Homework #3 • MATH 314 • Convergence of Fourier Series

- please respect page limits. Have each problem start at the top of a new page.
- submit your write-up into homework box #2 by 4pm, Friday 01 February.
- the webct discussions are working well, your participation has been noted. Thanks.
- please acknowledge collaborations & assistance from colleagues, TA and instructor.

- read the *Guideline* for assignments as posted on the class webpage.

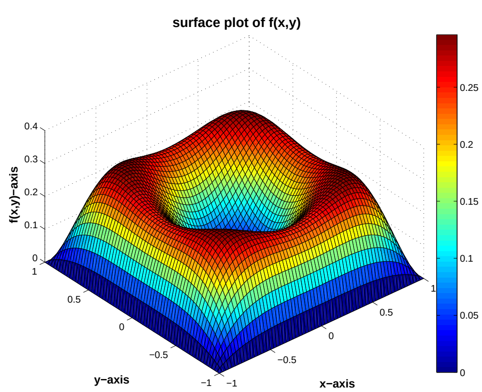
A) Least Square (2 pages max, 10pts) For an odd function, $f_{odd}(x)$, on the interval $[-L, +L]$, express the least-square error of an N-term sine series, $\tilde{f}_N(x)$, as an inner product formula. (From here, you should be able to derive the Fourier sine coefficients from a completion of squares derivation.) There is another derivation of the best Fourier sine coefficient $\tilde{b}_j$ that does <u>not</u> require expansion of the product of the sums. Take the partial derivative $\partial(\text{LSerror})/\partial\tilde{b}_j$, set it equal to zero, and solve. Give the value of the minimal least-square error, this however, requires an expansion of a product. (The text gives a generalized hint on page 61.)

In what way would this derivation of the $\tilde{b}_j$ become much more difficult if the basis functions were not mutually orthogonal?

B) Convergence (2 pages, 10pts) Calculate the Fourier series, $f_S(x)$, for the piecewise function

$$f(x) = \left\{ \begin{array}{cccc} 0 & \text{for } -\pi & \leq x \leq & -\pi/2 \\ 1+\cos(2x) & \text{for } -\pi/2 & \leq x \leq & +\pi/2 \\ 0 & \text{for } -\pi/2 & \leq x \leq & +\pi \end{array} \right\}.$$

By quoting the text, how does the series $f_S(x)$ converge to $f(x)$? State similarly the convergence of the first derivatives ($f_S'(x)$ and $f'(x)$), and second derivatives ($f_S''(x)$ and $f''(x)$). Wherever the convergence is not pointwise, give the limiting values of the Fourier series.



C) 2D Fourier Series (2 pages + 2 annotated plot pages, 15pts) In this problem, you will calculate the double Fourier series for the "volcano" function

$$V(x,y) = (x^2+y^2)(x^2-L_x^2)(y^2-L_y^2)$$

on the rectangular domain $\mathcal{R} = \{-L_x \leq x \leq +L_x, -L_y \leq y \leq +L_y\}$. But first, run the matlab script *w04volcano.m*. Observe that surface plot of the volcano shows some obvious symmetries, and verify these from the formula for the function $V(x,y)$.

Begin from the expression for the general double Fourier series on the rectangle $\mathcal{R}$

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \{a_{jk}C_j(x)C_k(y) + b_{jk}C_j(x)S_k(y) + c_{jk}S_j(x)C_k(y) + d_{jk}S_j(x)S_k(y)\}$$

using the Fourier basis functions

$$\{C_j(x) = \cos(j\pi/L_x)x, S_j(x) = \sin(j\pi/L_x)x, C_k(y) = \cos(k\pi/L_y)y, S_k(y) = \sin(k\pi/L_y)y\} \ .$$

It turns out that many of the Fourier coefficients are zero. Identify these, and explain why – without carrying out an explicit integration. Because of the particular form of $V(x, y)$, the important nonzero coefficients can be evaluated from knowing only two trig integral formulas! State these before giving a formula for the non-zero Fourier coefficients – simplify the expressions as far as is useful for accurate typing into a matlab plot script. The logic of your method is more important than the minute details of the integration.

Modify the volcano script to demonstrate that your coefficients are correct. Present two plots that provide the clearest evidence of the correctness of your calculations.

extra: How does the maximum error, $\max_{\mathcal{R}} |V(x, y) - \tilde{V}(x, y)|$, decrease with the number of terms in a partial sum?