Homework #2 • MATH 419 • Convergence of Trigonometric Series

- please respect page limits.
- submit your write-up Wednesday 25 May.
- remember that the webct discussion is an open forum.
- please annotate plots well.
- refer to *Guidelines for Reports*.

A) Weierstrauss M-Test (3 pages max) The trigonometric series ($C > 0$, a real constant)

$$S(x) = \sum_{j=1}^{\infty} e^{-Cj} \sin jx . \qquad (1)$$

appeared as an example Fourier series in the first lecture.

- Show that the Weierstrauss M-test can be applied to show the uniform convergence of the series. Clearly verify each of the premises required by the M-test.
- In fact, by a suitable choice for the Weierstrauss sequence $\{M_j\}$ you can give an explicit function $N(\epsilon)$ as used in the definition of uniform convergence.
- More generally, wiith a positive result for the M-test, concisely describe how one produces $N(\epsilon)$. (Note that a rigorous statement of the Weierstrauss M-test is attached. You are welcome to use another version as long as you attach a copy as an Appendix.)

B) Dirichlet Test (3 pages max) Consider the complex exponential series

$$F(x) = \sum_{j=1}^{\infty} \frac{e^{ijx}}{j} \qquad (2)$$

over the subinterval $I = [a, \pi - a]$ where a is a fixed positive real.

- Indicate why the Weierstrauss M-test cannot readily be applied to the uniform convergence of $F(x)$ over $x \in I$.
- An alternative to the M-test, is the Dirichlet test (copy also attached). Apply this test to prove that $F(x)$ converges uniformly on all closed subintervals of $[0, \pi]$ which do not contain the endpoints. (Summing a geometric series is useful here.) Again, carefully verify that the premises of the test are satisified.
- Plot the real and imaginary parts of the partial sums. Can you design a graphic which illustrates the uniform convergence and the possible non-uniform convergence of the series?

C) Half Parabola (3 pages max) Produce the series for Exercises 1.4.3 and 1.4.4 of the text (no need to show the calculus, but explain your thinking and/or methods). Plot the errors between the original function and partial sums of the Fourier series, $f(x) - F_N(x)$. Illustrate the near self-similarity of the errors (try contrasting N which are powers of two).

Note: two functions $f(x)$ and $g(x)$ are self-similar about $x = 0$ if there are constants a, b such that $g(x) = af(bx)$.

9.3 DEFINITION OF UNIFORM CONVERGENCE

Let $\{f_n\}$ be a sequence of functions which converges pointwise on a set S to a limit function f. This means that for each point x in S and for each $\varepsilon > 0$, there exists an N (depending on both x and ε) such that

$$n > N \quad \text{implies} \quad |f_n(x) - f(x)| < \varepsilon.$$

If the same N works equally well for *every* point in S, the convergence is said to be *uniform* on S. That is, we have

Definition 9.1. *A sequence of functions $\{f_n\}$ is said to converge uniformly to f on a set S if, for every $\varepsilon > 0$, there exists an N (depending only on ε) such that $n > N$ implies*

$$|f_n(x) - f(x)| < \varepsilon, \qquad \textit{for every } x \textit{ in } S.$$

We denote this symbolically by writing

$$f_n \to f \textit{ uniformly on } S.$$

When each term of the sequence $\{f_n\}$ is real-valued, there is a useful geometric interpretation of uniform convergence. The inequality $|f_n(x) - f(x)| < \varepsilon$ is then equivalent to the *two* inequalities

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon. \tag{3}$$

If (3) is to hold for all $n > N$ and for all x in S, this means that the entire graph of f_n (that is, the set $\{(x, y) : y = f_n(x),\ x \in S\}$) lies within a "band" of height 2ε situated symmetrically about the graph of f. (See Fig. 9.4.)

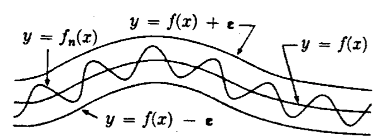

Figure 9.4

A sequence $\{f_n\}$ is said to be *uniformly bounded* on S if there exists a constant $M > 0$ such that $|f_n(x)| \le M$ for all x in S and all n. The number M is called a *uniform bound* for $\{f_n\}$. If each individual function is bounded and if $f_n \to f$ uniformly on S, then it is easy to prove that $\{f_n\}$ is uniformly bounded on S. (See Exercise 9.1.) This observation often enables us to conclude that a sequence is *not* uniformly convergent. For instance, a glance at Fig. 9.2 tells us at once that the sequence of Example 2 cannot converge uniformly on any subset containing a neighborhood of the origin. However, the convergence in this example *is* uniform on every compact subinterval not containing the origin.

9.6 UNIFORM CONVERGENCE OF INFINITE SERIES OF FUNCTIONS

Definition 9.4. *Given a sequence $\{f_n\}$ of functions defined on a set S. For each x in S, let*

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \qquad (n = 1, 2, \ldots). \tag{4}$$

If there exists a function f such that $s_n \to f$ uniformly on S, we say the series $\sum f_n(x)$ converges uniformly on S and we write

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \qquad (\text{uniformly on } S).$$

Theorem 9.5 (Cauchy condition for uniform convergence of series). *The infinite series $\sum f_n(x)$ converges uniformly on S if, and only if, for every $\varepsilon > 0$ there is an N such that $n > N$ implies*

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon, \qquad \text{for each } p = 1, 2, \ldots, \text{ and every } x \text{ in } S.$$

Proof. Define s_n by (4) and apply Theorem 9.3.

Theorem 9.6 (Weierstrass M-test). *Let $\{M_n\}$ be a sequence of nonnegative numbers such that*

$$0 \le |f_n(x)| \le M_n, \qquad \text{for } n = 1, 2, \ldots, \text{ and for every } x \text{ in } S.$$

Then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

Proof. Apply Theorems 8.11 and 9.5 in conjunction with the inequality

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| \le \sum_{k=n+1}^{n+p} M_k.$$

Theorem 8.11 (Cauchy condition for series). *The series $\sum a_n$ converges if, and only if, for every $\varepsilon > 0$ there exists an integer N such that $n > N$ implies*

$$|a_{n+1} + \cdots + a_{n+p}| < \varepsilon \qquad \text{for each } p = 1, 2, \ldots \tag{2}$$

9.11 SUFFICIENT CONDITIONS FOR UNIFORM CONVERGENCE OF A SERIES

The importance of uniformly convergent series has been amply illustrated in some of the preceding theorems. Therefore it seems natural to seek some simple ways of testing a series for uniform convergence without resorting to the definition in each case. One such test, the *Weierstrass M-test*, was described in Theorem 9.6. There are other tests that may be useful when the M-test is not applicable. One of these is the analog of Theorem 8.28.

Theorem 9.15 (Dirichlet's test for uniform convergence). *Let $F_n(x)$ denote the nth partial sum of the series $\sum f_n(x)$, where each f_n is a complex-valued function defined on a set S. Assume that $\{F_n\}$ is uniformly bounded on S. Let $\{g_n\}$ be a sequence of real-valued functions such that $g_{n+1}(x) \le g_n(x)$ for each x in S and for every $n = 1, 2, \dots$, and assume that $g_n \to 0$ uniformly on S. Then the series $\sum f_n(x)g_n(x)$ converges uniformly on S.*

Proof. Let $s_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$. By partial summation we have

$$s_n(x) = \sum_{k=1}^{n} F_k(x)(g_k(x) - g_{k+1}(x)) + g_{n+1}(x)F_n(x),$$

and hence if $n > m$, we can write

$$s_n(x) - s_m(x) = \sum_{k=m+1}^{n} F_k(x)(g_k(x) - g_{k+1}(x)) + g_{n+1}(x)F_n(x) - g_{m+1}(x)F_m(x).$$

Therefore, if M is a uniform bound for $\{F_n\}$, we have

$$\begin{aligned}|s_n(x) - s_m(x)| &\le M \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) + Mg_{n+1}(x) + Mg_{m+1}(x)\\ &= M(g_{m+1}(x) - g_{n+1}(x)) + Mg_{n+1}(x) + Mg_{m+1}(x)\\ &= 2Mg_{m+1}(x).\end{aligned}$$

Since $g_n \to 0$ uniformly on S, this inequality (together with the Cauchy condition) implies that $\sum f_n(x)g_n(x)$ converges uniformly on S.