20 Isometries of the Plane

We already have a theorem that gives a good analytic description of all isometries of $\mathbb{R}^n$. The isometries are precisely the set AGO_n of functions of the form $\mathbf{x} \to A\mathbf{x} + \mathbf{y}$ where A is an orthogonal matrix. Though powerful, this theorem may seem a bit opaque... what do these isometries really look like? In this section we will investigate isometries in $\mathbb{R}^2$ and come to an alternate description of these functions that gives a description making it a little clearer what these functions are.

Fixing the origin

The isometries that fix $\mathbf{0}$ are precisely those of the form $F(\mathbf{x}) = A\mathbf{x}$ where A is an orthogonal matrix. The set of all such isometries is GO_n. We will begin our investigation of the plane by investigating these isometries when $n = 2$; that is we will investigate the behaviour of GO_2, the 2×2 orthogonal matrices.

Lemma 20.1. *Every 2×2 orthogonal matrix has one of the following forms for some $\theta \in \mathbb{R}$.*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \textit{or} \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Proof. Let A be a 2×2 orthogonal matrix, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The first column $\begin{bmatrix} a \\ c \end{bmatrix}$ is a unit vector, so $a^2 + c^2 = 1$. Equivalently, the point (a, c) must lie on the unit circle. Therefore, we may choose θ so that $a = \cos\theta$ and $c = \sin\theta$. Now (b, d) must also lie on the unit circle, and since (b, d) is orthogonal to (a, c) they must make an angle of $\frac{\pi}{2}$. This leaves just two possibilities for (b, d)

1. $(b, d) = (\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})) = (-\sin\theta, \cos\theta)$, or
2. $(b, d) = (\cos(\theta - \frac{\pi}{2}), \sin(\theta - \frac{\pi}{2})) = (\sin\theta, -\cos\theta)$.

and the result follows immediately. $\square$

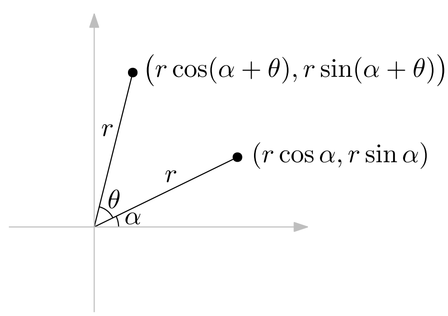
Lemma 20.2. *The functions in GO_2 are precisely those of the form*

1. *$R_{\mathbf{0},\theta}$ given by* $R_{\mathbf{0},\theta}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

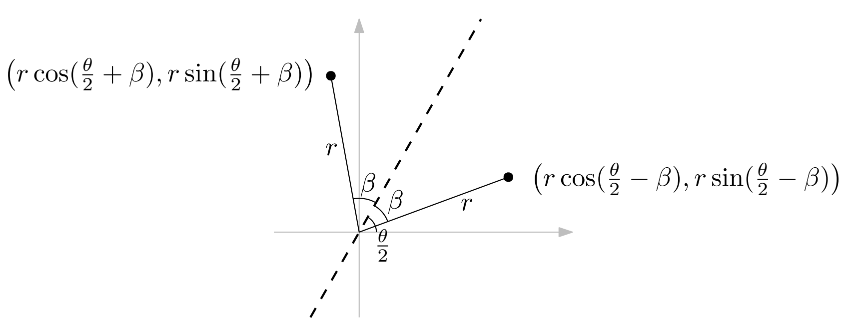
2. M_L where L is a line with angle $\frac{\theta}{2}$ through $\mathbf{0}$ given by $M_L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Proof. Consider a function in GO_2 of the form $\mathbf{x} \to A\mathbf{x}$. To see how these matrices act, it will be helpful to use polar coordinates for $\mathbf{x}$ so we let $\mathbf{x} = (r\cos\alpha, r\sin\alpha)$. By the previous lemma, the matrix A must have one of two forms that we consider separately.

1. $A\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\cos\alpha\sin\theta + r\cos\theta\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$ and we see that this matrix rotates the plane by an angle of θ around the origin. So, in other words, this function is precisely the rotation $R_{\mathbf{0},\theta}$ defied previously.



2. $A\mathbf{x} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\alpha + r\sin\theta\sin\alpha \\ r\cos\alpha\sin\theta - r\cos\theta\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos(\theta-\alpha) \\ r\sin(\theta-\alpha) \end{bmatrix}$
This transformation is a little easier to understand if we change variables. Define $\alpha = \frac{1}{2}\theta + \beta$. Then the point $\begin{bmatrix} r\cos(\frac{1}{2}\theta+\beta) \\ r\sin(\frac{1}{2}\theta+\beta) \end{bmatrix}$ maps to $\begin{bmatrix} r\cos(\frac{1}{2}\theta-\beta) \\ r\sin(\frac{1}{2}\theta-\beta) \end{bmatrix}$. So this transformation is precisely M_L where L is the line through the origin at an angle of $\frac{1}{2}\theta$.



$\square$

Fixing a different point

Suppose that F is an isometry and $F(\mathbf{y}) = \mathbf{y}$. If $\mathbf{y} = \mathbf{0}$ then we know that F is given by an orthogonal matrix, but what if $\mathbf{y}$ is another point? Next we show that these isometries are given by closely related affine transformations.

Proposition 20.3. *The functions in AGO_2 fixing $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are precisely*

1. *$R_{\mathbf{y},\theta}$ given by*

$$R_{\mathbf{y},\theta}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

2. *M_L where L is a line with angle $\frac{\theta}{2}$ through $\mathbf{y}$ given by*

$$M_L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1-\cos\theta & -\sin\theta \\ -\sin\theta & 1+\cos\theta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Proof. Let $F \in AGO_2$ satisfy $F(\mathbf{y}) = \mathbf{y}$ and consider the transformation

$$G = T_{-\mathbf{y}} F T_{\mathbf{y}}$$

The function G is an isometry since it is a product of three isometries, and isometries form a subgroup. Furthermore, $G(\mathbf{0}) = T_{-\mathbf{y}}(F(T_{\mathbf{y}}(\mathbf{0}))) = T_{-\mathbf{y}}(F(\mathbf{y})) = T_{-\mathbf{y}}(\mathbf{y}) = \mathbf{0}$. So G is an isometry fixing the origin. Therefore, there is an orthogonal matrix A so that $G(\mathbf{x}) = A\mathbf{x}$. Multiplying the equation $G = T_{-\mathbf{y}} F T_{\mathbf{y}}$ on the left by $T_{\mathbf{y}}$ and on the right by $T_{-\mathbf{y}}$ yields the equation $F = T_{\mathbf{y}} G T_{-\mathbf{y}}$. It follows that the function F is give by the rule

$$F(\mathbf{x}) = T_{\mathbf{y}} G T_{-\mathbf{y}}(\mathbf{x}) = T_{\mathbf{y}} G(\mathbf{x} - \mathbf{y}) = T_{\mathbf{y}} A(\mathbf{x} - \mathbf{y}) = \mathbf{y} + A(\mathbf{x} - \mathbf{y})$$

We see that the function F maps the point $\mathbf{y} + \mathbf{w}$ to the point $\mathbf{y} + A\mathbf{w}$. The result now follows by combining this with the analysis in the previous lemma. $\square$

Classifying all isometries

We have seen several different types of isometries of the plane, namely translations, rotations, and mirrors (the identity may be viewed as a trivial translation and as a trivial rotation).

However, there is one last type of isometry we have yet to introduce, and it's time to meet this last variety.

Definition. Let L be a line in the plane and let $\mathbf{v}$ be a vector in the same direction as L. We define the *glide reflection* (or just *glide* for short) $G_{L,\mathbf{v}}$ by the rule

$$G_{L,\mathbf{v}}(\mathbf{x}) = T_{\mathbf{v}} M_L.$$

So a glide reflection is just a mirror about a line L followed by a translation in the same direction as L. Note that (since $\mathbf{v}$ is nonzero) the glide reflection $G_{L,\mathbf{v}}$ will not have any fixed points. However $G_{L,\mathbf{v}}$ does map the line L to itself (i.e. $G_{L,\mathbf{v}}(L) = L$).

It is easy to see that glides are isometries since we know that translations and mirrors are isometries, and isometries are closed under products. However, it is not obvious that we have really listed all types of isometries yet. This fact we prove next.

Theorem 20.4. *Every isometry of $\mathbb{R}^2$ is either a translation, rotation, mirror, or glide.*

Proof. Consider an arbitrary isometry F of $\mathbb{R}^2$. By a theorem from before may choose an orthogonal matrix A and $\mathbf{w} \in \mathbb{R}^2$ so that

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{w}$$

We need to show that F is either a translation, rotation, mirror, or glide. Our analysis will break into cases depending on the matrix A.

Case 1: $A = I$

In this case we have $F(\mathbf{x}) = \mathbf{x} + \mathbf{w} = T_{\mathbf{w}}(\mathbf{x})$ so F is the translation $T_{\mathbf{w}}$.

Case 2: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some $0 < \theta < 2\pi$.

It follows from the previous proposition that for every $\mathbf{y} \in \mathbb{R}^2$ the rotation $R_{\mathbf{y},\theta}$ is given by

$$R_{\mathbf{y},\theta}(\mathbf{x}) = A\mathbf{x} + (I - A)\mathbf{y}$$

So, if there exists a vector $\mathbf{y}$ so that $(I - A)\mathbf{y} = \mathbf{w}$, then F is a rotation about $\mathbf{y}$. The matrix $I - A$ has determinant

$$\det(I - A) = \det\left(\begin{bmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{bmatrix}\right) = (1 - \cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta > 0$$

Therefore $I - A$ is invertible and there exists a vector $\mathbf{y}$ so that $(I - A)\mathbf{y} = \mathbf{w}$ and we conclude that F is a rotation about $\mathbf{y}$.

Case 3: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ for some $0 \le \theta < 2\pi$.

We claim that the vectors $\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$ and $\begin{bmatrix} 1 - \cos\theta \\ -\sin\theta \end{bmatrix}$ are orthogonal. To see this we compute the dot product using half-angle formulas from trig.

$$\begin{aligned}
\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 - \cos\theta \\ -\sin\theta \end{bmatrix} &= \cos\tfrac{\theta}{2} - \cos\tfrac{\theta}{2}\cos\theta - \sin\tfrac{\theta}{2}\sin\theta \\
&= \cos\tfrac{\theta}{2} - \cos\tfrac{\theta}{2}\left(\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\right) - \sin\tfrac{\theta}{2}\left(2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\right) \\
&= \cos\tfrac{\theta}{2}\left(1 - \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\right) \\
&= 0
\end{aligned}$$

Since these vectors are orthogonal, we can express $\mathbf{w}$ as follows

$$\mathbf{w} = s\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} + t\begin{bmatrix} 1 - \cos\theta \\ -\sin\theta \end{bmatrix}$$

Now we claim that the isometry F is the glide reflection $G_{L,\mathbf{v}}$ where $\mathbf{v} = s\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$ and L is the line through the point $(t, 0)$ at an angle of $\frac{\theta}{2}$. To check this, we evaluate this function at an arbitrary point $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\begin{aligned}
G_{L,\mathbf{v}}(\mathbf{x}) &= T_{\mathbf{v}}M_L(\mathbf{x}) \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 - \cos\theta & -\sin\theta \\ -\sin\theta & 1 + \cos\theta \end{bmatrix}\begin{bmatrix} t \\ 0 \end{bmatrix} + s\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix} \\
&= A\mathbf{x} + \mathbf{w}
\end{aligned}$$

□