Evolved Attitudes to Idiosyncratic and Aggregate Risk in Age-Structured Populations

Online Appendix: Proof of Uniform Convergence[1]

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We demonstrate that the convergence described in equation (26) in the paper can be taken to be uniform in the initial population and across states.

1 The Renewal Equation

Fix a state. Suppose the population distribution is described by $N_a(t)$ where $\int_0^A N_a(0)da = 1$. It is without loss of generality to assume the initial population is of size 1, given that the problem is linear. It is also without essential loss of generality to suppose that individuals die when their reproduction ceases. Individuals could, more generally, live beyond A, and would then swell the size of the population, but this cannot affect the growth rates.

Suppose $B(t)$ is the total flow of births at date t. Frauenthal [2, p. 131, Eq (12)] shows that the evolution of the population is governed by the following "renewal equation"

$$B(t) = G(t) + \int_0^t B(t-a)\varphi(a)da \quad \text{where} \quad \varphi(a) = p_a\mu_a$$

and

$$G(t) = \int_0^A N_a(0)\frac{\varphi(a+t)}{p_a}da.$$

Feller [1, Ch XI] provides an elegant analysis of the renewal equation. This can be rewritten as follows. Define $Z(t) = B(t)e^{-\lambda t}$ and $z(t) = G(t)e^{-\lambda t}$, where λ is the unique real root of the Euler-Lotka equation. It follows that the renewal equation becomes

$$Z(t) = z(t) + \int_0^t Z(t-a)f(a)da \quad \text{where} \quad f(a) = \varphi(a)e^{-\lambda a}.$$

Since $\int_0^A f(a)da = 1$, by the Euler-Lotka equation, f is taken as a pdf with cdf F, say. Feller shows that the unique solution to the renewal equation is

$$Z(t) = \int_0^t z(t-y)U\{dy\} \quad \text{where} \quad U = \sum_{n=0}^{\infty} F^{n*}.$$

[1] We are very grateful to Ken Wachter for illuminating discussions of these issues.

In this expression, F^{n*} denotes the n-fold convolution of F, that is, it is the cdf of the sum of n independent random variables with cdf's F. It follows that $U(t)$ can be interpreted as the expected number of total offspring at time t resulting from a single newborn at 0 (see Feller [1, Ch VI.6]). The Renewal Theorem (alternative form) Feller [1, p. 363], shows that

$$B(t)e^{-\lambda t} = Z(t) \to \frac{S}{R} = Q, \text{ where} \tag{1}$$

$$S = \int_0^A z(y)dy = \int_0^A G(t)e^{-\lambda t}dt \text{ and } R = \int_0^A a\varphi(a)e^{-\lambda a}da.$$

Since $N_a(t) = B(t-a)p_a$, this result shows that the population converges to steady state growth at rate λ. For a given state, this implies the result given as (26) in the paper, that is,

$$\frac{\ln P(\tau)}{\tau} \to \lambda, \text{ as } \tau \to \infty. \tag{2}$$

We first show that this convergence is uniform in the underlying initial distribution $N_a(0)$, or equivalently in the function z, given the fixed state. The straightforward argument that (1) implies (2) makes use of the fact that $\lim_{\tau\to\infty} \frac{\ln Q}{\tau} = 0$. Because the logarithm is not continuous at zero, establishing that the convergence in (2) is uniform requires establishing a lower bound on Q. It is thus sufficient to establish uniform convergence in the initial distribution to show 1) That $Q \geq \underline{Q} > 0$ where $\underline{Q}$ is uniform across all initial population distributions z (Section 2), and 2) The convergence $Z(t) \to Q$ is uniform in z (Section 3).

Since there are a finite number of states, it is straightforward to ensure this convergence is uniform in the states as well.

2 Uniform Lower Bound on Q

2.1 A Lower Bound on the Proportion of Young

We first show that there is an age $B < A$ and a constant $\eta > 0$ such that for any $t > A$, the proportion of the population of age less than B is at least $\eta/(1+\eta)$, regardless of the initial condition and regardless of the succession of states that arise in the interval $[0, t]$.

Choose $C \in (M, A)$ and $B \in (C, A)$ where B is close enough to A so that $A - B < M < C < B$. Suppose this construction of C and B is uniform across states. It follows that there exists $\hat{\mu} > 0$ such that $\mu_a \geq \hat{\mu}$ for $a \in [C, B]$ and all states.

Take any $a \in [B, A]$ and time $t > A$. Individuals of age a at t were born at $t - a$, so that $N_a(t) = N_0(t-a)p(a)$. At time $\tau = t - a + C$, these offspring reached age C; producing offspring at rate $\mu_{\tau-t+a}$ for $\tau \in [t-a+C, t]$. The offspring who are of age in $[0, B]$ at time t of individuals who are age a at time t is then at least[2]

$$\int_{t-a+C}^{t} \frac{N_a(t)}{p(a)} p(\tau - t + a)\mu_{\tau-t+a} p(t-\tau) d\tau. \tag{3}$$

Let $\hat{p}$ be a uniform lower bound for the $p(A)$ across states. Since $p(a) \leq 1$, $p(\tau - t + a) \geq p(A)$, $p(t - \tau) \geq p(A)$, and $\mu_{\tau-t+a} \geq \hat{\mu} > 0$ for $\tau - t + a \in [C, B]$, the contribution in (3) is, at least

$$(B - M)\hat{p}^2 \hat{\mu} N_a(t).$$

All of the offspring at time t of individuals who are aged in $[B, A]$ have ages in $[0, B]$ at time t. The oldest such offspring came from the individuals who are now age A. The date at which these individuals first had offspring is $t - A + M$. The age of these offspring at t is $t - (t - A + M) = A - M < B$ (because $A - B < M$).

The total contribution of all those of age in $[B, A]$ at t in producing those of age in $[0, B]$ at t is then at least

$$(B - C)\hat{p}^2 \hat{\mu} \int_B^A N_a(t) da,$$

and hence we have the desired inequality, with $\eta = (B - C)\hat{p}^2 \hat{\mu} > 0$.

Note that, although the condition need not be satisfied by $N_a(0)$, it is always satisfied for all $t > A$, regardless of the sequence of states.

2.2 Completing the Argument for a Lower Bound on Q

As discussed above, since we use a logarithmic criterion, we need to bound the limit Q away from zero.

The term describing the asymptotic behavior of the population is

$$Q = \frac{S}{R} \text{ where } R = \int_0^A a e^{-\lambda a} p_a \mu_a da$$

and

$$S = \int_0^A e^{-\lambda t} G(t) dt \qquad \text{for} \qquad G(t) = \int_0^A N_a(0) \frac{p_{a+t}}{p_a} \mu_{a+t} da,$$

where λ is the dominant root of the Euler-Lotka equation.

We thus need to show that S/R is uniformly bounded below by a strictly positive bound. Note that R is independent of the initial distribution. The average age in the steady state is then bounded above by some $\bar{R}$, across states.

Consider now a lower bound on S. We have

$$G(t) \geq e^{-\hat{\rho} t} \int_0^A N_a(0) \mu_{a+t} da \geq e^{-\hat{\rho} t} \int_0^B N_a(0) \mu_{a+t} da \ \text{ where } \ \hat{\rho} = \max_{a,s} \rho_a^s.$$

[2] This is a lower bound since it neglects the offspring produced in the age range $[M, C]$ and grandchildren.

and hence

$$S \geq \int_0^A e^{-(\lambda+\hat{\rho})t} \int_0^B N_a(0)\mu_{a+t} dadt = \int_0^B N_a(0) \int_0^A e^{-(\lambda+\hat{\rho})t} \mu_{a+t} dtda.$$

Consider $B \in (M, A)$ and $A' \in (B, A)$ so that $\mu_a > 0$ for $a \in [B, A']$. Make these choices independent of the state. Now consider t and $a \in [0, B]$ such that $a + t \in [B, A']$, so that $t \in [B - a, A' - a] \subseteq [0, A]$. It follows that

$$S \geq \int_0^B N_a(0) \int_{B-a}^{A'-a} e^{-(\lambda+\hat{\rho})A'} \hat{\mu} dtda$$

where $\hat{\mu} = \min_{s,a \in [B,A']} \mu_a^s > 0$. It follows then that

$$S \geq e^{-(\lambda+\hat{\rho})A'} \hat{\mu}(A' - B) \int_0^B N_a(0)da.$$

Our bound from Section 2 ensures that

$$\int_0^B N_a(0)da \geq \frac{\eta}{1+\eta} > 0,$$

yielding a uniform positive lower bound $\underline{S}$ on S, so that $\underline{S}/\overline{R} > 0$ is the desired uniform lower bound on S/R.

3 Uniform Convergence of $Z(t)$

Finally we need to show that $Z(t)$ converges to Q uniformly in the function z.

The function U in Feller [1, p. 360] is independent of the initial conditions. However, the Renewal Theorem (second form) [1, p. 363] involves the initial population. We argue that this convergence can also be taken to be uniform in the initial distribution, and hence in the finite number of states.

In the notation of Feller, we have that

$$Z(x) \to \frac{\int_0^\infty z(y)dy}{\mu} \text{ as } x \to \infty \text{ where } Z(x) = \int_0^x z(x-y)U(dy).$$

We that that z has support in $[0, A]$ and is bounded, so the set of such z is compact. We need to show that the convergence is uniform in z.

Suppose then this convergence is *not* uniform. It follows that there exists $\varepsilon > 0$ and a sequence $x_n \to \infty$ with associated z_n such that

$$\left| Z_n(x_n) - \frac{\int_0^A z_n(y)dy}{\mu} \right| > \varepsilon, \text{ for all } n.$$

Since the set of z is compact, there is a subsequence such that $z_n \to z^*$, say. Of course

$$\int_0^A z_n(y)dy \to \int_0^A z^*(y)dy.$$

Feller [1, Theorem 2, P. 367] shows that, given that F has a pdf that is "directly Riemann integrable", as is the case in our model[3], then the measure U has a density u and $u(t) \to \mu^{-1}$ as $t \to \infty$. It follows then that

$$Z_n(x_n) = \int_{x_n - A}^{x_n} z_n(x_n - y) u(y) dy = \int_0^A z_n(w) u(x_n - w) dw.$$

Hence

$$Z_n(x_n) \to \mu^{-1} \int_0^A z^*(w) dw,$$

providing the desired contradiction.

References

[1] William Feller. *An Introduction to Probability Theory and Its Applications, Vol. II.* John Wiley, New York, 2nd edition, 1971.

[2] James C. Frauenthal. Analysis of age-stucture models. In Thomas G. Hallam and Simon A. Levin, editors, *Mathematical Ecology*, pages 117–147. Springer-Verlag, New York, 1986.

[3] Direct Riemann integrability holds here since f is continuous and has compact support.