15 Transformation Subgroups

Subgroups

Informally, we have described a symmetry of an object as a "structure-preserving" transformation that sends the object back to itself. As an example of what this informal notion means, consider a square S in the plane. Rotating S by $90°$ around the centre of S is a symmetry of it. However, the transformation of $\mathbb{R}^2$ that fixes every point of the plane except for two opposite corners of S and interchanges these two is not a symmetry of S. To work with symmetries of such objects, we will not be working with all of the transformation group $\text{Trans}(\mathbb{R}^n)$, but rather a subset of it. This brings us to the following key notion.

Definition. A subset of transformations, $\mathcal{G} \subseteq \text{Trans}(X)$ is a *subgroup* if it satisfies:

- $I_X \in \mathcal{G}$. (Identity containment)
- If $F, G \in \mathcal{G}$, then $FG \in \mathcal{G}$. (Closure under products)
- If $F \in \mathcal{G}$, then $F^{-1} \in \mathcal{G}$. (Closure under inverses)

In this case we also call $\mathcal{G}$ a *transformation subgroup* or a *subgroup of transformations.*

Note: A subgroup $\mathcal{G} \subseteq \text{Trans}(X)$ has all of the key features of the original group (the identity, products, and inverses). Therefore, $\mathcal{G}$ is group on its own.

Continuing our informal discussion of symmetry, let us note that the set of symmetries of a square will naturally form a subgroup of $\text{Trans}(\mathbb{R}^2)$ since the identity function is a symmetry, the product of two symmetries is another symmetry (this just means performing one then the other) and the inverse of a symmetry is a symmetry.

Examples: Here are some subgroups of $\text{Trans}(\mathbb{R})$.

1. $\mathcal{F} = \{F \in \text{Trans}(\mathbb{R}) \mid F \text{ is continuous}\}$.
2. $\mathcal{G} = \{F \in \text{Trans}(\mathbb{R}) \mid F \text{ is differentiable}\}$.
3. $\mathcal{H} = \{F \in \text{Trans}(\mathbb{R}) \mid F(x) = cx \text{ for some } c \neq 0\}$.
4. $\mathcal{J} = \{F \in \text{Trans}(\mathbb{R}) \mid F(x) = x + c \text{ for some } c \in \mathbb{R}\}$.

Subgroups of S_n

Before diving into the world of subgroups of $\text{Trans}(\mathbb{R}^n)$ let us pause to prove one result concerning a subgroup of S_n.

Definition. Let $A_n = \{A \in S_n \mid A \text{ is even}\}$. The following lemma shows that A_n is a subgroup and we call it the *alternating group*.

Lemma 15.1. *A_n is a subgroup of S_n.*

Proof. To prove that a subset of a group is a subgroup, we need to check that it contains the identity, is closed under products, and is closed under inverses. We previously showed that the identity is an even permutation, so $I \in S_n$. We also proved that whenever A, B are even, the product AB is even, so A_n is closed under products. Finally, if A is even, then A^{-1} must also be even since $AA^{-1} = I$ is even. Therefore A_n is closed under inverses. □

Proposition 15.2. *The groups S_n and A_n have sizes $|S_n| = n!$ and $|A_n| = \frac{1}{2}n!$ for $n \geq 2$*

Proof. To see that the number of permutations in S_n is equal to $n!$, consider one row notation. There are n choices for the first position, $n-1$ for the second, $n-2$ for the third, and so on. The total number of elements is therefore $n(n-1)\ldots(2)(1) = n!$.
Define a function $F : S_n \to S_n$ by the rule $F(A) = (12)A$. The composition of this function with itself is the identity (since applying it twice has the effect of multiplying the input permutation on the left by $(12)(12) = I$). It follows from this that F is a bijection. If $A \in S_n$ is even, then $F(A)$ is odd, and if A is odd, then $F(A)$ is even. Therefore $|A_n| = |F(A_n)| = |S_n \setminus A_n|$ from which it follows that $|A_n| = \frac{1}{2}|S_n| = \frac{1}{2}n!$. □

Subgroups of $\text{Trans}(\mathbb{R}^n)$

Definition. For every $\mathbf{y} \in \mathbb{R}^n$ define the function $T_\mathbf{y} : \mathbb{R}^n \to \mathbb{R}^n$ by the rule $T_\mathbf{y}(\mathbf{x}) = \mathbf{x} + \mathbf{y}$. We call the function $T_\mathbf{y}$ a *translation* by $\mathbf{y}$. We let $\mathcal{T}_n = \{T_\mathbf{y} \mid \mathbf{y} \in \mathbb{R}^n\}$. The following lemma shows that $\mathcal{T}_n$ is a subgroup and we call it the *translation group*.

Lemma 15.3. *$\mathcal{T}_n$ is a subgroup of* $\text{Trans}(\mathbb{R}^n)$.

Proof. For every $\mathbf{y} \in \mathbb{R}^n$ the functions $T_{\mathbf{y}}$ and $T_{-\mathbf{y}}$ are inverse. It follows immediately from this that $\mathcal{T}$ is a subset of Trans($\mathbb{R}^n$). Furthermore, this observation shows that $\mathcal{T}$ is closed under inverses. The function $T_{\mathbf{0}}$ is the identity function since it maps $\mathbf{x}$ to $\mathbf{x} + \mathbf{0} = \mathbf{x}$. To prove closure under products, let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and observe that

$$T_{\mathbf{y}}T_{\mathbf{z}}(\mathbf{x}) = T_{\mathbf{y}}(\mathbf{x} + \mathbf{z}) = \mathbf{x} + \mathbf{z} + \mathbf{y} = T_{\mathbf{y}+\mathbf{z}}(\mathbf{x})$$

It follows that $T_{\mathbf{y}}T_{\mathbf{z}} = T_{\mathbf{y}+\mathbf{z}}$, so $\mathcal{T}$ is closed under products. It follows that $\mathcal{T}$ is a subgroup of $Trans(\mathbb{R}^n)$ as desired. □

Note: Usually when working with functions it is most convenient to give them names. For instance, we may define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = x^2$ and this gives the name f to the "squaring function". However, in other instances we might just write $y = x^2$ to describe this same function without giving it a name. Another variation of this is to write $x \to x^2$ to describe the same function. More generally, we will use the notation $\mathbf{x} \to$ (expression in terms of $\mathbf{x}$) to describe a functions of Euclidean space (in many such instances, the domain and codomain must be inferred from context).[1]

Definition. $GL_n = \{\mathbf{x} \to A\mathbf{x} \mid A \text{ is an invertible } n \times n \text{ matrix}\}$. The following lemma shows that GL_n is a subgroup and we call it the *general linear group*.

Lemma 15.4. *GL_n is a subgroup of* Trans($\mathbb{R}^n$).

Proof. For every invertible matrix A, the function $\mathbf{x} \to A\mathbf{x}$ has inverse function $\mathbf{x} \to A^{-1}\mathbf{x}$. Therefore GL_n is a subset of Trans($\mathbb{R}^n$). Furthermore, this observation shows that GL_n is closed under inverses. The identity function I is given by the identity matrix, so it is contained in GL_n. To prove closure under multiplication, let A, B be invertible matrices and note that $\det A \neq 0 \neq \det B$. Now the composition of $\mathbf{x} \to A\mathbf{x}$ and $\mathbf{x} \to B\mathbf{x}$ is the function $\mathbf{x} \to AB\mathbf{x}$. Since $\det AB = \det A \det B \neq 0$ the matrix AB is also invertible, and therefore, GL_n is closed under products. □

Definition. $AGL_n = \{\mathbf{x} \to A\mathbf{x} + \mathbf{w} \mid A \text{ is an invertible } n \times n \text{ matrix and } \mathbf{w} \in \mathbb{R}^n\}$. The following lemma shows that AGL_n is a subgroup and we call it the *general affine group*.

[1]Why would we want to describe a function but not name it? Naming an object gives it a certain signficance and status, so for clarity we wish to keep names to a minimum. Our choice not to give a name to a function is a decision to place emphasis elsewhere.

Lemma 15.5. *AGL_n is a subgroup of* $\mathrm{Trans}(\mathbb{R}^n)$.

Proof. Every function in AGL_n is the composition of a function in GL_n and a function in $\mathcal{T}_n$ so it is indeed a transformation. Therefore AGL_n is a subset of $\mathrm{Trans}(\mathbb{R}^n)$. The identity function is given by $\mathbf{x} \to I\mathbf{x} + \mathbf{0}$ so it is contained in AGL_n. Let A, B be invertible and let $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. The composition of $\mathbf{x} \to A\mathbf{x} + \mathbf{y}$ and $\mathbf{x} \to B\mathbf{x} + \mathbf{z}$ sends $\mathbf{x}$ to

$$B\left(A\mathbf{x} + \mathbf{y}\right) + \mathbf{z} = BA\mathbf{x} + \left(B\mathbf{y} + \mathbf{z}\right)$$

Since A, B are invertible, the matrix BA is also invertible and we find that AGL_n is closed under products. Finally we claim that the function $\mathbf{x} \to A\mathbf{x} + \mathbf{y}$ has inverse $\mathbf{x} \to A^{-1}\mathbf{x} + A^{-1}(-\mathbf{y})$. To verify this, note that the composition of these functions sends $\mathbf{x}$ to

$$A\left(A^{-1}\mathbf{x} + A^{-1}(-\mathbf{y})\right) + \mathbf{y} = \mathbf{x} + A^{-1}A(-\mathbf{y}) + \mathbf{y} = \mathbf{x}.$$

We conclude that AGL_n is closed under inverses, and is therefore a subgroup. □

It is straightforward to see that $GL_n, \mathcal{T}_n \subseteq AGL_n$. The only transformation in $\mathcal{T}_n$ which sends $\mathbf{0}$ to $\mathbf{0}$ is the identity, so $GL_n \cap \mathcal{T}_n = \{I\}$. This gives the Venn Diagram below.