7 Some Chromatic Numbers

Let $G = (V, E)$ be a graph, and let $\chi(G)$, $\omega(G)$ denote the chromatic, clique numbers of G.

Homomorphisms: If G, H are graphs a homomorphism from G to H is a map $\phi : V(G) \to V(H)$ with the property that $\phi(x)\phi(y)$ is an edge of H whenever xy is an edge of G. If such a map exists, we write $G \to H$. Note that $G \to K_n$ if and only if $\chi(G) \leq K_n$.

7.1 Fractional Colouring

Let $\mathcal{I}(G)$ denote the collection of all independent sets in G and let A be the $V \times \mathcal{I}(G)$ incidence matrix (i.e. $A_{v,S}$ is 1 if v is in the independent set S and 0 otherwise). Now we can express $\chi(G)$ and $\omega(G)$ with the following integer programs.

$$\chi(G) = \min\{1^\top x : x \in \mathbb{Z}_+^{\mathcal{I}(G)} \text{ and } Ax \geq 1\}$$
$$\omega(G) = \min\{y^\top 1 : y \in \mathbb{Z}_+^{V} \text{ and } y^\top A \leq 1\}$$

Relaxing these integrality constraints yield the *fractional clique number* ω_f, and *fractional chromatic number* χ_f, expressed as linear programs.

$$\chi_f(G) = \min\{1^\top x : x \in \mathbb{R}_+^{\mathcal{I}(G)} \text{ and } Ax \geq 1\}$$
$$\omega_f(G) = \min\{y^\top 1 : y \in \mathbb{R}_+^{V} \text{ and } y^\top A \leq 1\}$$

Now, by LP duality we have the following string of inequalities

$$\omega(G) \;\leq\; \omega_f(G) \;=\; \chi_f(G) \;\leq\; \chi(G).$$

Note: Although linear programs can be solved in polynomial time, the size of the matrix A is generally exponential in the size of G, so this does not allow us to compute χ_f efficiently. In fact, it is NP-hard to compute $\chi_f(G)$.

Kneser Graphs: Let $\binom{[n]}{k}$ denote the set of all k element subsets of $\{1, \ldots, n\}$. The Kneser graph $Kn(n, k)$ has vertex set $\binom{[n]}{k}$, with two vertices adjacent if they are disjoint.

Theorem 7.1 *$\chi_f(G) \leq \frac{p}{q}$ if and only if $G \to Kn(\ell p, \ell q)$ for some $\ell \in \mathbb{Z}$.*

Proof: "only if": If $\chi_f(G) \leq \frac{p}{q}$ then there is a rational vector $w \in \mathbb{Q}^{\mathcal{I}(G)}$ so that $\sum\{w_I : v \in I\} \geq 1$ for every $v \in V$ and $w^\top 1 = \frac{p}{q}$. Choose an integer ℓ so that $(\ell q)w \in \mathbb{Z}^{\mathcal{I}(G)}$. Now, form a sequence L of independent sets so that $(\ell q)w_I$ is the number of times I appears in L. By construction, the length of L is equal to the sum of the entries in $(\ell q)w$ which is precisely ℓp. Furthermore, for every $v \in V$ we have $\sum\{(\ell q)w_I : v \in I\} \geq \ell q$, so v appears in $\geq \ell q$ of the terms of L. By possibly removing v from some terms of L, we may arrange that every vertex appears in exactly ℓq terms. Now, let $\phi : V(G) \to \binom{[\ell p]}{\ell q}$ be given by the rule

$$\phi(v) = \{j \in \{1, \ldots, \ell p\} : v \in L_j\}$$

It follows that ϕ is a homomorphism from G to $Kn(\ell p, \ell q)$ as required.

"if": To prove this it suffices to show that $\chi_f(Kn(p,q)) \leq \frac{p}{q}$, since the composition of a homomorphism and a fractional $\frac{p}{q}$-colouring is another fractional $\frac{p}{q}$-colouring. A fractional $\frac{p}{q}$-colouring of $Kn(p,q)$ is given by assigning weight $\frac{1}{q}$ to each independent set of the form $T_i = \{S \in \binom{[p]}{q} : i \in S\}$. $\square$

7.2 Circular Colouring

For $t \in \mathbb{R}$, a *circular t-colouring* of G is a map $\phi : V \to S^1$ so that the angle between $\phi(x)$ and $\phi(y)$ is $\geq \frac{2\pi}{t}$ whenever x and y are adjacent. The *circular chromatic number* of G is

$$\chi_c(G) = \inf\{t \in \mathbb{R} : G \text{ has a circular } t\text{-colouring}\}.$$

$\mathbf{K_{n/k}}$: If n, k are positive integers, we let $K_{n/k}$ denote a graph consisting of n vertices in a cyclic order, with two vertices adjacent if they have distance $\geq k$ in this ordering.

Theorem 7.2

(i) $\lceil \chi_c(G) \rceil = \chi(G)$

(ii) *$\chi_c(G) \leq \frac{n}{k}$ if and only if $G \to K_{n/k}$.*

7.3 Vector Colouring

For $t \in \mathbb{R}$, a *vector t-colouring* of G is a mapping $x : V \to S^m$ with the property that $x(i) \cdot x(j) \leq -\frac{1}{t-1}$ whenever $ij \in E$. The *vector chromatic number* of G is

$$\chi_v(G) = \inf\{t \in \mathbb{R} : G \text{ has a vector } t\text{-colouring}\}.$$

Note: $\chi_v(G)$ can be computed (efficiently!) with the following SDP (here $X \in \mathbb{R}^{V \times V}$)

$$\min s$$

$$X \succeq 0$$

$$X_{ii} = 1 \text{ for every } i \in V$$

$$X_{ij} \leq s \text{ whenever } ij \in E$$

Theorem 7.3 $\omega(G) \leq \chi_v(G) \leq \chi_f(G) \leq \chi_c(G) \leq \chi(G)$

Proof: $\omega(G) \leq \chi_v(G)$: Let $x : V \to S^n$ be a vector t-colouring of G and choose a clique $S \subseteq V$ with $|S| = \omega(G)$. Now we have

$$0 \leq \left(\sum_{i \in S} x(i)\right) \cdot \left(\sum_{i \in S} x(i)\right) = |S| + 2 \sum_{i,j \in S\,:\, i \neq j} x(i) \cdot x(j) \leq |S| + |S|(|S| - 1)\left(-\frac{1}{t-1}\right)$$

from which we deduce $t \geq |S| = \omega(G)$ as desired.

$\chi_v(G) \leq \chi_f(G)$: It suffices to show that $\chi_v(Kn(n,k)) \leq \frac{n}{k}$ as the composition of a homomorphism from G to $Kn(n,k)$ with a vector t-colouring of $Kn(n,k)$ is a vector t-colouring of G. For every vertex $S \subseteq \{1, 2, \ldots, n\}$ in we define $w(S) \in \mathbb{R}^n$ as follows:

$$w(S)_i = \begin{cases} k - n & \text{if } i \in S \\ k & \text{if } i \notin S \end{cases}$$

Now w assigns each vertex S a vector $w(S)$ with $||w(S)||^2 = k(n-k)^2 + (n-k)k^2 = k(n-k)n$ and if S, T are adjacent vertices, then $w(S) \cdot w(T) = (2k)k(k-n) + (n-2k)(k^2) = -k^2 n$ so now setting $x = \frac{1}{\sqrt{k(n-k)n}} w$ we have that $||x(S)|| = 1$ for every S and $x(S) \cdot x(T) = \frac{-k}{n-k} = \frac{-1}{n/k-1}$ whenever S and T are adjacent, so x is a vector $\frac{n}{k}$ colouring.

$\chi_f(G) \leq \chi_c(G)$: It suffices to show that $\chi_f(K_{n/k}) \leq \frac{n}{k}$ since the composition of a homomorphism from G to $K_{n/k}$ and a fractional $\frac{n}{k}$-colouring of $K_{n/k}$ is a fractional $\frac{n}{k}$-colouring of G. A fractional $\frac{n}{k}$-colouring of $K_{n/k}$ is given by assigning weight $\frac{1}{k}$ to each of the n intervals of length k in the cyclic ordering of the vertices.

$\chi_c(G) \leq \chi(G)$: This follows immediately from Theorem 7.2. □