19 Orthogonal Matrices

Recall that in $\mathbb{R}^n$ an orthonormal basis is a set of vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ satisfying:

- $||\mathbf{u_i}|| = 1$ for every $1 \leq i \leq n$
- $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ whenever $i \neq j$.

Examples: The following are orthonormal bases of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ (check!)

- $\{(\frac{3}{5}, \frac{4}{5}), (\frac{4}{5}, -\frac{3}{5})\}$
- $\{(\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}), (-\frac{2}{7}, \frac{6}{7}, \frac{3}{7}), (\frac{3}{7}, -\frac{2}{7}, \frac{6}{7})\}$
- $\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$

Definition. We say that a square matrix U is an *orthogonal* matrix if the columns of U form an orthonormal basis.

Examples: The following are orthogonal matrices

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} \qquad \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{3}{7} & \frac{6}{7} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Definition. $GO_n = \{\mathbf{x} \to U\mathbf{x} \mid U \text{ is an orthogonal } n \times n \text{ matrix}\}$. The following theorem shows that GO_n is precisely the subgroup of isometries of $\mathbb{R}^n$ fixing the origin. We call GO_n the *general orthogonal group*.

Theorem 19.1. *The subgroup of isometries of* $\mathbb{R}^n$ *fixing* $\mathbf{0}$ *is equal to* GO_n.

Proof. First we prove every element of GO_n is an isometry fixing $\mathbf{0}$. To do so, consider an arbitrary function in GO_n of the form $\mathbf{x} \to A\mathbf{x}$ where $A = \begin{bmatrix} \mathbf{a_1} & \ldots & \mathbf{a_n} \end{bmatrix}$ is an orthogonal matrix. First we prove a key fact:

Claim: Every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ satisfy $(A\mathbf{v}) \cdot (A\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$

To prove this claim, let $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$. Then using the fact that $\mathbf{a_i} \cdot \mathbf{a_j} = 0$ if $i \neq j$ and $\mathbf{a_i} \cdot \mathbf{a_i} = 1$ we find that

$$\begin{aligned}(A\mathbf{v}) \cdot (A\mathbf{w}) &= (v_1\mathbf{a_1} + \ldots + v_n\mathbf{a_n}) \cdot (w_1\mathbf{a_1} + \ldots + w_n\mathbf{a_n}) \\ &= v_1w_1 + \ldots + v_nw_n \\ &= \mathbf{v} \cdot \mathbf{w}\end{aligned}$$

Now for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\begin{aligned}\text{dist}(A\mathbf{x}, A\mathbf{y}) &= ||A\mathbf{x} - A\mathbf{y}|| \\ &= ||A(\mathbf{x} - \mathbf{y})|| \\ &= \sqrt{A\big(\mathbf{x} - \mathbf{y}\big) \cdot A\big(\mathbf{x} - \mathbf{y}\big)} \\ &= \sqrt{\big(\mathbf{x} - \mathbf{y}\big) \cdot \big(\mathbf{x} - \mathbf{y}\big)} \\ &= ||\mathbf{x} - \mathbf{y}|| \\ &= \text{dist}(\mathbf{x}, \mathbf{y})\end{aligned}$$

and thus the function $\mathbf{x} \to A\mathbf{x}$ is an isometry.

To prove the other direction, let F be an isometry of $\mathbb{R}^n$ fixing $\mathbf{0}$. By a lemma proved earlier we have that $F(\mathbf{x}) = A\mathbf{x}$ for some $n \times n$ matrix $A = \begin{bmatrix}\mathbf{a_1} & \ldots & \mathbf{a_n}\end{bmatrix}$. Let $\mathbf{e_i}$ be the vector with a 1 in position i and 0 in every other position and note that $F(\mathbf{e_i}) = A\mathbf{e_i} = \mathbf{a_i}$. Now we have

$$||a_i|| = \text{dist}(\mathbf{0}, \mathbf{a_i}) = \text{dist}(F(\mathbf{0}), F(\mathbf{e_i})) = \text{dist}(\mathbf{0}, \mathbf{e_i}) = 1$$

Now assuming $i \neq j$ we have

$$\begin{aligned}2 &= ||\mathbf{e_i} - \mathbf{e_j}||^2 \\ &= (\text{dist}(\mathbf{e_i}, \mathbf{e_j}))^2 \\ &= (\text{dist}(\mathbf{a_i}, \mathbf{a_j}))^2 \\ &= ||\mathbf{a_i} - \mathbf{a_j}||^2 \\ &= (\mathbf{a_i} - \mathbf{a_j}) \cdot (\mathbf{a_i} - \mathbf{a_j}) \\ &= ||\mathbf{a_i}|| + ||\mathbf{a_j}|| - 2\mathbf{a_i} \cdot \mathbf{a_j} \\ &= 2 - 2\mathbf{a_i} \cdot \mathbf{a_j}\end{aligned}$$

It follows that $\mathbf{a_i} \cdot \mathbf{a_j} = 0$ and we deduce that A is an orthogonal matrix, as desired. $\square$

Definition. $AGO_n = \{\mathbf{x} \to U\mathbf{x} + \mathbf{w} \mid U$ is an orthogonal $n \times n$ matrix and $\mathbf{w} \in \mathbb{R}^n\}$. The following corollary shows that AGO_n is precisely the set of isometries of $\mathbb{R}^n$. We call AGO_n the *affine general orthogonal group*.

Corollary 19.2. *The set of isometries of* $\mathbb{R}^n$ *is precisely* AGO_n.

Proof. Every function in AGO_n is the composition of two isometries—a translation and a function in GO_n. Since the product of two isometries is another isometry, we deduce that AGO_n is a subset of the set of isometries.

To prove the other direction, let F be an isometry of $\mathbb{R}^n$ and let $\mathbf{y} = F(\mathbf{0})$. Now define the transformation $G = T_{-\mathbf{y}}F$. Note that G is an isometry since the product of two isometries is an isometry. Now $G(\mathbf{0}) = T_{-\mathbf{y}}(F(\mathbf{0})) = T_{-\mathbf{y}}(\mathbf{y}) = \mathbf{0}$. By the previous theorem there is an orthogonal matrix A so that G is the function $G(\mathbf{x}) = A\mathbf{x}$. Since $F = T_{\mathbf{y}}G$ we deduce that $F(\mathbf{x}) = A\mathbf{x} + \mathbf{y}$ and thus F is in AGO_n. $\square$