2 Semidefinite Programming

Linear Programming: Let $b, a^1, a^2, \ldots, a^m \in \mathbb{R}^n$, let $c \in \mathbb{R}^m$, and let $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ be variables. Consider the following linear program (LP) and its dual.

LP	Dual
maximize $c \cdot x$	minimize $y \cdot b$
s.t. $x_1 a^1 + x_2 a^2 + \ldots x_m a^m \leq b$	s.t. $y \geq 0$
	$y \cdot a^i = c_i$ for every $1 \leq i \leq m$

Define the following affine subspaces $U, V \subseteq \mathbb{R}^n$:

$$U = \{b - x_1 a^1 - x_2 a^2 \ldots - x_m a^m : x_1, \ldots x_m \in \mathbb{R}\}$$
$$V = \{y \in \mathbb{R}^n : y \cdot a^i = c_i \text{ for every } 1 \leq i \leq m\}$$

Here we see two (dual) ways to define an affine subspace: U is defined with generators, while V is defined by hyperplane constraints. In the dual program, we optimize a linear function over $V \cap \mathbb{R}^n_+$, while the LP may be viewed as optimizing a function (in terms of the generators) over $U \cap \mathbb{R}^n_+$. So, both problems are optimizations over the intersection of an affine subspace with the cone of nonnegative points.

Theorem 2.1 (LP Duality) *Assume both the LP and Dual problems are feasible. Then both optimums exist, and furthermore, if x^*, y^* are optimal for the LP and the Dual, then*

$$c \cdot x^* = y^* \cdot b$$

Definition: We let Σ_n denote the vector space of symmetric $n \times n$ matrices and we let $\Delta_n \subseteq \Sigma_n$ denote the cone of semidefinite matrices.

Semidefinite Programming: Fix $B, A^1, A^2, \ldots A^m \in \Sigma_n$ and $c \in \mathbb{R}^m$, and let $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $Y \in \Sigma_n$ be variables. We now have the following semidefinite program (SDP) and its dual.

SDP	Dual
maximize $c \cdot x$	minimize $Y \cdot B$
s.t. $x_1 A^1 + x_2 A^2 \ldots x_m A^m \preceq B$	s.t. $Y \succeq 0$
	$Y \cdot A^i = c_i$ for every $1 \leq i \leq m$

Define the following affine subspaces $U, V \subseteq \Sigma_n$:

$$U = \{B - x_1A^1 - x_2A^2 \ldots - x_mA^m : x_1, x_2, \ldots, x_m \in \mathbb{R}\}$$
$$V = \{Y \in \Sigma_n : Y \cdot A^i = c_i \text{ for every } 1 \leq i \leq m\}$$

As before, U is defined with generators, while V is defined by hyperplane constraints. In the dual we optimize a linear function over $V \cap \Delta_n$, while the SDP may be viewed as optimizing a function (in terms of generators) over $U \cap \Delta_n$. So, we may view both of these problems as optimizations over the intersection of an affine subspace with the cone Δ_n.

Note: If we let $B, A^1, A^2, \ldots, A^m$ be diagonal matrices with the vectors $b, a^1, a^2, \ldots, a^m$ on the diagonal, then our SDP reduces to our LP.

Proposition 2.2 (SDP Weak Duality) *If x is feasible in the SDP and Y is feasible in its Dual*

$$c \cdot x \leq Y \cdot B$$

Proof: Let $X = B - x_1A^1 - x_2A^2 \ldots - x_mA^m$. Then $X, Y \succeq 0$ so we have

$$\begin{aligned} 0 &\leq Y \cdot X \\ &= Y \cdot B - x_1c_1 - x_1c_2 \ldots - x_mc_m \\ &= Y \cdot B - x \cdot c. \qquad \square \end{aligned}$$

Theorem 2.3 (SDP Strong Duality) *Assume both the SDP and Dual are feasible. If there is a feasible matrix $X = B - x_1A^1 \ldots - x_mA^m$ (for the SDP) or Y (for the dual) which is definite, then there are optimal solutions x^* for the SDP and Y^* for the dual, and further*

$$c \cdot x^* = Y^* \cdot B.$$