18 Isometries

Basic Properties

Recall that the distance between two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is given by

$$\text{dist}(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}.$$

Definition. A function $F \in \mathrm{T}rans(\mathbb{R}^n)$ is an *isometry* if every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfy

$$\text{dist}(F(\mathbf{x}), F(\mathbf{y})) = \text{dist}(\mathbf{x}, \mathbf{y})$$

In words, we say that a function F is an isometry if it preserves distances. Isometries are also called *rigid* transformations and we view them as the natural family of transformations that preserve the "structure" of Euclidean space.

Example: In $\mathbb{R}^2$ (as we will prove) every rotation $R_{\mathbf{x},\theta}$ and every mirror M_L is an isometry.

Our goal is to develop an understanding of isometries. The next two lemmas take a couple of steps toward this goal.

Lemma 18.1. *Every translation is an isometry*

Proof. For the translation $T_{\mathbf{z}}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We have

$$\text{dist}(T_{\mathbf{z}}(\mathbf{x}), T_{\mathbf{z}}(\mathbf{y})) = \text{dist}(\mathbf{z} + \mathbf{x}, \mathbf{z} + \mathbf{y}) = ||(\mathbf{z} + \mathbf{x}) - (\mathbf{z} + \mathbf{y})|| = ||\mathbf{x} - \mathbf{y}|| = \text{dist}(\mathbf{x}, \mathbf{y}) \quad \square$$

Lemma 18.2. *The set of all isometries of* $\mathbb{R}^n$ *is a subgroup of* $\text{Trans}(\mathbb{R}^n)$

Proof. We need to show identity containment and closure under multiplication and inverses.
(identity) It is immediate from the definition that the identity is an isometry.
(mult. closure) If F, G are isometries of $\mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then

$$\text{dist}(FG(\mathbf{x}), FG(\mathbf{y})) = \text{dist}(F(G(\mathbf{x})), F(G(\mathbf{y}))) = \text{dist}(G(\mathbf{x})), G(\mathbf{y})) = \text{dist}(\mathbf{x}, \mathbf{y}).$$

It follows that FG is an isometry, thus establishing closure under multiplication.
(inverses) Let F be an isometry and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Since F is an isometry

$$\text{dist}(F^{-1}(\mathbf{x}), F^{-1}(\mathbf{y})) = \text{dist}(F(F^{-1}(\mathbf{x})), F(F^{-1}(\mathbf{y}))) = \text{dist}(\mathbf{x}, \mathbf{y})$$

and it follows that F^{-1} is also an isometry. This establishes closure under inverses. $\square$

Lemma 18.3. *Let $F \in \text{Trans}(\mathbb{R}^n)$ be an isometry. If L is a line in $\mathbb{R}^n$, then $F(L)$ is a line.*

Proof. To prove this lemma, it suffices to show that whenever $\mathbf{x}, \mathbf{y}, \mathbf{z}$ lie on a line, then $F(\mathbf{x}), F(\mathbf{y}), F(\mathbf{z})$ lie on a line. So assume $\mathbf{x}, \mathbf{y}, \mathbf{z}$ lie on a line with $\mathbf{y}$ between $\mathbf{x}$ and $\mathbf{z}$. Then

$$\begin{aligned} &\text{dist}(\mathbf{x}, \mathbf{z}) = \text{dist}(\mathbf{x}, \mathbf{y}) + \text{dist}(\mathbf{y}, \mathbf{z}) \\ \Rightarrow\ &\text{dist}(F(\mathbf{x}), F(\mathbf{z})) = \text{dist}(F(\mathbf{x}), F(\mathbf{y})) + \text{dist}(F(\mathbf{y}), F(\mathbf{z})) \\ \Rightarrow\ &F(\mathbf{x}), F(\mathbf{y}), F(\mathbf{z}) \text{ lie on a line with } \mathbf{y} \text{ between } \mathbf{x} \text{ and } \mathbf{z} \end{aligned}$$

giving the desired conclusion. □

Symmetry

Definition. A *symmetry* of a set $S \subseteq \mathbb{R}^n$ is an isometry $F \in \text{Trans}(\mathbb{R}^n)$ so that $F(S) = S$.

Proposition 18.4. *For every $S \subseteq \mathbb{R}^n$, the set of symmetries of S is a subgroup of* $\text{Trans}(S)$.

Proof. Let $\mathcal{G}$ be the set of symmetries of S. We need to prove that $\mathcal{G}$ contains the identity, is closed under products, and closed under inverses.

(identity) The identity I satisfies $I(S) = S$ so $I \in \mathcal{G}$.

(mult. closure) If $F, G \in \mathcal{G}$ then $F(S) = S$ and $G(S) = S$ we find that $FG(S) = F(G(S)) = F(S) = S$ and thus $FG \in \mathcal{G}$.

(inverses) Finally, if $F \in \mathcal{G}$ then $F(S) = S$ so F maps S bijectively to S. It follows that F^{-1} also maps S bijectively to S, so $F^{-1} \in \mathcal{G}$. □

Linearity

Definition. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if it satisfies the following properties:

(1) For every $\mathbf{x} \in \mathbb{R}^n$ and every $t \in \mathbb{R}$ we have $F(t\mathbf{x}) = tF(\mathbf{x})$.

(2) For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have $F(\mathbf{x} + \mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y})$.

Note: If F is linear, then there exists an $m \times n$ matrix A so that the function F is given by the rule $F(\mathbf{x}) = A\mathbf{x}$.

Definition. We say that a function F *fixes* $\mathbf{x}$ if $F(\mathbf{x}) = \mathbf{x}$.

Lemma 18.5. *Every isometry that fixes* $\mathbf{0}$ *is linear.*

Proof. Let $F \in \text{Trans}(\mathbb{R}^n)$ be an isometry that satisfies $F(\mathbf{0}) = \mathbf{0}$. We will show that F satisfies (1) and (2) in the definition of linear.

To prove (1) let $\mathbf{x} \in \mathbb{R}^n$ and let $t \in \mathbb{R}$. If $\mathbf{x} = \mathbf{0}$ then $F(t\mathbf{x}) = F(\mathbf{0}) = \mathbf{0} = t\mathbf{0}$. So, we may assume $\mathbf{x} \neq \mathbf{0}$. Let $\mathbf{y} = F(\mathbf{x})$ and observe that since F is an isometry fixing $\mathbf{0}$ we must have

$$||\mathbf{x}|| = \text{dist}(\mathbf{x}, \mathbf{0}) = \text{dist}(\mathbf{y}, \mathbf{0}) = ||\mathbf{y}||$$

Define $L_{\mathbf{x}}$ to be the line $\text{Span}(\mathbf{x})$ and $L_{\mathbf{y}}$ to be the line $\text{Span}(\mathbf{y})$ and note that Lemma 18.3 shows that $F(L_{\mathbf{x}}) = L_{\mathbf{y}}$. Now, $t\mathbf{x}$ is the unique point on $L_{\mathbf{x}}$ that has distance $||t\mathbf{x}||$ to $\mathbf{0}$ and distance $||(t-1)\mathbf{x}||$ to $\mathbf{x}$. Similarly, $t\mathbf{y}$ is the unique point on $L_{\mathbf{y}}$ that has distance $||t\mathbf{y}||$ to $\mathbf{0}$ and distance $||(t-1)\mathbf{y}||$ to $\mathbf{y}$. It follows that $F(t\mathbf{x}) = t\mathbf{y} = tF(\mathbf{x})$ as desired.

To prove (2) let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$. Since F is an isometry, it must map the midpoint between $\mathbf{x}$ and $\mathbf{x}'$ to the midpoint between their images $F(\mathbf{x})$ and $F(\mathbf{x}')$, so

$$F(\tfrac{1}{2}\mathbf{x} + \tfrac{1}{2}\mathbf{x}') = \tfrac{1}{2}F(\mathbf{x}) + \tfrac{1}{2}F(\mathbf{x}').$$

It follows from (1) that

$$\tfrac{1}{2}F(\mathbf{x} + \mathbf{x}') = F(\tfrac{1}{2}\mathbf{x} + \tfrac{1}{2}\mathbf{x}').$$

Combining these equations gives $F(\mathbf{x} + \mathbf{x}') = F(\mathbf{x}) + F(\mathbf{x}')$ as desired. □

Lemma 18.6. *Every isometry is an affine transformation.*

Proof. Let $F \in \text{Trans}(\mathbb{R}^n)$ be an isometry and let $\mathbf{y} = F(\mathbf{0})$. Now we may define the transformation $G = T_{-\mathbf{y}}F$ and we have

$$G(\mathbf{0}) = T_{-\mathbf{y}}F(\mathbf{0}) = T_{-\mathbf{y}}(\mathbf{y}) = \mathbf{0}$$

It follows from Lemma 18.5 that G is linear, so we may choose a matrix A so that $G(\mathbf{x}) = A\mathbf{x}$. Now $F = (T_{-\mathbf{y}})^{-1}G = T_{\mathbf{y}}G$ so F is given by the rule $F(\mathbf{x}) = A\mathbf{x} + \mathbf{y}$, so F is an affine transformation. □