Homework #7 • MATH 419 • Orthogonal Polynomials

- submit your write-up on **Wednesday** 13 July.

A) Differential Equations (3 pages) Beginning from the following basic definitions of the Legendre and Chebyshev polynomials,

$$\begin{aligned} P_N(x) &= \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \\ T_N(x) &= \cos(N \cos^{-1} x) \end{aligned} \tag{1}$$

derive the second-order differential equation associated with each. (Note, there are standard forms which can be looked up.) Clearly state the recipe by which the result is achieved. Simplicity and clarity counts.

B) Chebyshev Interpolation (3-4 pages) Given values of any function $f(x)$ at the set of points $\{x_1, x_2 \ldots x_N\}$, the *Lagrange interpolant* is the degree $N-1$ polynomial

$$L(x) = \sum_{j=1}^{N} f(x_j) \prod_{k \neq j} \frac{(x - x_k)}{(x_j - x_k)} \tag{2}$$

and has the property that $L(x_j) = f(x_j)$. This is unique among degree $N-1$ polynomials provided all $\{f(x_j)\}$ are not zero. The Chebyshev interpolation method chooses the nodes $\{x_j\}$ to be the N zeros of polynomial $T_N(x)$ — the result is a reasonably uniform polynomial approximation to a given function $f(x)$.

Illustrate numerically that the Chebyshev nodes do result in a fairly uniform approximation by comparing the graphs of the Lagrange interpolants for several other choices for the interpolation nodes $\{x_j\}$. For this exercise, you should choose an interesting (non-polynomial) function $f(x)$, and an N of significant size. (Hint: search the term *Runge phenomenon*.)