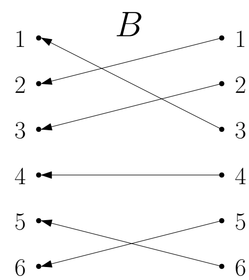
13 Permutations

Definition. For every positive integer n we define $S_n = Trans(\{1, 2, \ldots, n\})$. An element of S_n is called a *permutation*. So, a permutation is a bijection from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. In the example below we define a permutation that we carry through this section.

Example: A permutation $B \in S_6$.



Two Row Notation. We may denote $A \in S_n$ by the two row array

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ A(1) & A(2) & \ldots & A(n) \end{pmatrix}.$$

Here each element in the top row maps to the one below it.

Example Continued: Two row notation for our example permutation B is

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$$

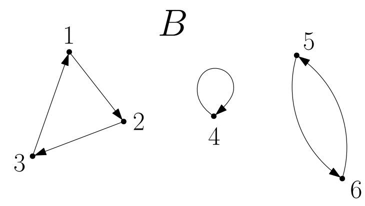
One Row Notation. If $A \in S_n$ then we may represent A by the sequence $(A(1), A(2), \ldots, A(n))$. So, the i^{th} term of this sequence is $A(i)$. This is just two row notation without the top row.

Example Continued: One row notation for our example permutation B is

$$B = (231465)$$

Graphic Representation. We can denote a permutation $A \in S_n$ by drawing a dot labelled i in the plane for $1 \leq i \leq n$ and then drawing an arrow from i to j if $A(i) = j$.

Example Continued: We can describe B with the following graphic representation.



Definition. Let $A \in S_n$. A *cycle of* A is a sequence $(a_1 a_2 \ldots a_k)$ of distinct numbers from the set $\{1, \ldots, n\}$ where $A(a_i) = a_{i+1}$ for $1 \leq i \leq k-1$ and $A(a_k) = a_1$. We call k the length of the cycle.

Example Continued: The permutation B has cycles (123), (4), and (56). Note: (231) is also a cycle of B, but we treat this as the same as (123)

Extended Cycle Notation: The *extended cycle representation* of a permutation $A \in S_n$ is a list of cycles of A so that every element in $\{1, 2, \ldots, n\}$ appears in exactly one of these cycles.

Example Continued: Our permutation B has extended cycle notation

$$B = (123)(4)(56)$$

Cycle Notation: We modify the extended cycle notation by omitting cycles of length one.

Example Continued: Our permutation B has cycle notation

$$B = (123)(56)$$

Composing permutations: To compose two permutations in cycle notation, write them down and then work right to left to see where each element goes. For instance taking $B = (123)(56)$ and $C = (1356)(24)$ we find

$$BC = (123)(56)(1356)(24) = (1)(2436)(5) = (2436)$$