For B.

THEOREM 5.3 Independence of Eigenvectors

Let A be an $n \times n$ matrix. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors of A corresponding to *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent and A is diagonalizable.

PROOF Suppose that the conclusion is false, so the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent. Then one of them is a linear combination of its predecessors. (See Exercise 37, page 203.) Let $\mathbf{v}_k$ be the first such vector, so that

$$\mathbf{v}_k = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_{k-1}\mathbf{v}_{k-1} \tag{2}$$

and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ is independent. Multiplying Eq. (2) by λ_k, we obtain

$$\lambda_k\mathbf{v}_k = d_1\lambda_k\mathbf{v}_1 + d_2\lambda_k\mathbf{v}_2 + \cdots + d_{k-1}\lambda_k\mathbf{v}_{k-1}. \tag{3}$$

On the other hand, multiplying both sides of Eq. (2) on the left by the matrix A yields

$$\lambda_k\mathbf{v}_k = d_1\lambda_1\mathbf{v}_1 + d_2\lambda_2\mathbf{v}_2 + \cdots + d_{k-1}\lambda_{k-1}\mathbf{v}_{k-1}, \tag{4}$$

because $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$. Subtracting Eq. (4) from Eq. (3), we see that

$$\mathbf{0} = d_1(\lambda_k - \lambda_1)\mathbf{v}_1 + d_2(\lambda_k - \lambda_2)\mathbf{v}_2 + \cdots + d_{k-1}(\lambda_k - \lambda_{k-1})\mathbf{v}_{k-1}.$$

This last equation is a dependence relation because not all the coefficients are zero. (Not all d_i are zero because of Eq. (2) and because the λ_i are distinct.) But this contradicts the linear independence of the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$. We conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is independent. That A is diagonalizable follows at once from Corollary 1 of Theorem 5.2. ▲

EXAMPLE 1 Diagonalize the matrix $A = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$, and compute A^k in terms of k.

SOLUTION We compute

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 5 \\ -2 & 4-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$. For $\lambda_1 = 2$, we have

$$A - 2I = \begin{bmatrix} -5 & 5 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

which yields an eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -1$, we have

$$A + I = \begin{bmatrix} -2 & 5 \\ -2 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix},$$