda(w)$ is given by (7).

Evolution will then select for preferences over aggregate lotteries that correspond to maximizing (8).

4.2 Implications for Risk-Taking

We say that an individual is arbitrarily averse to a lottery if they would prefer every certain outcome w inducing fertility $g(w) > 0$ to such a lottery, no matter how small is $g(w)$. Equivalently, we say that such an individual exhibits extreme risk aversion.

First, to take a simple example, suppose that $M = 0$, so fertility begins immediately, and that fertility is age-independent up to age A, given simply by $g(w)$. Suppose there is an age-independent fertility threshold, $\underline{w}$, say, so $g(w) = 0$ for $w \le \underline{w}$. We can then drop the argument a from the function g and write the Euler-Lotka (7) equation as

$$1 = \int_0^A g(w) e^{-\lambda a} da = \begin{cases} g(w)\frac{1 - e^{-\lambda A}}{\lambda} & \text{if } \lambda \ne 0 \\ g(w)A & \text{if } \lambda = 0 \end{cases} \tag{9}$$

It follows, as is true in general, that there exists a unique λ solving (9) for each $g(w) > 0$. Further, if $g(w) > 1/A$, then $\lambda > 0$; if $g(w) = 1/A$ then

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[6]These assumptions ensure that the convergence of the population to its steady-state age distribution is uniform in the initial population distribution and across states of the environment, and are needed to establish (8). The intuitive interpretation of these assumptions is that different environments may have a significant effect on the expected number of offspring at each age, but have a limited effect on the range of ages for which expected fertility is positive. The Appendix provides details of the argument.

$\lambda = 0$; and, if $g(w) < 1/A$, then $\lambda < 0$. It is immediate that, if $g(w) \downarrow 0$, then $\lambda \to -\infty$.

The finding that $\lambda \to -\infty$ as $g(w) \downarrow 0$ ensures that the individual will be arbitrarily averse to lotteries that attach positive probability to consumption levels that fall below the fertility threshold.

Now consider a more general case, in which we may have $M > 0$ and fertility need not be constant over the interval $[M, A]$. Let

$$
\begin{aligned}
\underline{w} &= \min_{a \in [M,A]} w_a \\
\overline{w} &= \max_{a \in [M,A]} w_a
\end{aligned}
$$

define the minimum (across ages) and maximum fertility thresholds. Notice that $\underline{w} \leq \overline{w}$. Then extreme risk aversion arises as follows:

Proposition 2 *Suppose $\underline{w} > 0$. Then:*

[2.1] The growth rate $\lambda(w)$ approaches $-\infty$ as w approaches $\underline{w} > 0$. Thus agents are arbitrarily averse to aggregate lotteries placing mass below $\underline{w}$.

[2.2] Preferences will be risk averse for aggregate lotteries for which $w \geq \overline{w}$.

Proof [2.1] follows by contradiction. That is, if this were not true, there exists a $\underline{\lambda} \in (0, -\infty)$ and a sequence $w_n \to \underline{w}$ with associated $\lambda_n \geq \underline{\lambda}$. Now

$$
1 = \int_M^A g(w_n, a) e^{-\lambda_n a} da \leq \int_M^A g(w_n, a) e^{-\underline{\lambda} a} da \to 0,
$$

a contradiction.

To establish [2.2], suppose $w > \overline{w}$, the maximum fertility threshold. Then we can differentiate the Euler-Lotka equation (7) to obtain

$$
\int_M^A g_w(w, a) e^{-\lambda a} da - \int_M^A a g(w, a) e^{-\lambda a} da \frac{d\lambda}{dw} = 0 \tag{10}
$$

and hence

$$
\int_M^A g_{ww}(w, a) e^{-\lambda a} da - 2 \int_M^A a g_w(w, a) e^{-\lambda a} da \frac{d\lambda}{dw} - \int_M^A a g(w, a) e^{-\lambda a} da \frac{d^2\lambda}{dw^2} = 0. \tag{11}
$$

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It follows from these two equations that

$$
\begin{aligned}
sgn\frac{d^2\lambda}{dw^2} &= sgn\left(\int_M^A g_{ww}e^{-\lambda a}da \int_M^A age^{-\lambda a}da - 2\int_M^A ag_we^{-\lambda a}da \int_M^A g_we^{-\lambda a}da\right) \\
&= -1.
\end{aligned}
$$

This ensures that individuals will be risk averse over lotteries all of whose consequences remain above the fertility threshold. ∎

To contrast these results with idiosyncratic risk, suppose consumption is subject to idiosyncratic risk and is given by $\tilde{c}$ at age a, with realized fertility given by $g(\tilde{c}, a)$. The effect of this idiosyncratic uncertainty is fully captured by taking the expectation of fertility. That is, the optimal choice of idiosyncratic lottery at age a will maximize $\mathbb{E}g(\tilde{c}, a)$, since idiosyncratic risk concerning fertility should be evaluated at its mean and the left hand side of the Euler-Lotka equation (7) must be maximized at the maximum growth rate λ (see Robson and Samuelson [21]). Attitudes toward idiosyncratic risk thus remain moderate, governed by the curvature of the function g.

An analogous conclusion holds for intertemporal tradeoffs. Suppose that consumption at any age b is increased by a spike so that overall consumption is $c + \varepsilon_b\delta(a - b)$, where δ is the Dirac function. From the Euler-Lotka equation (7), it follows that

$$
\frac{d\lambda}{d\varepsilon_b} = \frac{g_c(c, b)e^{-\lambda b}}{\int_M^A g(c, a)ae^{-\lambda a}da} \quad \text{at } \varepsilon_b = 0 \tag{12}
$$

so that the marginal rate of substitution between consumption at times b and d is

$$
MRS_{bd} = \frac{\frac{d\lambda}{d\varepsilon_b}}{\frac{d\lambda}{d\varepsilon_d}} = \frac{g_c(c, b)e^{-\lambda b}}{g_c(c, d)e^{-\lambda d}} \quad \text{at } \varepsilon_b = \varepsilon_d = 0. \tag{13}
$$

That is, attitudes to idiosyncratic risk and intertemporal tradeoffs are linked together and both derive from the criterion $\int_M^A g(c, a)e^{-\lambda a}da$. Attitudes toward idiosyncratic risk will then be shaped by the curvature of the function g, and attitudes toward intertemporal tradeoffs will be shaped by the curvature of g and the population growth rate λ. Both will remain at modest levels, in contrast to the arbitrarily large aversion to aggregate lotteries with even small probabilities of sufficiently low outcomes.

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5 Resource Allocation

We now incorporate a resource allocation problem that involves social transfers, finding that basic features of human life histories emerge endogenously. We restrict the analysis to the steady state.[7]

5.1 The Evolutionary Setting

We begin with no uncertainty. Suppose an individual of age a produces output $y(a) \geq 0$, for all $a \in [0, A]$. These resources can be used to reduce mortality or promote fertility, and, although no storage is possible, such resources are freely transferable across the age cohorts that are present at a particular date. We let $s(a) \geq 0$ be the flow of resources devoted to reducing morality at age a, with $c(a) \geq 0$ devoted to promoting fertility at age a.

Various frictions may impede resource flows across age. We exclude these from the model, noting that this may be a reasonable approximation for hunter-gatherer societies, where resource transfers are significant (see Kaplan, Hill, Hurtado and Lancaster [12] and Kaplan and Robson [13]).

We assume the population is characterized by a steady state with growth rate λ.

We let $p(a)$ denote the probability that an individual survives until age a. In the steady state, the ratio of individuals of age a to individuals of age $\tilde{a} < a$ is then given by $e^{-\lambda a} p(a) / e^{-\lambda \tilde{a}} p(\tilde{a})$. The steady state social budget constraint is then

$$\int_0^A e^{-\lambda a} p(a)(y(a) - c(a) - s(a))da = 0,$$

indicating that the society's total resources ($\int_0^A e^{-\lambda a} p(a) y(a) da$) are divided between morality reduction ($\int_0^A e^{-\lambda a} p(a) s(a) da$) and fertility promotion ($\int_0^A e^{-\lambda a} p(a) c(a) da$).

We let $g(c, a)$ be the fertility of an agent of age who devotes c resources to fertility. We again adopt a threshold formulation, so that $g(c, a) = 0$

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[7]Since it is not straightforward to model social transfers outside the steady state, we leave for future work the full dynamic examination of age-structured populations with inter-age resource transfers. This examination would involve extending the results of Section 4 that underpin equation (8), which are basically those provided by Robson and Samuelson [21] for an age-structured population *without* resource transfers.

for $c \leq \bar{c}_a$, and $g(c, a)$ is twice continuously differentiable, with $g_c(c, a) > 0$ and $g_{cc}(c, a) < 0$, for all $c > \bar{c}_a > 0$. Further, $g_c(c, a) \to 0$ as $c \to \infty$ for all $a \in [M, A]$. As we shall see, this formulation naturally gives rise to endogenous menarche and menopause.

It follows that

$$\xi_a^* = \max_{c \geq \bar{c}_a} \frac{g(c, a)}{c}$$

is well-defined and continuous in $a \in [M, A]$. Further, it will useful to note that

$$\arg \max_c \{g(c, a) - \xi c\} = \begin{cases} 0 & \text{if} \quad \xi > \xi_a^* \\ \{0, c_a^*\} \text{ where } c_a^* > \bar{c}_a & \text{if} \quad \xi = \xi_a^* \\ > c_a^* & \text{if} \quad \xi < \xi_a^* \end{cases} . \quad (14)$$

We assume that ξ_a^* is hump shaped in a—first increasing and then decreasing. This captures the reproductive advantage of young adults over children and the elderly. For example, this is satisfied if $g(c, a)$ is replaced by $h(a)g(c)$, where h is first increasing and then decreasing.

Suppose the mortality rate of an individual of age a who uses resources $s \geq 0$ is $r(s, a)$ which twice continuously differentiable in $s \geq 0$ and $a \in [0, A]$, with $r_s(s, a) < 0$ and $r_{ss}(s, a) > 0$. Further, $r_s(s, a) \to -\infty$ as $s \downarrow 0$, for all $a \in [0, A]$. We have that the probability of survival to age a satisfies

$$\frac{dp}{da} = -pr(s, a).$$

The basic evolutionary problem is then

$$\max_{c, s} \lambda \quad (15)$$

subject to

$$\int_0^A e^{-\lambda a} p(a) g(c, a) da = 1 \quad (16)$$

$$\int_0^A e^{-\lambda a} p(a)(y - c - s) da = 0 \quad (17)$$

$$\frac{dp}{da} = -pr(s, a). \quad (18)$$

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5.2 Equilibrium

It is convenient to consider the following auxiliary problem, for each fixed $\lambda \in (-\infty, \infty)$:

$$\max_{c,s} \int_0^A e^{-\lambda a} p(a) g(c, a) da \equiv V(\lambda) \qquad (19)$$

subject to the constraints (17) and (18).

How does the solution to the auxiliary problem as in (19) generate a solution to the original problem as in (15)? It is immediate that

$$V(\lambda) \to \begin{cases} \infty & \text{if} \quad \lambda \to -\infty \\ 0 & \text{if} \quad \lambda \to \infty. \end{cases}$$

Since $V(\lambda)$ is continuous (indeed, differentiable by the envelope theorem, see LaFrance and Barney [14]), we can then define

$$\lambda^* = \max\{\lambda | V(\lambda) = 1\}.$$

It follows that λ^* is the maximal growth rate for the basic problem (15) with the solution for the controls and states from the auxiliary problem for λ^*. If not, there exists some feasible $\lambda^{**} > \lambda^*$. Since $V(\lambda^{**}) < 1$ and $V(\lambda^{**})$ is the maximum feasible V given λ^{**}, this is a contradiction.

The necessary conditions for maximizing (19) follow from the maximum principle. The Hamiltonian is then

$$\mathcal{H} = e^{-\lambda a} p(a) g(c, a) + \xi e^{-\lambda a} p(a) (y - c - s) - \psi p(a) r(s, a), \qquad (20)$$

where ξ and ψ are the associated costate variables for the constraints (17) and (18) respectively. Since (17) is "isoperimetric", the multiplier ξ is constant. In addition,

$$\frac{d\psi}{da} = -\frac{d\mathcal{H}}{dp} = -e^{-\lambda a} g - \xi e^{-\lambda a} (y - c - s) + \psi r. \qquad (21)$$

It follows that

$$\frac{d(p\psi)}{da} = -p(g + \xi(y - c - s)) e^{-\lambda a},$$

so that, integrating,

$$e^{\lambda a} \psi = \frac{\int_a^A p\left(g + \xi(y - c - s)\right) e^{-\lambda a'} da'}{p(a) e^{-\lambda a}}, \qquad (22)$$

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using the transversality condition that $\psi(A) = 0$. Hence $e^{\lambda a}\psi$ is the reproductive value of an individual of age a generalized to allow for future contributions to economic output as well as to future fertility.

Even if there is no fertility remaining, such an individual will have positive reproductive value if economic contributions still lie ahead. Indeed, maximizing $\mathcal{H}$ over $s \geq 0$ implies that, since there must be an interior solution for all $a \in [0, A]$,

$$-\psi e^{\lambda a} r_s(s, a) = \xi,$$

so that it pays to invest in the survival of anyone with $\psi > 0$, even if that individual has no remaining fertility.

The optimal choice of c satisfies

$$\max_c g(c, a) - \xi c.$$

Given the threshold formulation for g, it follows from (14) that

$$
\begin{aligned}
c > 0 \text{ and } g_c(c, a) = \xi, & \quad \text{if} \quad \xi < \xi_a^* \\
c = 0 \text{ or } c_a^* & \quad \text{if} \quad \xi = \xi_a^* \\
c = 0 & \quad \text{if} \quad \xi > \xi_a^*.
\end{aligned}
$$

That is, the marginal product of resources used for fertility should be constant across age, for all interior solutions. Given that ξ_a^* is hump shaped in a, the general solution involves then an initial range where $\xi > \xi_a^*$ and $c = 0$, followed by an intermediate range where $\xi < \xi_a^*$ and $c > 0$, and a final range where $\xi > \xi_a^*$ and $c = 0$ again.

This gives us necessary conditions for a solution. Because the maximized Hamiltonian $\mathcal{H}$ is linear in the only nontrivial state variable p and hence concave in that state variable, sufficiency follows from the results of Seierstad and Sydsaeter [23, Theorem 7].

5.3 Implications

We first note that menarche and menopause arise endogenously as part of the equilibrium. Reproduction occurs over an interval of ages $[\underline{a}, \overline{a}]$. Outside of this interval, no resources are devoted to reproduction. Resources are still devoted to survival, among the young because their survival to reproductive age is valuable, and among the old because they still contribute resources that can be transferred to others. The optimality of the latter, commonly referred

to as the grandmother effect, is a common explanation for why human life histories, relatively atypically among species, exhibit significant longevity after the maximum age of reproduction (e.g., Hawkes and Coxworth [10], Hawkes, O'Connell, Blurton Jones, Alvarez and Charnov [8, 9]).

Next, consider risk attitudes. Idiosyncratic risks to output $y(a)$ at age a will be evaluated at their expected value. This is a reflection of our assumption that resources are freely transferable. These transfers allow the agents to effectively diversify any idiosyncratic risks.

Now consider aggregate risks. Let w denote an environment, interpreted as a draw of the aggregate uncertainty, and let $y(a, w)$ denote the income of an agent of age a in environment w. Let $\lambda(w)$ denote the corresponding growth rate. As the environment becomes sufficiently harsh, the population growth rate becomes arbitrarily small.

Proposition 3 *Let* $\lim_{w \to 0} y(a, w) = 0$, *uniformly in a. Then* $\lim_{w \to 0} \lambda(w) = -\infty$, *and hence agents will become arbitrarily risk averse in the limit as a lottery places mass increasingly close to 0.*

Proof Suppose not. Then there exists $\bar{\lambda} > -\infty$ such that $\lambda \geq \bar{\lambda}$ for a sequence of $w \to 0$. Hence

$$\int_0^A p(a, w)c(a, w)e^{-\lambda a}da \leq \int_0^A p(a, w)y(a, w)e^{-\lambda a}da \leq \int_0^A y(a, w)e^{-\bar{\lambda}a}da \to 0.$$

Hence

$$1 = \int_0^A pge^{-\lambda a}da \leq \int_0^A p\xi_a^* ce^{-\lambda a}da \leq \bar{\xi}\int_0^A pce^{-\lambda a}da \to 0,$$

where $\bar{\xi} = \max_{a \in [0,A]} \xi_a^* < \infty$ This establishes the desired contradiction. ∎

This implies that under the approximation given by (8), agents will be arbitrarily averse to aggregate lotteries that put nonzero weight on sufficiently adverse outcomes, no matter what the remaining composition of the lottery.

The ability to transfer resources across ages allows evolution to moderate the effects of adverse outcomes. In particular, evolution moderates aggregate shocks by adjusting the period of fertility. As a result, aversion to aggregate risks, while still extreme, is less dramatic than in the previous sections. Previously, driving the growth rate to $-\infty$ required only a lottery

with realizations c below the fertility threshold, i.e., low enough (but still positive) that $g(c) = 0$. In the current setting, intergenerational transfers modify the picture. Even if all income levels are shrinking, it never pays for any age to choose a consumption $c > 0$ below the fertility threshold. Rather optimality requires that either $c = 0$ or $c > \bar{c}_a$. The response to shrinking incomes will be then to shrink the set of ages at which $c > \bar{c}_a$, transferring all of the resources devoted to fertility to an ever narrower set of ages, in order to preserve some reproduction.[8] The population growth rate approaches $-\infty$ only as the environment becomes sufficiently bleak as to extinguish all consumption. However, it remains true that low levels of aggregate resources lead to arbitrarily low growth rates and that agents will be arbitrarily averse to aggregate lotteries that place even a small probability on sufficiently adverse outcomes.

The discount factor for resources, as derived from the resource constraint (17), is $e^{-\lambda a}p(a)$. Expressed as a rate of time discount this becomes $\lambda + r$, the sum of the growth rate and the mortality rate (as in Robson and Samuelson [19]). With only rare changes in the environment, the observed rate of time preference is then linked in a standard fashion to the characteristics of each steady state, and will be moderate. Hence the model can again imply an arbitrarily large aversion to aggregate risk, while maintaining plausible attitudes to idiosyncratic risk and to intertemporal tradeoffs.

6   Discussion

The message of this analysis is that evolution will select for arbitrarily strong aversion to aggregate risks that place mass on low outcomes. These risk attitudes evolved in an evolutionary environment in which there was no money, no financial markets, essentially no heritable wealth and no inequality. How do we expect these risk attitudes to be reflected in our contemporary environment?

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[8]This positive relationship between the window of fertility and aggregate resources is consistent with life history data. Thomas, Renaud, Bénéfice, de Meeûs and Guegan [28] conduct a meta-analysis of studies of menarche, finding that age at menarche is positively related to a collection of factors characteristic of rich environments. Weil [30] complies data from a variety or sources showing that the age of menarche decreases as counties become wealthier. Schoenaker, Jackson, Rowlands and Mishra [22] conduct a meta-analysis of studies of menopause, finding later ages of menopause in wealthier societies.

If evolution is to induce different behavior in the face of idiosyncratic and aggregate risks, then evolution must select people to distinguish these risks. Some cases are obvious. The possibility of the Earth's colliding with a massive asteroid is clearly an aggregate risk. The possibility of an appliance failure in one's home is an idiosyncratic risk. But some cases are ambiguous. A standard finding in psychological studies of risk attitudes is that a feeling of control is important in inducing people to be comfortable with risk.[9] Risks arising out of situations in which people feel themselves unable to affect the outcome cause considerably more apprehension than risks arising out of circumstances people perceive themselves to control. People who fear flying think nothing about undertaking a much more dangerous drive home from the airport. From evolution's point of view, "control" may be a convenient stand-in for an idiosyncratic risk.

The risks arising out of asset markets may then trigger attitudes shaped by evolution for aggregate risk, both because financial crises tend to affect either everyone or no one, and because of a feeling that market outcomes are driven by mysterious factors beyond one's control. People may then be especially averse to financial portfolios whose distribution of outcomes exhibits too long a lower tail, even if there is minuscule probability in that lower tail.

The possible implications of these risk attitudes point to a number of topics for further research. Suppose the economy features a distribution of wealth levels. Let people have the opportunity to invest some proportion of their wealth into an asset, such as the stock market, that exhibits a random rate of return. Suppose the mean return is high, even perhaps quite lucrative, but there lurks in the background the specter of a catastrophic "black swan" event. A first implication is that many people may prefer to avoid the market, in the process seemingly revealing degrees of risk aversion vastly higher than seen in other risky decisions. In addition, this effect will be particularly powerful for people at low wealth levels. It may then be that people with wealth levels below some threshold will invest nothing in the asset, with agents with higher wealth levels investing more. This is consistent with the observation that low-wealth people tend to not participate in the stock market, while those with higher wealth do participate.[10] This generates a

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[9]See Slovic, Fishhoff and Lichtenstein [26] for an early development of this idea and Slovic [25] for a more recent discussion.

[10]The driving force behind this pattern is the fertility technology g that is first convex and then concave. When composed with the log function appropriate for evaluating ag-

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force for increasing inequality, as those with low wealth remain trapped at the bottom, while those with higher wealth exploit the stock market and other opportunities to make their wealth grow. We thus have what appears to be a puzzlingly high equity premium leading to behavior that pushes people out of the middle of the income distribution toward either end. Had our risk preferences evolved in an environment exhibiting financial markets and a nontrivial distribution of heritable wealth, our risk preferences might have evolved differently. As it is, preferences well-suited an environment devoid of inequality may exacerbate inequality in our current environment.

The extreme aversion to aggregate risks is driven by the possibility of a realization so low as to preclude reproduction. We suspect that few people in the modern world are subject to adverse shocks so severe as to preclude reproduction, and that this was perhaps true during much of our evolutionary history. Once again, however, we must recognize that evolution faces a challenge in inducing the appropriate risk attitudes. If evolution could simply design people to maximize the (appropriately quality adjusted) quantity of surviving offspring, there would be no difficulty in people adopting the appropriate risk attitudes. However, the computational challenges of this optimization problem forces evolution to induce utilities for intermediate objections such as income or consumption. How, however, evolution faces a challenge in determining the appropriate levels of consumption and income. The solution to this problem is likely to involve placing weight on relative consumption or income levels. As a result, aggregate risks generating outcomes significantly lower than habitual levels may elicit extreme risk aversion.

Our results emerge from an analysis of the case in which aggregate shocks are rare. In particular, an adverse aggregate shock may persist for a period of time longer than the period of fertility. The more quickly do aggregate transitions occur, the less extreme will be the induced risk aversion. We view our analysis as a convenient approximation of cases in which aggregate shocks tend to be long lived. What is required for our analysis to hold is that aggregate shocks persist long enough to have some effect on reproduction. The baby boom after the second World War and the observation that experience with the Great Depression shaped behavior throughout people's remaining lives are examples. Of course, our model captures this an an extreme form,

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gregate lotteries, this technology induces extreme risk aversion to lotteries with weight on low outcomes. In contrast, Friedman and Savage [6] assume the utility function is first concave and then convex (perhaps with a subsequent concave region), causing low-income people to prefer idiosyncratic lotteries that might lift them out of the concave range.

yielding results that serve as analytically convenient approximations.

The models we have examined accommodate a combination of aggregate and idiosyncratic risk. Consider the model of Section 2, previously examined by Robson [17]. Aggregate uncertainty is captured by a sequence of independent (across periods) and identically distributed environmental states $\{w_t\}_{t=0}^{\infty}$. Given each such state, idiosyncratic uncertainty is captured by independent (across agents) and identically distributed random variables determining the number of offspring of each agent. The latter uncertainty typically becomes only implicit upon being replaced by the appropriate mean. The theory of branching processes (Athreya and Ney [1]), which considers the detailed growth of an initially-finite population, provides the foundation for such models. In the present case, the result is that if the population avoids extinction, it grows to infinity at a limiting exponential rate, a rate that we focus upon here.

A relaxation of the independent-and-identically-distributed nature of the w_t that allows for $\{w_t\}_{t=0}^{\infty}$ to satisfy merely "exchangeability'—a particular form of symmetry—has been considered by Athreya and Karlin [?]. They show that if the process avoids extinction, it grows at a limiting rate determined by the product of the mean offspring levels conditional on the realized $\{w_t\}_{t=0}^{\infty}$. These results could be further generalized to allow for an age structured population, though the interpretation of the results in terms of behavior is less straightforward.

Our basic structure involves an aggregate random variable at each date t together with independent draws from a common distribution conditional on the realized state. This structure is more general than it might first appear. It seems reasonable to require that the *ex ante* offspring distributions at any date t satisfy "exchangeability" in the sense of de Finetti. This is the requirement that all permutations of any finite set of individuals at date t have the same joint distribution of offspring. This captures a notion of symmetry or anonymity across individuals that is appropriate in the present biological context. For infinite sets of individuals and Bernoulli distributions, de Finetti [?] showed this implies there is an underlying aggregate state variable with offspring then drawn independently from identical distributions with mean determined by the state. (See Hewitt and Savage [?] for an extension of de Finetti's result to more general distributions than Bernoulli.) These results will not hold exactly if the set of individuals is finite, as will be true in the branching model in general. However, Diaconis and Freedman [?] show these results hold *approximately* if there is a *large* number of individuals, as there

would ultimately be with a limiting positive growth rate. That is, they show that there exists a joint offspring distribution with an underlying aggregate state variable and independent and identically distributed offspring draws given the state that, for large population size n, is close to any given exchangeable joint distribution of offspring at t.[11] Since the limiting growth rate depends primarily on the large t distributions, this suggests that the current results are robust in a reasonable sense. This is a topic for further research.

7 Online Appendix: Proof of Proposition 1

In this appendix we prove Proposition 1, showing that the criterion given by (8) provides a good approximation of the population growth rate when the Poisson arrivals heralding a change in the aggregate state are rare.[12]

Our argument is that each state induces a limiting growth rate that characterizes the population if that state persists for a sufficiently long time, and then that the overall limiting growth rate (as changes become rare) is the expectation of these limiting growth rates in each state. The idea is that when changes are rare, the population will spend almost all of its time close to the limiting growth rate of whatever happens to be the current state, with transitions between state prompting adjustments that have an insignificant effect on the overall growth rate. The basic argument is provided by Proposition 1 of Robson and Samuelson [21]. For this argument to work, we require that for any state and any possible initial population configuration, the population converges to the limiting growth rate for that state (provided the state persists) at a rate that is uniform across states and possible initial population configurations. This appendix establishes the required uniformity, via an argument adapted from and following very closely the argument presented in the online appendix to Robson and Samuelson [21].

The Renewal Equation

Fix a state, and consider a population that enters this state at time 0. Suppose the population distribution is described by $N_a(t)$ where $\int_0^A N_a(0)da = 1$.

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[11]To be precise, the two joint distributions are within a distance of order k/n, where k is the size of the set which is subjected to permutation.

[12]We are very grateful to Ken Wachter for illuminating discussions of these issues.

It is without loss of generality to assume the initial population is of size 1, given that the problem is linear. It is also without essential loss of generality to suppose that individuals die when their reproduction ceases. Individuals could, more generally, live beyond A, and would then swell the size of the population, but this cannot affect the growth rates.

We first develop a description of the evolution of the population. Footnote 4 notes that the fertility rate $g(a)$ (suppressing notation for the fixed state) can be interpreted as the production of the probability of surviving to age a and fertility conditional on survival. Here, for clarity, we separate the two, letting p_a denote the former and μ_a the latter.

Suppose $B(t)$ is the total flow of births at date t. Frauenthal [5, p. 131, Eq (12)] shows that the evolution of the population is governed by the following "renewal equation"

$$B(t) = G(t) + \int_0^t B(t-a)g(a)da \quad \text{where} \quad g(a) = p_a\mu_a$$

and

$$G(t) = \int_0^A N_a(0)\frac{g(a+t)}{p_a}da.$$

Feller [4, Ch XI] provides an elegant analysis of the renewal equation. This can be rewritten as follows. Define $Z(t) = B(t)e^{-\lambda t}$ and $z(t) = G(t)e^{-\lambda t}$, where λ is the unique real root of the Euler-Lotka equation. It follows that the renewal equation becomes

$$Z(t) = z(t) + \int_0^t Z(t-a)f(a)da \quad \text{where} \quad f(a) = g(a)e^{-\lambda a}.$$

Since $\int_0^A f(a)da = 1$, by the Euler-Lotka equation, f is taken as a pdf with cdf F, say. Feller shows that the unique solution to the renewal equation is

$$Z(t) = \int_0^t z(t-y)U\{dy\} \quad \text{where} \quad U = \sum_{n=0}^\infty F^{n*}.$$

In this expression, F^{n*} denotes the n-fold convolution of F, that is, it is the cdf of the sum of n independent random variables with cdf's F. It follows that $U(t)$ can be interpreted as the expected number of total offspring at

time t resulting from a single newborn at time 0 (See Feller [4, Ch VI.6]). The Renewal Theorem (alternative form) Feller [4, p. 363], shows that

$$B(t)e^{-\lambda t} = Z(t) \to \frac{S}{R} = Q, \tag{23}$$

where

$$S = \int_0^A z(y)dy = \int_0^A G(t)e^{-\lambda t}dt \text{ and } R = \int_0^A ag(a)e^{-\lambda a}da.$$

Since $N_a(t) = B(t-a)p_a$, this result shows that the population converges to steady state growth at rate λ. For a given state, this implies that

$$\frac{\ln P(\tau)}{\tau} \to \lambda, \text{ as } \tau \to \infty, \tag{24}$$

where $P(\tau)$ is the total population at date τ.

We first show that this convergence is uniform in the underlying initial distribution $N_a(0)$ or equivalently in the function z, given the fixed state. The straightforward argument that (23) implies (24) makes use of the fact that $\lim_{\tau\to\infty} \frac{\ln Q}{\tau} = 0$. Because the logarithm is not continuous at zero, establishing that the convergence in (24) is uniform requires establishing a lower bound on Q. It is thus sufficient to establish uniform convergence in the initial distribution to show that $Q \geq \underline{Q} > 0$ where $\underline{Q}$ is uniform across all initial population distributions z (Section 7), and then show that the convergence $Z(t) \to Q$ is uniform in z (Section 7). Since there are a finite number of states, it is straightforward to ensure this convergence is uniform in the states as well.

Uniform Lower Bound on Q

Lower Bound on the Proportion of Young

We first show that there is an age $B < A$ and a constant $\eta > 0$ such that for any $t > A$, the proportion of the population of age less than B is at least $\eta/(1+\eta)$, regardless of the initial population structure and regardless of the succession of states that arise in the interval $[0, t]$.

Let $\max_{w\in W} \underline{a}(w) =: \underline{\alpha}$ and $\min_{w\in W} \overline{a}(w) =: \overline{\alpha}$. We assume that $\underline{\alpha} < \overline{\alpha}$. Fertility in all states is then strictly positive on $(\underline{\alpha}, \overline{\alpha})$. Choose $C \in (\underline{\alpha}, A)$ and $B \in (C, A)$ with $B < \overline{\alpha}$, where B is close enough to A so that $A - B < M < C < B.$[13] This construction of C and B is uniform across states. Using

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[13]The assumption that $A - M < \bar{\alpha}$ ensures this is possible.

the abbreviation for fertility that $\mu_a = g(w, a)$, it follows that there exists $\hat\mu > 0$ such that $\mu_a \geq \hat\mu$ for $a \in [C, B]$ and all states.

Take any $a \in [B, A]$ and time $t > A$, which ensures that no-one currently alive was also alive in the initial population. Individuals of age a at t were born at $t - a$, so that $N_a(t) = N_0(t - a)p(a)$. At time $\tau = t - a + C$, these offspring reached age C; producing offspring at rate $\mu_{\tau - t + a}$ for $\tau \in [t - a + C, t]$. The number of offspring who are of age in $[0, B]$ at time t of individuals who are age a at time t is then at least[14]

$$\int_{t-a+C}^{t} \frac{N_a(t)}{p(a)}p(\tau - t + a)\mu_{\tau - t + a}p(t - \tau)d\tau. \qquad (25)$$

Let $\hat p$ be a uniform lower bound for the $p(A)$ across states. Since $p(a) \leq 1$, $p(\tau - t + a) \geq p(A)$, $p(t - \tau) \geq p(A)$, and $\mu_{\tau - t + a} \geq \hat\mu > 0$ for $\tau - t + a \in [C, B]$, the contribution in (25) is, at least

$$(B - C)\hat p^2 \hat\mu N_a(t).$$

All of the offspring at time t of individuals who are aged in $[B, A]$ have ages in $[0, B]$ at time t, because the oldest possible such offspring came from the individuals who are now age A. The date at which these individuals could possibly first have had offspring is $t - A + M$. The age of these offspring at t is $t - (t - A + M) = A - M < B$ (because $A - B < M$).

The total contribution of all those of age in $[B, A]$ at t in producing those of age in $[0, B]$ at t is then at least

$$(B - C)\hat p^2 \hat\mu \int_{B}^{A} N_a(t)da,$$

and hence we have the desired inequality, with $\eta = (B - C)\hat p^2 \hat\mu > 0$.

We have made no assumptions about the structure of the initial population, so this initial population $N_a(0)$ need not satisfy the condition that at least proportion $\eta/(1 + \eta)$ of the population is below age B. However, this condition is then satisfied for all $t > A$, and so for all initial populations after the very first, regardless of the random sequence of changes in states.

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[14]This is a lower bound since it neglects the offspring produced in the age range $[M, C]$ and grandchildren.

Completing the Argument for a Lower Bound on Q

As discussed above, since we use a logarithmic criterion, we need to bound the limit Q away from zero.

The term describing the asymptotic behavior of the population is

$$Q = \frac{S}{R} \quad \text{where} \quad R = \int_0^A ae^{-\lambda a}p_a\mu_a da$$

and

$$S = \int_0^A e^{-\lambda t}G(t)dt \quad \text{for} \quad G(t) = \int_0^A N_a(0)\frac{p_{a+t}}{p_a}\mu_{a+t}da,$$

where λ is the dominant root of the Euler-Lotka equation.

We thus need to show that S/R is uniformly bounded below by a strictly positive bound. Note that R is independent of the initial distribution. The average age in the steady state is then bounded above by some $\bar{R}$, across states.

Consider now a lower bound on S. Where ρ_a^w is mortality in state w at age a we have

$$G(t) \geq e^{-\hat{\rho}t}\int_0^A N_a(0)\mu_{a+t}da \geq e^{-\hat{\rho}t}\int_0^B N_a(0)\mu_{a+t}da$$

where $\hat{\rho} = \max_{a,w}\rho_a^w$ is assumed to be finite.

Hence

$$S \geq \int_0^A e^{-(\lambda+\hat{\rho})t}\int_0^B N_a(0)\mu_{a+t}dadt = \int_0^B N_a(0)\int_0^A e^{-(\lambda+\hat{\rho})t}\mu_{a+t}dtda.$$

Recall $B \in (M, \overline{\alpha})$ and take $A' \in (B, \overline{\alpha})$ so that $\mu_a > 0$ for $a \in [B, A']$. Make these choices independent of the state. Now consider t and $a \in [0, B]$ such that $a + t \in [B, A']$, so that $t \in [B - a, A' - a] \subseteq [0, A]$. It follows that

$$S \geq \int_0^B N_a(0)\int_{B-a}^{A'-a} e^{-(\lambda+\hat{\rho})A'}\hat{\mu}dtda$$

where $\hat{\mu} = \min_{w,a\in[B',A']}\mu_a^w > 0$. It follows then that

$$S \geq e^{-(\lambda+\hat{\rho})A'}\hat{\mu}(A' - B)\int_0^B N_a(0)da.$$

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Our bound from Section 7 ensures that

$$\int_0^B N_a(0)da \geq \frac{\eta}{1+\eta} > 0 \text{ since } \int_0^A N_a(0)da = 1,$$

yielding a uniform positive lower bound $\underline{S}$ on S, so that $\underline{S}/\overline{R} > 0$ is the desired uniform lower bound on S/R.

Uniform Convergence of $Z(t)$

Finally we need to show that $Z(t)$ converges to Q uniformly in the function z.

The function U in Feller [4, p. 360] is independent of the initial conditions. However, the Renewal Theorem (second form) [4, p. 363] involves the initial population. We argue that this convergence can also be taken to be uniform in the initial distribution, and hence in the finite number of states.

In the notation of Feller, we have that

$$Z(x) \to \frac{\int_0^\infty z(y)dy}{\mu} \text{ as } x \to \infty \text{ where } Z(x) = \int_0^x z(x-y)U(dy).$$

We that that z has support in $[0, A]$ and is bounded, so the set of such z is compact. We need to show that the convergence is uniform in z.

Suppose then this convergence is *not* uniform. It follows that there exists $\varepsilon > 0$ and a sequence $x_n \to \infty$ with associated z_n such that

$$\left| Z_n(x_n) - \frac{\int_0^A z_n(y)dy}{\mu} \right| > \varepsilon, \text{ for all } n.$$

Since the set of z is compact, there is a subsequence such that $z_n \to z^*$, say. Of course

$$\int_0^A z_n(y)dy \to \int_0^A z^*(y)dy.$$

Feller [4, Theorem 2, P. 367] shows that, given that F has a pdf that is "directly Riemann integrable", as is the case in our model[15], then the measure U has a density u and $u(t) \to \mu^{-1}$ as $t \to \infty$. It follows then that

$$Z_n(x_n) = \int_{x_n-A}^{x_n} z_n(x_n-y)u(y)dy = \int_0^A z_n(w)u(x_n-w)dw.$$

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[15]Direct Riemann integrability holds here since f is continuous and has compact support.

Hence
$$Z_n(x_n) \to \mu^{-1} \int_0^A z^*(w)dw,$$
providing the desired contradiction.

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