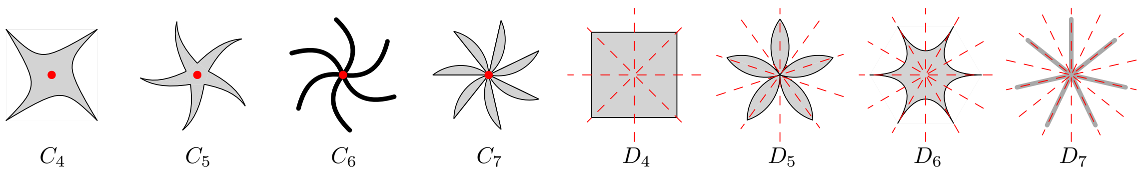
22 Symmetries

Finite subgroups of isometries

We have already seen various shapes in the plane with symmetry group isomorphic to C_n or D_n. For instance, the figure below shows a handful of shapes with a small cyclic or dihedral symmetry group. For the cyclic groups we have highlighted the rotation point. The dihedral group D_n is indicated by n equally spaced mirror lines.



In this section we will prove that every finite subgroup of isometries of $\mathbb{R}^2$ is isomorphic to either C_n or D_n for some positive integer n.

Lemma 22.1. *Let $\mathbf{y} \in \mathbb{R}^2$, let $\mathcal{G}$ be a finite subgroup of even isometries of $\mathbb{R}^2$, and assume that every $F \in \mathcal{G}$ has $\mathbf{y}$ as a fixed point. Then there is a positive integer n so that*

$$\mathcal{G} = \{R_{\mathbf{y},\theta} \mid \theta \textit{ is a multiple of } \tfrac{2\pi}{n}\}.$$

Proof. Every transformation in $\mathcal{G}$ is an even isometry fixing $\mathbf{y}$ so it must be a (possibly trivial) rotation around $\mathbf{y}$. Choose the smallest positive real number θ so that $R_{\mathbf{y},\theta} \in \mathcal{G}$ (since $R_{\mathbf{y},2\pi} = I$ such a number must exist). Suppose (for a contradiction) that there exists an integer k so that $k\theta < 2\pi < (k+1)\theta$. Then the rotation $R_{\mathbf{y},(k+1)\theta} = (R_{\mathbf{y},\theta})^{k+1}$ is in $\mathcal{G}$, but $0 < (k+1)\theta - 2\pi < \theta$ so this is a rotation by a smaller angle than θ, and this is a contradiction. It follows that $\theta = \frac{2\pi}{n}$ for some positive integer n. Since $\mathcal{G}$ is a group it must contain all powers of $R_{\mathbf{y},\frac{2\pi}{n}}$, so $\{R_{\mathbf{y},\theta} \mid \theta \text{ is a multiple of } \frac{2\pi}{n}\} \subseteq \mathcal{G}$. Suppose (for a contradiction) that $\mathcal{G}$ also contains a rotation not in this set, say a rotation by the angle θ' where $j\frac{2\pi}{n} < \theta' < (j+1)\frac{2\pi}{n}$. In this case $\mathcal{G}$ must contain the rotation $R_{\mathbf{y},\theta' - j\frac{2\pi}{n}}$. However, $0 < \theta' - j\frac{2\pi}{n} < \frac{2\pi}{n}$ so this contradicts the choice of θ. This completes the proof. □

Lemma 22.2. *Let $\mathbf{y} \in \mathbb{R}^2$, let $\mathcal{G}$ be a finite subgroup of isometries of $\mathbb{R}^2$, and assume that every $F \in \mathcal{G}$ has $\mathbf{y}$ as a fixed point. Then $\mathcal{G}$ is isomorphic to C_n or D_n for some positive integer n.*

Proof. Let $\mathcal{H}$ be the subgroup given by the intersection of $\mathcal{G}$ and ASO_2 (the subgroup of even isometries). By the previous lemma there exists a positive integer n so that

$$\mathcal{H} = \{R_{\mathbf{y},\theta} \mid \theta \text{ is a multiple of } \tfrac{2\pi}{n}\}.$$

If $\mathcal{G} = \mathcal{H}$ then $\mathcal{G}$ is isomorphic to C_n and there is nothing left to prove. Otherwise, $\mathcal{G}$ must contain a mirror M_L where L is a line through $\mathbf{y}$. Since $\mathcal{G}$ is a subgroup it must contain

$$M_L, \quad M_L R_{\mathbf{y},\frac{2\pi}{n}}, \quad M_L R_{\mathbf{y},2\frac{2\pi}{n}} M_L, \quad \ldots \quad M_L R_{\mathbf{y},(n-1)\frac{2\pi}{n}}$$

Each entry in this list is an odd isometry fixing $\mathbf{y}$ so it is a mirror about a line through L. It follows that every entry in this list is its own inverse. Now, the product of two consecutive entries in this list is

$$M_L R_{\mathbf{y},j\frac{2\pi}{n}} M_L R_{\mathbf{y},(j+1)\frac{2\pi}{n}} = \left(M_L R_{\mathbf{y},j\frac{2\pi}{n}} M_L R_{\mathbf{y},j\frac{2\pi}{n}}\right) R_{\mathbf{y},\frac{2\pi}{n}} = R_{\mathbf{y},\frac{2\pi}{n}}$$

so the mirror lines for consecutive mirrors in this list form an angle of $\frac{\pi}{n}$. If $\mathcal{G}$ were to contain a mirror not in this list, the product of this mirror with another mirror in the list would give a rotation by an angle that is not a multiple of $\frac{2\pi}{n}$ contradicting the definition of $\mathcal{H}$. It follows that $\mathcal{G}$ consists of $\mathcal{H}$ together with the above list of mirrors, and it follows that $\mathcal{G}$ is isomorphic to D_n. $\square$

Theorem 22.3 (Leonardo Da Vinci[1])**.** *Every finite subgroup of isometries of* $\mathbb{R}^2$ *is isomorphic to* C_n *or* D_n *for some* $n \geq 1$.

Proof. Let $\mathcal{G}$ be a finite subgroup of isometries of $\mathbb{R}^2$ and choose a point $\mathbf{y} \in \mathbb{R}^2$. Define the set $Y = \{F(\mathbf{y}) \mid F \in \mathcal{G}\}$ and note that Y is finite. We claim that every isometry $G \in \mathcal{G}$ maps each point in Y to another point in Y. To see this, let $\mathbf{z}$ be an arbitrary point in the set Y and choose $F \in \mathcal{G}$ so that $F(\mathbf{y}) = \mathbf{z}$. Now $G(\mathbf{z}) = GF(\mathbf{y}) \in X$ since $GF \in \mathcal{G}$. It follows that every $G \in \mathcal{G}$ satisfies $G(Y) = Y$. Now let $Y = \{\mathbf{y_1}, \ldots, \mathbf{y_k}\}$ and define the point

$$\mathbf{y}^* = \tfrac{1}{k}\mathbf{y_1} + \tfrac{1}{k}\mathbf{y_2} \ldots \tfrac{1}{k}\mathbf{y_k}$$

[1]This theorem was discovered independently by numerous theorists, making it a formidable if not impossible task to determine who proved it first. We have attributed it to Leonardo Da Vinci since he is among the most famous people to prove it.

(this is known as the barycentre of $\mathbf{y_1}, \ldots, \mathbf{y_k}$). We claim that every $G \in \mathcal{G}$ satisfies $G(\mathbf{y}^*) = \mathbf{y}^*$. To see this, express G as an affine orthogonal function by choosing an orthogonal matrix A and $\mathbf{w} \in \mathbb{R}^2$ so that G is the function $\mathbf{x} \to A\mathbf{x} + \mathbf{w}$. Now we compute:

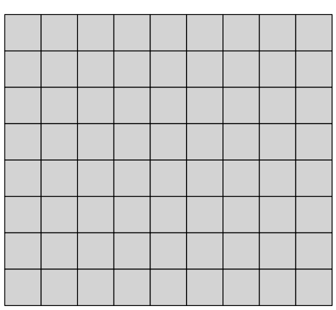
$$\begin{aligned}
G(\mathbf{y}^*) &= A\mathbf{y}^* + \mathbf{w} \\
&= A(\tfrac{1}{k}\mathbf{y_1} \ldots + \tfrac{1}{k}\mathbf{y_k}) + \mathbf{w} \\
&= \tfrac{1}{k}\,(A\mathbf{y_1} + \mathbf{w}) + \ldots + \tfrac{1}{k}\,(A\mathbf{y_k} + \mathbf{w}) \\
&= \tfrac{1}{k}G(\mathbf{y_1}) + \ldots + \tfrac{1}{k}G(\mathbf{y_k}) \\
&= \tfrac{1}{k}\mathbf{y_1} + \ldots + \tfrac{1}{k}\mathbf{y_k} \\
&= \mathbf{y}^*
\end{aligned}$$

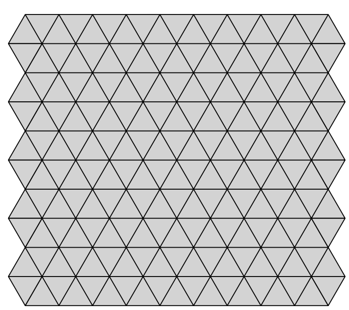
So, every function in $\mathcal{G}$ fixes the point $\mathbf{y}^*$ and now the result follows from the Lemma 22.2. □

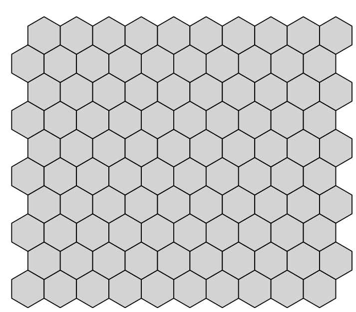
Wallpaper

Definition. A *wallpaper pattern* is a pattern in the Euclidean plane with the property that there are translations by two linearly independent vectors that are symmetries of the pattern. A *wallpaper group* is the symmetry group of a wallpaper pattern. More formally, one may define a wallpaper group to be a subgroup of isometries of $\mathbb{R}^2$ that contains translations by two linearly independent vectors, but not translations by arbitrarily small vectors.

Example: When we introduced the Platonic solids, we saw that assembling equilateral triangles (of side length 1) with three at each vertex gives a Tetrahedron, four at each vertex gives an Octahedron, and five at each vertex gives an Icosahedron. If you put six at each vertex you don't get a solid—you get a tiling of the plane! One can also tile the plane with squares and regular hexagons as shown below. All of these are wallpaper patterns.



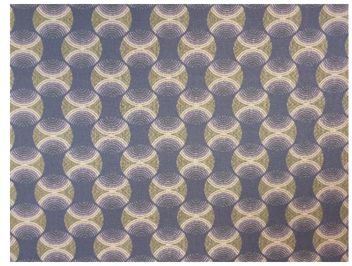




Example: There are many other kinds of wallpaper pattern, here are some more.

Ancient Persian pattern

Artwork by M. C. Escher



Maggie Benson Centre at SFU

Definition. When working with wallpaper patterns, we say that two points $\mathbf{x}$ and $\mathbf{x}'$ are *equivalent* if there is an symmetry of the pattern that maps $\mathbf{x}$ to $\mathbf{x}'$. Similarly, we say that two mirror lines (or glide lines) are *equivalent* if there is a symmetry of the pattern sending one line to the other.

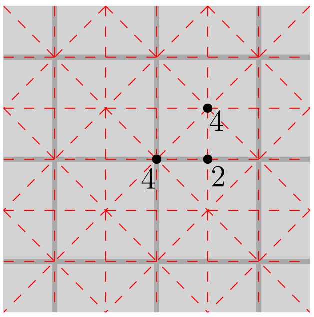
Note: By convention, when considering symmetries of a pattern such as the above artwork by Escher, we *do* allow a symmetry to interchange colours.

Wallpaper Groups

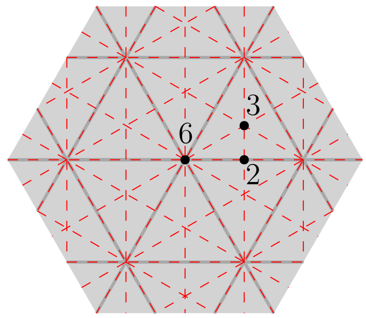
To describe the wallpaper groups, we will give each one a name consisting of symbols, where each symbol indicates a different feature. These features include points with cyclic symmetry (C_n for some $n \geq 2$), dihedral symmetry (D_m for $m \geq 2$), mirror lines, and glides (as well as one extra feature for patterns with only translations as symmetries). We will assign a symbol to the features (given by the chart below) and then combine these symbols (in order) to produce a name for the group. Each feature also has an associated cost (for reasons soon to be revealed!).

Symbol	Cost	Feature
an n before any $*$	$\frac{n-1}{n}$	A point with C_n symmetry
$*$	1	A mirror line
an m after a $*$	$\frac{m-1}{2m}$	A point with D_m symmetry (m mirrors)
$\times$	1	A glide
$\circ$	2	A wallpaper group with only translations

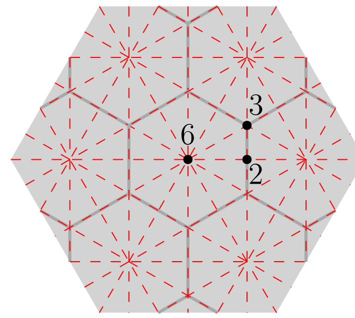
Example: For the tiling of the plane by squares, the mirror lines divide the plane into isosceles right triangles. The corners of each such triangle have two inequivalent points with D_4 symmetry (4 mirror lines) and one with D_2 symmetry (2 mirror lines). This symmetry group is called $*442$ where the $*$ indicates the presence of mirrors, and the 442 (following the $*$) indicate all of the different types of points with dihedral symmetry.



$*442$



$*632$

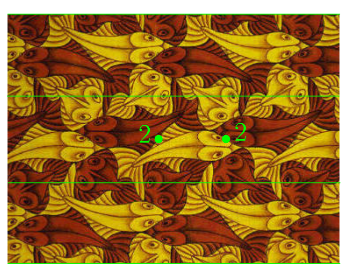


$*632$

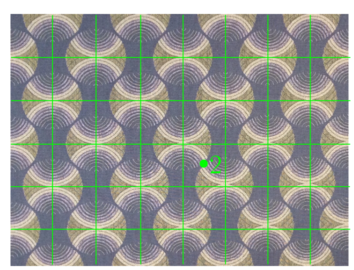
The triangle and hexagon tilings are dual to one another and therefore have the same symmetry group. Here the mirror lines divide the plane into $30-60-90$ triangles and each such triangle has one point with D_6 symmetry, one with D_3, and one with D_2. Hence these tilings have symmetry group $*632$.

Example: The Ancient Persian pattern has no mirrors and is described by the three types of rotation points, 6, 3, 2 so it has name 632. The M.C. Escher work has no mirror lines, two types of 2-fold rotation point plus a glide reflection. Therefore, it has name $22\times$. Finally, the bench from the Maggie Benson Centre has horizontal and vertical mirror lines dividing the plane into small squares. At the centre of each such square is a 2-fold rotation point, and the corners of the square have two inequivalent points with D_2 symmetry. So this symmetry group is called $2*22$.

632



$22\times$



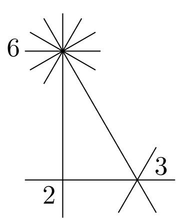
$2*22$

Theorem 22.4 (Magic Theorem)**.** *There are exactly 17 wallpaper groups (up to isomorphism). They are precisely the 17 possible names with total cost 2 as shown below.*

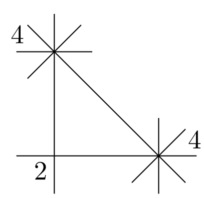
$*632$	$*333$	$*442$	$*2222$
632	333	442	2222
$4{*}2$	$3{*}3$	$2{*}22$	
$22{*}$	$22\times$		
$**$	$*\times$	$\times\times$	
$\circ$			

In the statement of the above theorem, we have arranged the 17 wallpaper groups in 6 rows. Below we provide a brief description of the groups in each row.

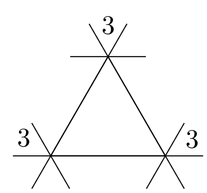
$*$632, **$*$442**, **$*$333**, **$*$2222** Mirror lines divide the plane into triangular or rectangular (possibly square!) regions and there is no point with cyclic symmetry. The corners of a region are inequivalent points with dihedral symmetry and the group name has a $*$ for the mirrors and a number m after the $*$ for each corner with D_m symmetry.



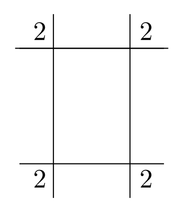
$*$632



$*$442

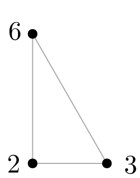


$*$333

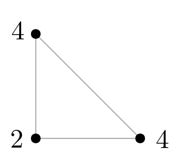


$*$2222

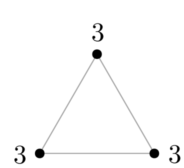
632, 442, 333, 2222 There are no mirror lines, but there are at least three inequivalent rotation points. The group name indicates all of the different types of rotation points. (Note: for 632, 442, and 333, you will see small triangles formed by nearby rotation points of the three inequivalent types. This triangle will always be a 30-60-90 triangle for 632, 45-45-90 for 442, and 60-60-60 for 333.)



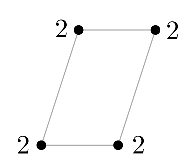
632



442

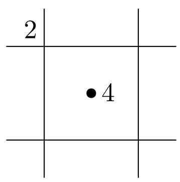


333

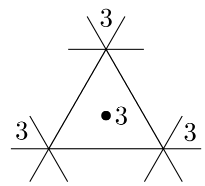


2222

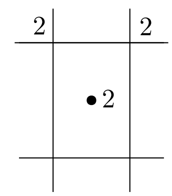
4∗2, 3∗3, 2∗22 Mirror lines divide the plane into triangular or rectangular (possibly square!) regions, and at the centre of each such region is a point with cyclic symmetry, say C_n. The group name has an n for this point with C_n symmetry, then a $*$ for the mirrors, and then an m for each inequivalent point with D_m symmetry. These dihedral points are corners of the region, but some corners are equivalent (due to cyclic symmetries).



4∗2

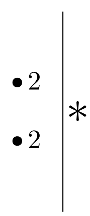


3∗3

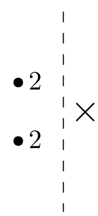


2∗22

22∗, 22× There are two inequivalent points with C_2 symmetry and in addition there is either a mirror line or a glide line.

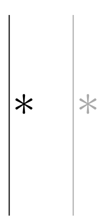


22∗

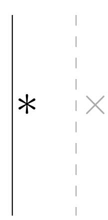


22×

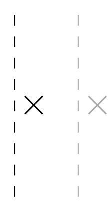
∗∗, ∗×, ×× There are no points with cyclic symmetry, just mirror lines and glide lines, all of which are parallel. There are either two inequivalent mirrors, one mirror and one glide, or no mirrors but two inequivalent glides.



∗∗

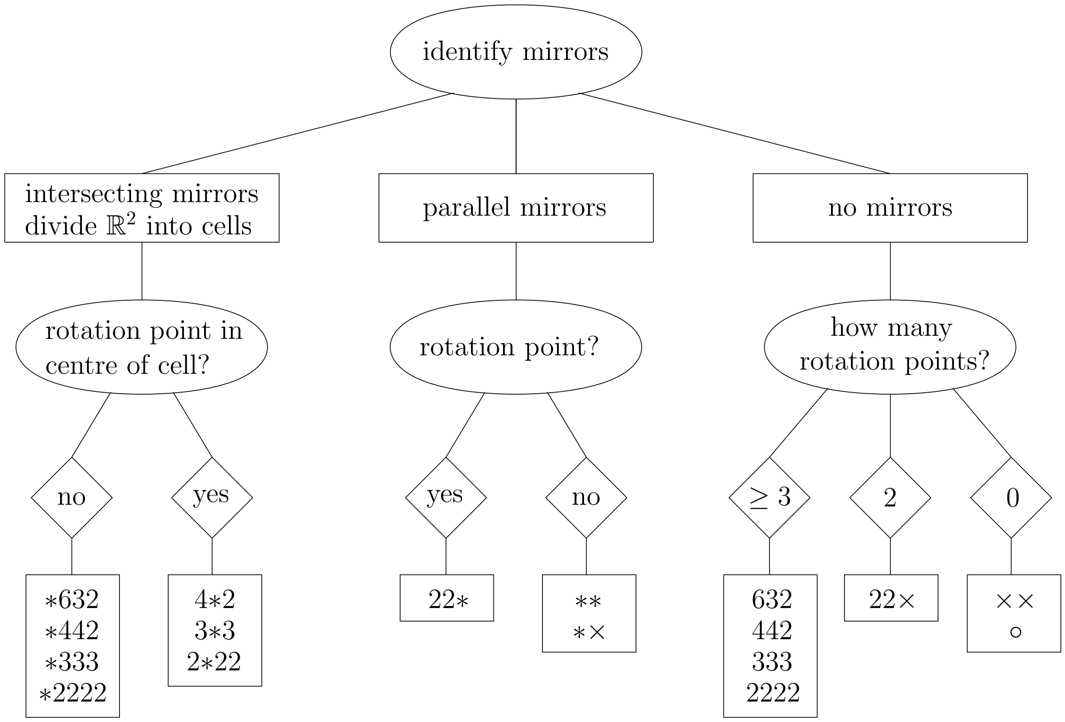


∗×



××

○ Here there are only translational symmetries, no mirrors, cyclic symmetries, or glides.


identify mirrors
intersecting mirrors divide $\mathbb{R}^2$ into cells
parallel mirrors
no mirrors
rotation point in centre of cell?
rotation point?
how many rotation points?
no
yes
yes
no
≥ 3
2
0
$*632$
$*442$
$*333$
$*2222$
$4*2$
$3*3$
$2*22$
$22*$
$**$
$*\times$
632
442
333
2222
$22\times$
$\times\times$
$\circ$